ON “GAUCHE” POLYGONS IN CENTRAL SURFACES OF THE SECOND ORDER

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Sir William Rowan Hamilton gave an account of some geometrical reasonings, tending to explain and confirm certain results to which he had been previously conducted by the method of quaternions, respecting the inscription of gauche polygons in central surfaces of the second order.

1. It is a very well known property of the conic sections, that if three of the four sides of a plane quadrilateral inscribed in a given plane conic be cut by a rectilinear transversal in three given points, the fourth side of the same variable quadrilateral is cut by the same fixed right line in a fourth point likewise fixed. And whether we refer to the relation of involution discovered by Desargues, or employ other principles, it is easy to extend this property to surfaces of the second order, so far as the inscription in them of plane quadrilaterals is concerned. If then we merely wish to pass from one point \( p \) to another point \( r \) of such a surface, under the condition that some other point \( q \) of the same surface shall exist, such that the two successive and rectilinear chords, \( pq \) and \( qr \), shall pass respectively through some two given guide-points, \( A \) and \( B \), internal or external to the surface; we are allowed to substitute, for this pair of guide-points, another pair, such as \( B' \) and \( A' \), situated on the same straight line \( AB \); and may choose one of these two new points anywhere upon that line, provided that the other be then suitably chosen. In fact, if \( C \) and \( C' \) be the two (real or imaginary) points in which the surface is crossed by the given transversal \( AB \), we have only to take care that the three pairs of points \( AA' \), \( BB' \), \( CC' \), shall be in involution. And it is important to observe, that in order to determine one of the new guide-points, \( B' \) and \( A' \), when the other is given, it is by no means necessary to employ the points \( C \), \( C' \), of intersection of the transversal with the surface, which may be as often imaginary as real. We have only to assume at pleasure a point \( p \) upon the given surface; to draw from it the chords \( PAQ \), \( QBR \); and then if \( A' \) be given, and \( B' \) sought, to draw the two new chords \( RA'S \), \( SB'P \); or else if \( A' \) is to be found from \( B' \), to draw the chords \( PB'S \), \( SA'R \). For example, if we choose to throw off the new guide-point \( B' \) to infinity, or to make it a guide-star, in the direction of the given line \( AB \), we have only to draw, from the assumed initial and superficial point \( P \), a rectilinear chord \( PS \) of the surface, which shall be parallel to \( AB \), and then to join \( SR \), and examine in what point \( A' \) this joining line crosses the given line \( AB \). The point \( A' \) thus found will be entirely independent of the assumed initial point \( P \), and will satisfy the condition required: in such a matter that if, from any other assumed superficial point \( P' \), we draw the chords \( P'AQ' \), \( Q'BR' \), and the parallel \( P'S' \) to \( AB \), the chord \( R'S' \) shall pass through the same point \( A' \). All this follows easily from principles perfectly well known.
2. Since then for two given guide-points we may thus substitute the system of a guide-star and a guide-point, it follows that for three given guide-points we may substitute a guide-star and two guide-points; and, therefore, by a repetition of the same process, may substitute anew a system of two stars and one point. And so proceeding, for a system of $n$ given guide-points, through which $n$ successive and rectilinear chords of the surface are to pass, we may substitute a system of $n - 1$ guide-stars, and of a single guide-point. The problem of inscribing, in a given surface of the second order, a gauche polygon of $n$ sides, which are required to pass successively through $n$ given points, is, therefore, in general, reducible, by operations with straight lines alone, to the problem of inscribing in the same surface another gauche polygon, of which the last side shall pass through a new fixed point, while all its other $(n - 1)$ sides shall be parallel to so many fixed straight lines. And if the first $n$ sides of an inscribed polygon of $n + 1$ sides, $P_1P_2 \ldots P_n$, be obliged to pass, in order, through $n$ given points, $A_1A_2 \ldots A_n$, namely, the side or chord $PP_1$ through $A_1$, &c., it will then be possible, in general, to inscribe also another polygon $PQ_1Q_2 \ldots Q_n$, having the same first and $n$th points, $P$ and $P_n$, and therefore the same final or closing side $PnP$, but having the other $n$ sides different, and such that the $n - 1$ first of these sides, $PQ_1, Q_1Q_2, \ldots Q_{n-2}Q_{n-1}$, shall be respectively parallel to $n - 1$ given right lines, while the $n$th side $Q_{n-1}Q_n$ shall pass through a fixed point $B_n$. The analogous reductions for polygons in conic sections have long been familiar to geometers.

