ON THE NATURE AND PROPERTIES OF THE ACONIC FUNCTION OF SIX VECTORS

By

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Sir William Rowan Hamilton entered into some explanatory details respecting the nature and properties of that aconic function of six vectors, of which he had spoken in a recent communication with reference to a certain generalization or extension of Pascal's theorem, conducting to a relation between ten points on a surface of the second order.

In the Proceedings of the Royal Irish Academy for July 20, 1846, it was remarked by Sir W. Rowan Hamilton, that the theorem of Pascal might, in the calculus of quaternions, be expressed by the following general equation of cones of the second degree:

\[ S \cdot \beta \beta' \beta'' = 0, \]

where

\[
\beta = V(V \cdot \alpha^{I} \cdot V \cdot \alpha^{III} \alpha^{IV}), \]
\[
\beta' = V(V \cdot \alpha^{I} \alpha^{II} \cdot V \cdot \alpha^{IV} \alpha^{V}), \]
\[
\beta'' = V(V \cdot \alpha^{II} \alpha^{III} \cdot V \cdot \alpha^{V} \alpha); \]

\( \alpha, \alpha^{I}, \alpha^{II}, \alpha^{III}, \alpha^{IV}, \alpha^{V} \) being any six homoconic vectors, and the letters S and V being the characteristics of the operations of taking respectively the scalar and vector parts of a quaternion. Now it is precisely that function of six vectors \( \alpha \ldots \alpha^{V} \), which was denoted in that communication of 1846, by \( S \cdot \beta \beta' \beta'' \), to which it has since appeared to Sir W. Rowan Hamilton convenient to give the name of the aconic (or heteroconic) function of those six vectors; because in the more general case, when they are not sides of any common cone of the second degree, this function no longer vanishes, but acquires some positive or negative value.

One of the most important properties of this aconic function is, that it changes its sign without otherwise changing its value, whenever any two of the six vectors on which it depends change places among themselves. Admitting this property, which there are many ways of easily proving by the general rules of quaternions, and observing that the following function of four vectors, \( \alpha^{VI}, \alpha^{VII}, \alpha^{VIII}, \alpha^{IX} \), namely

\[ S \cdot (\alpha^{VI} - \alpha^{VII})(\alpha^{VII} - \alpha^{VIII})(\alpha^{VIII} - \alpha^{IX}), \]

can be shewn to change sign in like manner, for any binary interchange among the vectors on which it depends, and to vanish when any two of them are equal; denoting also, for conciseness, the former function by 012345, the latter by 6789, and their product by 012345 . 6789;
Sir W. Rowan Hamilton proceeds to form, by binary transpositions of these figures, or of the vectors which they denote, from one factor of each product to the other, accompanied with a change of the algebraic sign prefixed to each such product as a term, for every such binary interchange, a system of 210 terms, namely

\[ + 012345 \cdot 6789 - 012346 \cdot 5789 \\
+ 012347 \cdot 5689 - 012348 \cdot 5679 \\
+ 012349 \cdot 5678 - 012359 \cdot 4678 \\
+ 012358 \cdot 4679 - 012357 \cdot 4689 \\
+ 012356 \cdot 4789 - 012376 \cdot 4589 \\
+ \text{(a hundred other products)} - \text{(a hundred other products)}; \]

these remaining terms being easily formed in succession, according to the lately mentioned law. And to the algebraic sum of all these 210 terms, of which each separately is a positive or negative number,—its positive or negative character depending of course not alone on the prefixed sign + or −, but also on the positive or negative characters of the factors of the product, which enters with that sign prefixed into the term,—Sir W. Rowan Hamilton proposes to give the name of the heterodeuteric, or (more shortly) the adeuteric function of the ten vectors \( \alpha \ldots \alpha^9 \), for a reason which will presently appear.

