ON ANHARMONIC CO-ORDINATES

By

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Sir William Rowan Hamilton.

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[April 9th, 1860]

1. Let \( ABC \) be any given triangle; and let \( O, P \) be any two points in its plane, whereof \( O \) shall be supposed to be given or constant, but \( P \) variable. Then, by a well-known theorem, respecting the six segments into which the sides are cut by right lines drawn from the vertices of a triangle to any common point the three following anharmonics of pencils have a product equal to positive unity:—

\[
(A \cdot PCOB) \cdot (B \cdot PAOC) \cdot (C \cdot PBOA) = +1.
\]

It is, therefore, allowed to establish the following system of three equations, of which any one is a consequence of the other two:—

\[
\frac{y}{z} = (A \cdot PCOB); \quad \frac{z}{x} = (B \cdot PAOC); \quad \frac{x}{y} = (C \cdot PBOA);
\]

and, when this is done, I call the three quantities \( x, y, z \), or any quantities proportional to them, the Anharmonic Co-ordinates of the Point \( P \), with respect to the given triangle \( ABC \), and to the given point \( O \). And I denote that point \( P \) by the symbol,

\[
P = (x, y, z); \quad \text{or,} \quad P = (tx, ty, tz); \quad &c.
\]
2. When the variable point \( p \) takes the given position \( o \), the three anharmonics of pencils above mentioned become each equal to unity; so that we may write then,

\[
x = y = z = 1.
\]

The given point \( o \) is therefore denoted by the symbol,

\[
o = (1, 1, 1);
\]

on which account I call it the *Unit-Point*.

3. When the variable point \( p \) comes to coincide with the given point \( a \), so as to be at the vertex of the first pencil, but on the second ray of the second pencil, and on the fourth ray of the third, without being at the vertex of either of the two latter pencils, then the first anharmonic becomes indeterminate, but the second is equal to zero, and the third is infinite. We are, therefore, to consider \( y \) and \( z \), but not \( x \), as vanishing for this position of \( p \); and consequently may write,

\[
a = (1, 0, 0).
\]

In like manner,

\[
b = (0, 1, 0), \quad \text{and} \quad c = (0, 0, 1);
\]

and on account of these simple representations of its three corners, I call the given triangle \( \triangle abc \) the *Unit-Triangle*.

![Diagram](image-url)

4. Again, let the sides of this given triangle \( \triangle abc \) be cut by a given transversal \( A'B'C' \), and by a variable transversal \( LMN \). Then, by another very well-known theorem respecting segments, we shall have the relation,

\[
(LBA'C) \cdot (MCB'A) \cdot (NAC'B) = +1;
\]

it is therefore permitted to establish the three equations,

\[
\frac{m}{n} = (LBA'C), \quad \frac{n}{l} = (MCB'A), \quad \frac{l}{m} = (NAC'B);
\]
where \( l, m, n \), or any quantities proportional to them, are what I call the *Anharmonic Coordinates of the Line* \( LMN \), with respect to the given triangle \( ABC \), and the given transversal \( \Lambda'B'C' \). And I *denote* the line \( LMN \) by the symbol,

\[
LMN = [l, m, n].
\]

For example, if this variable line come to coincide with the given line \( \Lambda'B'C' \), then

\[
l = m = n;
\]

so that this given line may be thus denoted,

\[
\overline{\Lambda'B'C'} = [1, 1, 1];
\]

on which account I call the *given transversal* \( \Lambda'B'C' \) the *Unit-Line* of the Figure. The *sides*, \( BC, \&c. \), of the given triangle \( ABC \), take on this plan the symbols \([1, 0, 0], [0, 1, 0], [0, 0, 1]\).

