ON A NEW SYSTEM OF TWO GENERAL EQUATIONS OF CURVATURE

By

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NOTE ON THE TEXT

This edition is based on the original text published posthumously in volume 9 of the Proceedings of the Royal Irish Academy.

The following obvious typographical errors have been corrected:—
before equation (a), a full stop (period) has been changed to a colon;
in equation (k), ‘(Z – x)’ has been corrected to ‘(Z – z)’;
in equation (o), ‘C = eE″ – e′E’ has been corrected to ‘C = eE′ – e′E’;
equation (q) in the original text was given as

\[(eR^{-1} - eK^{-1})(e''R^{-1} - e''K^{-1}) = (e'R^{-1} - eK^{-1})^2;\]

in equation (r), ‘= 0’ has been appended to the polynomial ‘R^{-2} – FR^{-1} + G’;
in equation (w), the equality sign = has been added.

David R. Wilkins
Dublin, March 2000
ON A NEW SYSTEM OF TWO GENERAL EQUATIONS OF CURVATURE,

Including as easy consequences a new form of the Joint Differential Equation of the Two Lines of Curvature, with a new Proof of their General Rectangularity; and also a new Quadratic for the Joint Determination of the Two Radii of Curvature: all deduced by Gauss’s Second Method, for discussing generally the Properties of a Surface; and the latter being verified by a Comparison of Expressions, for what is called by him the Measure of Curvature.

Sir William Rowan Hamilton

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1. Notwithstanding the great beauty and importance of the investigations of the illustrious GAUSS, contained in his Disquisitiones Generales circa Superficies Curvas, a Memoir which was communicated to the Royal Society of Göttingen in October, 1827, and was printed in Tom. vi. of the Commentationes Recentiores, but of which a Latin reprint has been since very judiciously given, near the beginning of the Second Part (Deuxième Partie, Paris, 1850) of LIOUVILLE’S Edition* of MONGE, it still appears that there is room for some not useless Additions to the Theory of Lines and Radii of Curvature, for any given Curved Surface, when treated by what Gauss calls the Second Method of discussing the General Properties of Surfaces. In fact, the Method here alluded to, and which consists chiefly in treating the three co-ordinates of the surface as being so many functions of two independent variables, does not seem to have been used at all by Gauss, for the determination of the Directions of the Lines of Curvature; and as regards the Radii of Curvature of the Normal Sections which touch these Lines of Curvature, he appears to have employed the Method, only for the Product, and not also for the Sum, of the Reciprocals, of those Two Radii.

2. As regards the notations, let \(x, y, z\) be the rectangular co-ordinates of a point \(P\) upon a surface \((S)\), considered as three functions of two independent variables, \(t\) and \(u\); and let the 15 partial derivatives, or 15 partial differential coefficients, of \(x, y, z\) taken with respect to \(t\) and \(u\), be given by the nine differential expressions:

\[
\begin{align*}
\frac{dx}{dt} &= x' dt + x_t du; \\
\frac{dy}{dt} &= y' dt + y_t du; \\
\frac{dz}{dt} &= z' dt + z_t du; \\
\frac{dx}{du} &= x' du + x_u dt; \\
\frac{dy}{du} &= y' du + y_u dt; \\
\frac{dz}{du} &= z' du + z_u dt.
\end{align*}
\]

(a)

* The foregoing dates, or references, are taken from a note to page 505 of that Edition.
3. Writing also, for abridgment,

\[ e = x'^2 + y'^2 + z'^2; \quad e' = x'x' + y'y' + z'z'; \quad e'' = x'' + y'' + z'' \]

we shall have

\[ ee'' - e'^2 = K^2, \]

if

\[ K^2 = L^2 + M^2 + N^2; \]

and

\[ L = y'z' - z'y'; \quad M = z'x' - x'z'; \quad N = x'y' - y'x'; \]

so that

\[ Lx' + My' + Nz' = 0, \quad Lx' + My' + Nz' = 0. \]

Hence \( K^{-1}L, K^{-1}M, K^{-1}N \) are the direction-cosines of the normal to the surface \((S)\) at \(P\); and if \(x, y, z\) be the co-ordinates of any other point \(Q\) of the same normal, we shall have the equations

\[ K(X - x) = LR; \quad K(Y - y) = MR; \quad K(Z - z) = NR; \]

with

\[ R^2 = (X - x)^2 + (Y - y)^2 + (Z - z)^2; \]

where \(R\) denotes the normal line \(PQ\), considered as changing sign in passing through zero.