To make the formation of this function of ten vectors more completely clear, it may be observed, that the function of four vectors, which has been above denoted by the symbol 6789, is easily found to represent the sextupled volume of the pyramid, whose corners are the terminations of the four vectors (all drawn from one common origin); this form being regarded as positive or negative, according to the character (as right handed or left handed) of a certain rotation; which character or direction is reversed when any two of the four vectors, and therefore, also, their terminations, are made to change places with each other. On this account the lately mentioned function of four vectors may be called their pyramidal function; and then the foregoing rule for the composition of the adeuteric function may be expressed in words as follows:—Starting with any one set of four vectors, form their pyramidal function, and multiply it by the aconic function of the remaining six, out of the proposed ten vectors, arranging the vectors of each set in any one selected order. Choose any vector of the four, and any other of the six, and interchange these two vectors, without altering the arrangement of the rest, so as to form a new group of four vectors, and another new group of six; and multiply the pyramidal function of the former group by the aconic function of the latter. Proceeding thus, we can gradually and successively form all the 210 possible groups or sets of four vectors, accompanied each with another set of six; and the four or the six vectors in each set will have an arrangement among themselves, determined by the foregoing process; so that the 210 pyramidal and the 210 aconic functions have each a determined value, including a known positive or negative sign or character. Each of the 210 products, thus obtained, is therefore itself also determinate, as being equal to some one positive or negative number, of which the sign as well as the absolute value can be definitely found, and may be considered as being known, before we introduce or employ any rule for combining or incorporating these various products among themselves, by any additions or subtractions. But if we now employ, for such incorporation, the rule that all those products
which have been formed by any even number of binary interchanges, from the product first assumed, which we may still suppose to be

012345 . 6789,

are to be algebraically added thereto; while, on the contrary, all which are formed from that original product by any odd number of binary interchanges are to be algebraically subtracted from it: we shall complete (as was before more briefly stated) the determination of that function of ten vectors, 0 to 9, which was lately called the adeuteric.

Indeed, it may for a moment still appear that this function is in some degree indeterminate, because there may be many different ways of passing, by successive binary interchanges, from one given set of six, and a companion set of four vectors, to a second given set of six, with its own companion set of four. For example, we passed from the first to the tenth of the products already written, by a succession of nine binary interchanges, which may be indicated thus:

56, 67, 78, 89, 45, 98, 87, 76, 57.

But we might also have passed from the same first product,

+012345 . 6789

by the two binary interchanges 47, 56, to this other product and sign,

+012376 . 5489,

where the sign + is prefixed, on account of their being now an even number (two) of such changes. On the other hand, the odd number (nine), of binary interchanges above described, had given the term

−012376 . 4589.

But because, by the properties of the pyramidal function of four vectors above referred to, we have

+5489 = −4589,

the two terms thus obtained differ only in appearance from each other. And similar reductions will in every other case hold good, in virtue of the properties of the pyramidal and aconic functions, combined with a principle respecting transpositions of symbols (which probably is well known): namely, that if a set of \( n \) symbols (as here the ten figures from 0 to 9) be brought in any two different ways, by any two numbers \( l \) and \( m \) of binary interchanges, to any one other arrangement, the difference \( m - l \) of these two numbers is even.

The value (including sign) of the foregoing adeuteric function, of any ten determined vectors, is therefore itself completely determined, if we fix (as before) the arrangement of the ten vectors in the first of the 210 terms from which the others are to be derived: because the value of each separate term becomes then fixed, although the forms of these various terms may undergo considerable variations, by interchanges conducted as above. If then we choose any two of the ten vectors, suppose those numbered 4 and 7, we may prepare the expression of the adeuteric function as follows. We may first collect into one group the 70 terms in
which these two vectors both enter into one common aconic function; and may call the sum of all these terms, Polynome I. We may next collect into a second group all those other terms, in number 28, for each of which the two selected vectors both enter into the composition of one common pyramidal function; and may call the sum of these 28 terms, Polynome II. And finally, we may arrange (after certain permitted transpositions) the remaining 112 terms into 56 pairs, such as

\[ +012345 \cdot 6789 - 012375 \cdot 6489, \]

and

\[ -012346 \cdot 5789 + 012376 \cdot 5489. \]

and may call the sum of these 56 pairs of terms, Polynome III; the rule of pairing being here, that the two selected vectors (in the present case 4 and 7) shall be interchanged in passing from any one term of the pair to the other, with a change of sign as before. But when the expression of the adeuteric has been thus prepared, it becomes clear that each of its three partial polynomes is changed to its own negative, when the two selected vectors are interchanged. In fact, each term of the first polynome changes sign, by this interchange, in virtue of the properties of the aconic function of six vectors. Again, each term of the second polynome in like manner changes sign, on account of the properties of the pyramidal function of four vectors. And finally, each pair of terms in the third polynome changes sign, from the manner in which that pair is composed. On the whole then we must infer, that the sum of these three polynomes, or the function above is called the adeuteric, changes sign without otherwise changing value, when any two of the ten vectors on which it depends are made to change places with each other: whence it is very easy to infer, that this adeuteric function vanishes, when any two of its ten vectors become equal.