5. Suppose now that the *unit-point* and the *unit-line* are related to each other, as being (in a known sense) *pole* and *polar*, with respect to the given or *unit-triangle*; or, in other words, let the lines \( OA, OB, OC \) be supposed to meet the sides \( BC, CA, AB \) of that given triangle, in points \( A'', B'', C'' \), which are, with respect to those sides, the harmonic conjugates of the points \( A', B', C' \), in which the same sides are cut by the given transversal \( \Lambda'B'C' \). Also, let the variable *point* \( P \) be situated upon the variable *line* \( LMN \); and let \( Q, R, S \) be the intersection of \( AP, BP, CP \) with \( BC, CA, AB \). Then, because

\[
(ba'ca'') = (cb'ab'') = (ac'bc'') = -1,
\]

\[
\]
we have
\[
\begin{align*}
\frac{-m}{n} &= (LBA''C), \quad \frac{-n}{l} = (MCA''A), \quad \frac{-l}{m} = (NAC''B), \\
\frac{-n}{m} &= (LCA''B), \quad \frac{-l}{n} = (MAB''C), \quad \frac{-m}{l} = (NBC''A);
\end{align*}
\]
as well as
\[
\begin{align*}
\frac{y}{z} &= (QCA''B), \quad \frac{z}{x} = (RAB''C), \quad \frac{x}{y} = (SBC''A), \\
\frac{z}{y} &= (QBA''C), \quad \frac{x}{z} = (RCB''A), \quad \frac{y}{x} = (SAC''B);
\end{align*}
\]
and therefore,
\[
\frac{-lx}{nz} = (MARC); \quad \frac{-my}{nz} = (LBQC).
\]
But, by the pencil through \( P \),
\[
(MARC) = (LQBC);
\]
and by the definition of the symbol \((ABCD)\), for any four collinear points,
\[
(ABCD) = \frac{AB}{BC} \cdot \frac{CD}{DA},
\]
which is here throughout adopted, we have the identity,
\[
(ABCD) + (ACBD) = 1;
\]
therefore
\[
(MARC) + (LBQC) = 1,
\]
or,
\[
lx + my + nz = 0.
\]

6. We arrive then at the following Theorem, which is of fundamental importance in the present system of Anharmonic Co-ordinates:—

“If the unit-point \( O \) be the pole of the unit-line \( A'B'C' \), with respect to the unit-triangle \( ABC \), and if a variable point \( P \), or \((x, y, z)\), be situated anywhere on a variable right line \( LMN \), or \([l, m, n]\), then the sum of the products of the corresponding co-ordinates of point and line is zero.”

7. It may already be considered as an evident consequence of this Theorem, that any homogeneous equation of the \( p \)th dimension,
\[
f_p(x, y, z) = 0,
\]
represents a curve of the \( p \)th order, considered as the locus of the variable point \( P \); and that any homogeneous equation of the \( q \)th dimension, of the form
\[
F_q(l, m, n) = 0,
\]
may in like manner be considered as the tangential equation of a curve of the \( q \)th class, which is the envelope of the variable line \( LMN \). But any examples of such applications must be reserved for a future communication. Meantime, I may just mention that I have been, for some time back, in possession of an analogous method for treating Points, Lines, Planes, Curves, and Surfaces in Space, by a system of Anharmonic Co-ordinates.
8. As regards the advantages of the Method which has been thus briefly sketched, the first may be said to be its geometrical interpretability, in a manner unaffected by perspective. The relations, whether between variables or between constants, which enter into the formula of this method, are all projective; and because they all depend upon, and are referred to, anharmonic functions, of groups or of pencils.

9. In the second place, we may remark that the great principle of geometrical Duality is recognized from the very outset. Confining ourselves, for the moment (as in the foregoing articles), to figures in a given plane, we have seen that the anharmonic co-ordinates of a point, and those of a right line, are deduced by processes absolutely similar, the one from a system of four given points, and the other from a system of four given right lines. And the fundamental equation \((lx + my + nz = 0)\) which has been found to connect these two systems of co-ordinates, is evidently one of the most perfect symmetry, as regards points and lines. An analogous symmetry will show itself afterwards, in relation to points and planes.

10. The third advantage of the anharmonic method may be stated to consist in its possessing an increased number of disposable constants. Thus, within the plane, trilinear co-ordinates give us only six such constants, corresponding to the three disposable positions of the sides of that assumed triangle, to the perpendicular distances from which the co-ordinates are supposed to be proportional; but anharmonics, by admitting an arbitrary unit-point, enable us to treat two other constants as disposable, the number of such constants being thus raised from six to eight. Again, in space, whereas quadriplanar co-ordinates, considered as the ratios of the distances from four assumed planes, allow of only twelve disposable constants, corresponding to the possible selection of the four planes of reference, anharmonic co-ordinates, on the contrary, which admit either five planes or five points as data, and which might, therefore, be called quinquiplanar or quinquipunctual, permit us to dispose of no fewer than fifteen constants as arbitrary, in the general treatment of surfaces.