4. The following, however, is for some purposes a more convenient form (comp. (f)) of the Equations of the Normal;

\[ (X - x)x' + (Y - y)y' + (Z - z)z' = 0; \]

\[ (X - x)x' + (Y - y)y' + (Z - z)z' = 0. \]

Differentiating these, as if \(X, Y, Z\) were constant, that is, treating the point \(Q\) as an intersection of two consecutive normals, we obtain these two other equations,

\[ \left\{ \begin{array}{l} (X - x) dx' + (Y - y) dy' + (Z - z) dz' = x' dx + y' dy + z' dz; \\ (X - x) dx' + (Y - y) dy' + (Z - z) dz' = x' dx + y' dy + z' dz. \end{array} \right. \]

If, then, we write, for abridgment,

\[ v = du : dt; \quad E = Lx'' + My'' + Nz''; \]

\[ E' = Lx' + My' + Nz'; \quad E'' = Lx' + My' + Nz'; \]

we shall have, by (a) (b) (g), the two important formulæ:

\[ R(E + E'v) = K(e + e'v); \quad R(E''v) = K(e' + e''v); \]

which we propose to call the two general Equations of Curvature.
5. In fact, by elimination of $R$, these equations (m) conduct to a quadratic in $v$, of which the roots may be denoted by $v_1$ and $v_2$, which first presents itself under the form,

\[(e + e'v)(E' + E''v) = (e' + e''v)(E + E'v),\]

but may easily be thus transformed,

\[
\begin{cases}
Av^2 - Bv + C = 0, \\
\text{with } A = e'E'' - e''E', \\
B = e''E - eE'', \\
C = eE' - e'E;
\end{cases}
\]

so that we have the following general relation,

\[eA + e'B + e''C = 0,\]

(of which we shall shortly see the geometrical signification), between the coefficients, $A$, $B$, $C$, of the joint differential equation of the system of the two Lines of Curvature on the surface.

6. The root $v_1$ of the quadratic (o) determines the direction of what may be called the First Line of Curvature, through the point $P$ of that surface; and the First Radius of Curvature, for the same point $P$, or the radius $R_1$ of curvature of the normal section of the surface which touches that first line, may be obtained from either of the two equations (m), as the value of $R$ which corresponds in that equation to the value $v_1$ of $v$. And in like manner, the Second Radius of Curvature of the same surface at the same point has the value $R_2$, which answers to the value $v_2$ of $v$, in each of the same two Equations of Curvature (m). We see, then, that this name for those two equations is justified by observing that when the two independent variables $t$ and $u$ are given or known; and therefore also the seven functions of them, above denoted by $e$, $e'$, $e''$, $E$, $E'$, $E''$, and $K$. The equations (m) are satisfied by two (but only two) systems of values, $v_1$, $R_1$, and $v_2$, $R_2$, of (I.) the differential quotient $v$, or $\frac{du}{dt}$, which determines the direction of a line of curvature on the surface; and (II.) the symbol $R$, which determines (comp. No. 4) at once the length and the direction, of the radius of curvature, corresponding to that line.

7. Instead of eliminating $R$ between the two equations (m), we may begin by eliminating $v$; a process which gives the following quadratic in $R^{-1}$ (the curvature):—

\[(eR^{-1} - EK^{-1})(e''R^{-1} - E''K^{-1}) = (e'R^{-1} - E'K^{-1})^2;\]

or

\[R^{-2} - FR^{-1} + G = 0; \text{ where (because } ee'' - e'^2 = K^2),\]

\[F = R_1^{-1} + R_2^{-1} = (eE'' - 2e'E' + e''E)K^{-3}, \text{ and}\]

\[G = R_1^{-1}R_2^{-1} = (EE'' - E'^2)K^{-4}.\]
We ought, therefore, as a **First General Verification**, to find that this last expression, which may be thus written,

\[(u)\quad G = R_1^{-1}R_2^{-1} = \frac{EE'' - E'E'}{(L^2 + M^2 + N^2)^2},\]

agrees with that reprinted in page 521 of Liouville’s Monge, for what Gauss calls the **Measure of Curvature** \((k)\) of a **Surface**; namely,

\[(v)\quad k = \frac{DD'' - D'D'}{(AA + BB + CC)^2};\]

which accordingly it evidently does, because our symbols \(LMNABC\) represent the combinations which he denotes by \(ABCDD\).

8. As a **Second General Verification**, we may observe that if \(I\) be the **inclination** of any **linear element**, \(du = vdt\), to the **element** \(du = 0\), at the point \(P\), then

\[(w)\quad \tan I = \frac{Kv}{e + e'v};\]

and therefore, that if \(H\) be the **angle** at which the **second crosses the first**, of any two lines represented **jointly** by such an equation as

\[(x)\quad Av^2 - Bv + C = 0, \text{ with } v_1 \text{ and } v_2 \text{ for roots, then}\]

\[(y)\quad \tan H = \tan(I_2 - I_1) = \frac{K(B^2 - 4AC)^{\frac{1}{2}}}{eA + e'B + e''C};\]

so that the **Condition of Rectangularity** \((\cos H = 0)\), for any two such lines, may be thus written:

\[(z)\quad eA + e'B + e''C = 0.\]

But this **condition** \((z)\) had already occurred in No. 5, as an equation \((p)\) which is satisfied generally by the **Lines of Curvature**; we see therefore anew, by this analysis, that those **lines** on **any surface** are in general (as is indeed well known) **orthogonal** to each other.

9. Finally, as a **Third General Verification**, we may assume \(x\) and \(y\) themselves (instead of \(t\) and \(u\)), as the two independent variables of the problem, and then, if we use **Monge’s Notation** of \(p, q, r, s, t\), we shall easily recover all his leading results respecting **Curvatures of Surfaces**, but by transformations on which we cannot here delay.