Now the aconic function is of the second degree, with respect to each of the six vectors on which it depends; while the pyramidal function is easily shewn to be only of the first degree, with respect to each of the four other vectors which enter into its composition. Hence each of the 210 terms of the adeuteric rises no higher than the second degree; and if we equate this adeuteric function to zero, we thereby oblige any one of the ten vectors to terminate on a given surface of the second order, if the other nine vectors be given. But it has been seen, that the adeuteric vanishes, when any two of its ten vectors are made equal to each other; the surface which is thus the locus of the extremity of the tenth vector, must, therefore, pass through the nine points in which the nine other vectors respectively terminate. On this account the ten vectors, or their extremities, may be said to be, under this condition, homodeuteric, as belonging all to one common surface of the second order. And thus we at once justify, by contrast, the foregoing appellation of the adeuteric function, and also see that to equate (as above) this adeuteric to zero, is to establish what may be called the equation of homodeuterism, as in fact it was so called in a recent communication to the Academy; while, as an abbreviation of the recent notation, we may now write that equation as follows:

\[ \Sigma(\pm012345 \cdot 6789) = 0; \]

where the sum in the left-hand member represents the adeuteric function.

What has been shewn respecting the composition of this adeuteric, may naturally produce a wish to possess some geometrical rule for constructing the aconic function (012345),
of any six given vectors; and the quaternion expression for that function enables us easily to assign such a rule. For this purpose, let A, B, C, D, E, F be the six points at which the six vectors lately numbered as 0, 1, 2, 3, 4, 5 terminate, being supposed to be all drawn from some assumed and common origin O; while G, H, I, K may denote the four other points, through which the surface of the second order passes, when the equation of homodeuterism is satisfied, and which are the terminations of the four other vectors above numbered as 6, 7, 8, 9. The aconic function, above denoted by 012345, of the six vectors OA, OB, OC, OD, OE, OF, which terminate generally at the six corners of a gauche hexagon ABCDEF, may now be concisely expressed by the symbol

\[ O \cdot \text{ABCDEF}; \]

or even simply by ABCDEF, the reference to an origin being understood. To construct it, Sir W. Rowan Hamilton constructs first the six vectors

\[ V \cdot \alpha \alpha^I, \quad V \cdot \alpha^I \alpha^II, \quad V \cdot \alpha^II \alpha^III, \quad V \cdot \alpha^III \alpha^IV, \quad V \cdot \alpha^IV \alpha^V, \quad V \cdot \alpha^V \alpha, \]

and then the three other vectors \( \beta, \beta', \beta'' \), which depend on these, in order to form thence that scalar \( S \cdot \beta \beta' \beta'' \), which, by what was stated near the commencement of the present Abstract, is the aconic function required. It will be seen that all the steps of the following construction of that function are in this way obvious consequences from the quaternion expression above given. The construction itself was communicated to a few scientific friends of his about the end of August and beginning of September, 1849, and has since been publicly stated at the Edinburgh Meeting of the British Association in 1850, although it has not hitherto been printed.