[May 28th, 1860]

11. To myself, it naturally appears as a fourth advantage of the anharmonic method, that it is found to harmonize well with the method of quaternions, and was in fact suggested thereby; though not without suggestions from other methods previously known.

12. Thus, if \(\alpha, \beta, \gamma\) denote the given vectors, OA, OB, OC, from a given origin O, while \(a, b, c\) are three given and constant scalars, but \(t, u, v\) are three variable scalars, subject to the condition that their sum is zero, 

\[ t + u + v = 0; \]

then the equation, 

\[ \text{OP} = \rho = \frac{t^r a\alpha + u^r b\beta + v^r c\gamma}{t^r \alpha + u^r \beta + v^r c}, \]

in which \(\rho\) is any positive and whole exponent, expresses general that the locus of the point \(P\) is a curve of the \(r^{th}\) order, in the given plane of \(ABC\); which curve has the property, that it is met in \(r\) coincident points, by any one of the three sides of the given triangle \(ABC\). But the coefficients \(t^r, u^r, v^r\) are examples here of what have been above called anharmonic co-ordinates.
13. Proceeding to space, let $a$, $b$, $c$, $d$ be the four corners of a given triangular pyramid, and let $e$ be any fifth given point, which is not on any one of the four faces of that pyramid. Let $p$ be any sixth point of space; and let $xyzw$ be four positive or negative numbers, such that

$$\frac{x}{w} = (bc \cdot AEP), \quad \frac{y}{w} = (ca \cdot BDP), \quad \frac{z}{w} = (ab \cdot CEP);$$

the right-hand member of these equations representing *anharmonics of pencils of planes*, in a way which is easily understood, with the help of the definition (5) of the symbol $(ABCD)$. Then I call $x$, $y$, $z$, $w$ (or any numbers proportional to them), the *Anharmonic Co-ordinates* of the Point $p$, with respect to what may be said to be the *Unit-Pyramid*, $abcd$, because its corners may (on the present plan) be thus denoted,

$$a = (1, 0, 0, 0); \quad b = (0, 1, 0, 0); \quad c = (0, 0, 1, 0); \quad d = (0, 0, 0, 1);$$

and with respect to that fifth given point $e$, which may be called the *Unit-Point*, because its symbol, in the present system, may be thus written:—

$$e = (1, 1, 1, 1).$$

And I denote the general or variable point by the symbol,

$$p = (x, y, z, w).$$

14. When we have thus five given points, $a \ldots e$, of which no four are situated in any common plane, we can connect any two of them by a right line, and the three others by a plane, and determine the point in which these last intersect each other, deriving in this way as system of ten lines, ten planes, and ten points, whereof the latter may be thus denoted:

$$A' = bc \cdot ADE = (0, 1, 1, 0), \quad B' = &c., \quad C' = &c.;$$
$$A_1 = AE \cdot BCD = (0, 1, 1, 1), \quad B_1 = &c., \quad C_1 = &c.;$$
$$A_2 = AD \cdot BCE = (1, 0, 0, 1), \quad B_2 = &c., \quad C_2 = &c.;$$
$$D_1 = DE \cdot ABC = (1, 1, 1, 0);$$

and the harmonic conjugates of these last points, with respect to the ten given lines on which they are situated, may on the same plan be represented by the following symbols:—

$$A'' = (0, 1, -1, 0), \quad B'' = &c., \quad C'' = &c.;$$
$$A'_1 = (2, 1, 1, 1), \quad B'_1 = &c., \quad C'_1 = &c.;$$
$$A'_2 = (1, 0, 0, -1), \quad B'_2 = &c., \quad C'_2 = &c.;$$
$$D_1 = (1, 1, 1, 2);$$

so that

$$(BA'CA'') = \ldots = (EA_1AA'_1) = \ldots = (DA_2AA'_2) = \ldots = (ED_1DD'_1) = -1.$$
15. Let any plane $\Pi$ intersect the three given lines $DA, DB, DC$, in points $Q, R, S$; and let $l m n r$ be any positive or negative numbers, such that

$$\frac{l}{r} = (DA'_{2}AQ), \quad \frac{m}{r} = (DB'_{2}BR), \quad \frac{n}{r} = (DC'_{2}CS);$$

then I call $l, m, n, r$, or any numbers proportional to them, the Anharmonic Co-ordinates of the Plane $\Pi$; which plane I also denote by the Symbol,

$$\Pi = [l, m, n, r].$$

In particular the four faces of the unit pyramid come thus to be denoted by the symbols,