Regarding the given and gauche hexagon, ABCDEF, as a sort of base of a hexahedral angle, of which the vertex is the assumed point o, Sir W. Rowan Hamilton represents the doubled areas of the six plane and triangular faces of this angle, namely,

\[ \text{AOB, BOC, COD, DOE, EOF, FOA,} \]

by six right lines from the vertex,

\[ \text{OL, OM, ON, OL', OM', ON'}, \]

which are respectively normals to the six faces, and are distinguished from their own opposites by a simple and uniform rule of rotation: for example, the line OL contains as many linear units as the doubled area of the triangle AOB (to the plane of which it is perpendicular) contains units of area; and the notation round OL from OA to OB is right-handed. The doubled areas of the three new triangles,

\[ \text{LOL', MOM', NON'}, \]

are next to be represented, on the same general plan, by three new lines from the vertex,

\[ \text{OL'', OM'', ON''}; \]

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which three lines will thus be the intersection of the three pairs of opposite faces of the hexahedral angle, and consequently will, by Pascal’s theorem, be situated in one common plane, if the given hexagon ABCDEF can be inscribed in a cone of the second degree, with the point o for its vertex. But in the more general case, when the given hexagon cannot be so inscribed, in any such cone with that assumed point for vertex, we can construct a parallelepipedon with the three last lines, OL", OM", ON", for three adjacent edges: and the volume of this solid is the geometrical representation which Sir W. Rowan Hamilton’s method assigns for what he calls (as above) the aconic function of the six given vectors, or of the six given points A, B, C, D, E, F, in which those vectors terminate, or of the (generally gauche) hexagon of which those points are corners. And with respect to the sign of this function, it is to be regarded as being positive or negative, according as the rotation round ON" from OM" towards OL", is to the right hand or to the left.

Such then is the construction of the aconic function, 012345, or ABCDEF; and it is still more easy to construct the pyramidal function 6789, which may also be denoted by the symbol GHIK; since the absolute value of this function is constructed (as above remarked) by the sextuple volume of the pyramid, which has the four points G, H, I, K for corners, or by the volume of the parallelepipedon which has GH, GI, GK, for edges; while the quaternion expression assigned near the commencement of this Abstract, admits of being thus written,

\[ S \cdot (\alpha^IX - \alpha^VI)(\alpha^VII - \alpha^VI)(\alpha^VII - \alpha^VI), \]

and conducts to the regarding this volume, or the function 6789, or GHIK, as being positive when the rotation round GH from GI towards GK is right-handed, but negative in the contrary case. And the aconic and pyramidal functions having thus been separately constructed, they have only to be combined with each other, according to the law already stated, in order to assign a geometrical signification to each term of the adeuteric function, namely, the sum,

\[ \Sigma(\pm ABCDEF \cdot GHIK); \]

and also to the equation of homodeuterism, which may now be written thus (as in a recent communication to the Academy),

\[ \Sigma(\pm ABCDEF \cdot GHIK) = 0, \]

and which expresses that the ten points, A, B, . . . , K, are situated upon one common surface of the second order. And if we place the arbitrary origin o at one of the ten points, the number of terms in the adeuteric function, or in the equation of homodeuterism, is easily seen to reduce itself, then, from 210 to 84.

If the thirty co-ordinates of the ten points were substituted in the function above called the adeuteric, the resulting expression could doubtless only differ by some numerical coefficient from that determinant which might otherwise be found, as the result of the elimination of the nine coefficients A, B, C, D, E, F, G, H, I, between the equations,

\[ Ax_0^2 + By_0^2 + Cz_0^2 + Dy_0z_0 + Ez_0x_0 + Fx_0y_0 + Gx_0 + Hy_0 + Iz_0 + 1 = 0, \]

\[ \ldots \ldots \]

\[ Ax_9^2 + By_9^2 + Cz_9^2 + Dy_9z_9 + Ez_9x_9 + Fx_9y_9 + Gx_9 + Hy_9 + Iz_9 + 1 = 0. \]
And Sir W. Rowan Hamilton has much pleasure in referring to a paper by Mr. Cayley, printed near the commencement of the Fourth Volume of the Cambridge Mathematical Journal, on Pascal’s Theorem considered in connexion with determinants, which paper had not been noticed by the present writer till his attention was called to it by a friend to whom he had communicated the above-stated construction. But while gladly acknowledging the great mathematical learning and originality exhibited in that and every paper by Mr. Cayley, Sir W. Rowan Hamilton thinks it right to state, that he was led to his own results, respecting the relation (above assigned) between ten points on the surface of the second order, not by any system of co-ordinates, but by the considerations of vectors, and by seeking to extend to ellipsoids the results respecting cones, which he had submitted to the Academy in July, 1846, and had also published in the Philosophical Magazine for the following month, as derived from the Calculus of Quaternions.