- $BCD = [1, 0, 0, 0]$,
- $CAD = [0, 1, 0, 0]$,
- $ABD = [0, 0, 1, 0]$,
- $ABC = [0, 0, 0, 1]$;

and the six planes through its edges, and through the unit point, are denoted thus:

- $BCE = [1, 0, 0, -1]$,
- $CAE = [0, 1, 0, -1]$,
- $ABE = [0, 0, 1, -1]$,
- $ADE = [0, 1, -1, 0]$,
- $BDE = [-1, 0, 1, 0]$,
- $CDE = [1, -1, 0, 0]$;

in connexion with which last planes it may be remarked that we have, generally, as a consequence of the foregoing definitions, the formulae,

$$\frac{n}{m} = (BA''CL), \quad \frac{l}{n} = (CB''AM), \quad \frac{m}{l} = (AC''BN),$$

if $L, M, N$ be the points in which a variable plane $\Pi$ intersects the sides $BC, &c.,$ of the given triangle $ABC$: as we have also, generally,

$$\frac{z}{y} = (AD \cdot CEBP), \quad \frac{x}{z} = (BD \cdot AECP), \quad \frac{y}{x} = (CD \cdot BEAP).$$

16. If a point, $P = (x, y, z, w)$, be situated on a plane, $\Pi = [l m n r]$, then I find that the following relation between their co-ordinates exists, which is entirely analogous to that already assigned (6) for the case of a point and line in a given plane, and is of fundamental importance in the application of the present Anharmonic Method to space:

$$lx + my + nz + rw = 0;$$

or in words, “the sum of the products of corresponding co-ordinates, of point and plane, is zero.”

For example, all planes through the unit point $(1, 1, 1, 1)$ are subject to the condition,

$$l + m + n + r = 0,$$

as may be seen for the six planes $(15)$ already drawn through that point $E$; and the six points $A''B''C'' A''_{2}B''_{2}C''_{2}$ $(14)$, in which the six edges $BC, CA, AB, DA, DB, DC$, of the given or unit pyramid $ABCD$, intersect the six corresponding edges of the inscribed and homologous pyramid $A_{1}B_{1}C_{1}D_{1}$, with the unit point $E$ for their centre of homology, are all ranged on one common plane of homology, of which the equation and the symbol may be thus written,

$$x + y + r + w = 0, \quad [E] = [1, 1, 1, 1],$$

and which may be called (comp. 4) the Unit-Plane.
17. Any four collinear points, \( p_0, p_1, p_2, p_3 \), have their anharmonic symbols connected by two equations of the forms,

\[
(p_1) = t(p_0) + u(p_2), \quad (p_3) = t'(p_0) + u'(p_2)
\]
each including four ordinary linear equations between the co-ordinates of the four points, such as

\[
x_1 = tx_0 + ux_2, \quad y_1 = ty_0 + uy_2, \quad \&c.;
\]
and the anharmonic of their group is then found to be expressed by the formula,

\[
(p_0, p_1, p_2, p_3) = \frac{ut'}{tu'}.
\]

And similarly, if any four planes \( \Pi_0 \ldots \Pi_3 \) be collinear (that is, if they have any one right line common to them all), their symbols satisfy two linear equations of the corresponding forms,

\[
[I_1] = t[I_0] + u[I_2], \quad [I_3] = t'[I_0] + u'[I_2];
\]
and the anharmonic of the pencil is,

\[
(\Pi_0\Pi_1\Pi_2\Pi_3) = \frac{ut'}{tu'}.
\]

18. If \( \phi(xyzw) \) be any rational fraction, the numerator and denominator of which are any two given homogeneous and linear functions of the co-ordinates of a variable point; and if we determine a line \( \Lambda \), and three planes \( \Pi_0, \Pi_1, \Pi_2 \) through that line, by the four local equations,

\[
\phi = 0, \quad \phi = \infty, \quad \phi = 1, \quad \phi = 0;
\]
then I find that the function \( \phi \) may be expressed as the anharmonic of a pencil of planes, as follows:

\[
\phi(xyzw) = (\Pi_0 \Pi_1 \Pi_2 \Pi);
\]
where \( \Pi \) is the variable plane \( \Lambda \Pi \), which passes through the fixed line \( \Lambda \), and through the variable point \( p = (xyzw) \).

19. And in like manner, as the geometrical dual (9) of this last theorem (18), if \( \Phi(lmnr) \) be any rational fraction, of which the numerator and denominator are any two given functions, homogeneous and linear, of the co-ordinates of a variable plane; and if we determine a line \( \Lambda \), and three points \( p_0, p_1, p_2 \), on that line, by the four tangential equations,

\[
\Phi = 0, \quad \Phi = \infty, \quad \Phi = 1, \quad \Phi = 0;
\]
I find that the proposed function \( \Phi \) may then be thus expressed as the anharmonic of a group of points.

\[
\Phi(lmnr) = (p_0\ p_1\ p_2\ p);
\]
p here denoting the variable point \( \Lambda \cdot \Pi \), in which the fixed line \( \Lambda \) intersects the variable plane \( \Pi = [lmnr] \).
20. All problems respecting *intersections of lines with planes*, &c., are resolved, with the help of the Fundamental Theorem (16) respecting the relation which exists between the anharmonic co-ordinates of point and plane, as easily by the present method, as by the known method of *quadriplanar co-ordinates* (10); and indeed, by the very same mechanism, of which it is therefore unnecessary here to speak.

But it may be proper to say a few words respecting the application of the anharmonic method to *Surfaces* (7); although here again the known mechanism of *calculation* may in great part be preserved unchanged, and only the *interpretations* need be new.

21. In general, it is easy to see (comp. 7) that, in the present method, as in older ones, the *order* of a curved surface is denoted by the *degree* of its *local equation*, \( f(xyzw) = 0 \); and that the *class* of the same surface is expressed, in like manner, by the degree of its *tangential equation*, \( F(lmnz) = 0 \): because the former degree (or dimension) determines the *number of points* (distinct or coincident, and real or imaginary), in which the surface, considered as a locus, is intersected by an arbitrary right line; while the latter degree determines the *number of planes* which can be drawn through an arbitrary right line, so as to touch the same surface, considered as an *envelope*. It may be added, that I find the *partial derivatives of each* of these two functions, \( f \) and \( F \), to be proportional to the *co-ordinates* which enter as variables into the *other*; thus we may write

\[
[D_x f, D_y f, D_z f, D_w f],
\]
as the symbol (15) of the *tangent plane* to the *locus* \( f \), at the point \((xyzw)\); and

\[
(D_l F, D_m F, D_n F, D_r F),
\]
as a symbol for the *point of contact* of the *envelope* \( F \), with the plane \([l m n r]\); whence it is easy to conceive how problems respecting the *polar reciprocals* of *surfaces* are to be treated.

22. As a very simple *example*, the surface of the *second order* which passes through the *nine points*, above called \(ABCDA_2'\) \(C'D_2\), is easily found to have for its *local equation* \( 0 = f = xz - yw \); whence the co-ordinates of its tangent plane are, \( l = D_x f = z, m = D_y f = -w, n = D_z f = x, r = D_w f = -y \), and its *tangential equation* is, therefore, \( 0 = F = ln - mr \), so that it is also a surface of the *second class*. In fact it is the *hyperboloid* on which the gauche quadrilateral \(ABCD\) is superscribed, and which passes also through the point \(E\); and the known *double generation*, and *anharmonic properties*, of this surface, may easily be deduced from either of the foregoing forms of its anharmonic equation, whereof the first may (by 13, 15) be expressed as an equality between the anharmonic functions of two *pencils of planes*, in either of the two following ways:— \((BC.AEDP) = (DA.BECP)\); \((AB.CEDP) = (CD.BEAP)\).