3. Let us now consider the inscribed gauche quadrilateral $PQ_1Q_2Q_3$, of which the four corners coincide with the four points of the last-mentioned polygon. In the plane $Q_1Q_2Q_3$ of the second and third sides of this gauche quadrilateral, draw a new chord $Q_1R_2$, which shall have its direction conjugate to the direction of $PQ_1$, with respect to the given surface. This new direction will itself be fixed, as being parallel to a fixed plane, and conjugate to a fixed direction, not generally conjugate to that plane; and hence in the plane inscribed quadrilateral $R_2Q_1Q_2Q_3$, the three first sides having fixed directions, the fourth side $Q_3R_2$ will also have its direction fixed: which may be proved, either as a limiting form of the theorem referred to in (1), respecting four points in one line, or from principles still more elementary. And there is no difficulty in seeing that because $PQ_1$ and $Q_1R_2$ have fixed and conjugate directions, the chord $PR_2$ is bisected by a fixed diameter of the surface, whose direction is conjugate to both of their’s; or in other words, that if $O$ be the centre of the surface, and if we draw the variable diameter $PON$, the variable chord $NR_2$ will then be parallel to the fixed diameter just mentioned. So far, then, as we only concern ourselves to construct the fourth or closing side $Q_3P$ of the gauche quadrilateral $PQ_1Q_2Q_3$, whose three first sides have given or fixed directions, we may substitute for it another gauche quadrilateral $PQ_1Q_2Q_3$, inscribed in the same surface, and such that while its first side $PN$ passes through the centre $O$, its second and third sides, $NR_2$ and $R_2Q_3$, are parallel to two fixed right lines. In other words, we may substitute, for a system of three guide-stars, a system of the centre and two stars, as guides for the three first sides; or, if we choose, instead of drawing successively three chords, $PQ_1, Q_1Q_2, Q_2Q_3$, parallel to three given lines, we may draw a first chord $PR_2$, so as to be bisected by a given diameter, and then a second chord $R_2Q_3$, parallel to a given right line.

4. Since, for a system of three stars, we may substitute a system of the centre and two stars, it follows that for a system of four stars we may substitute a system of the centre and
three stars; or, by a repetition of the same process, may substitute a system of the centre, the same centre again, and two stars; that is, ultimately, a system of two stars may be substituted for a system of four stars, the two employments of the centre as a guide having simply neutralized each other, as amounting merely to a return from N to P, after having gone from P to the diametrically opposite point N. For five stars we may therefore substitute three; and for six stars we may substitute four, or two. And so proceeding we perceive that, for any proposed system of guide-stars, we may substitute two stars, if the proposed number be even; or three, if that number be odd. And by combining this result with what was found in (2), we see that for any given system of n guide-points we may substitute a system of two stars and a point, if n be odd; or if n be even, then in that case we may substitute a system of three stars and a point: which may again be changed, by (3), to a system of the centre, two stars, and one point.

5. Let us now consider more closely the system of two guide-stars, and one guide-point; and for this purpose let us conceive that the two first sides PQ₁ and Q₁Q₂ of an inscribed gauche quadrilateral PQ₁Q₂P₃ are parallel to two given right lines, while the third side Q₂P₃ is obliged to pass through a fixed point B₃; the first point P, and therefore also the quadrilateral itself, being in other respects variable. In the plane PQ₁Q₂ of the two first sides, which is evidently parallel to a fixed plane, inscribe a chord Q₂S, whose direction shall be conjugate to that of the fixed line OB₃, and therefore shall itself also be fixed, O being still the centre of the surface; and draw the chord PS. Then, in the plane inscribed quadrilateral PQ₁Q₂S, the three first sides have fixed directions, and therefore, by (3), the direction of the fourth side SP is also fixed. In the plane SQ₂P₃, which contains the given point B₃, draw through that point an indefinite right line B₃C₃, parallel to SQ₂; the line so drawn will have a given position, and will be intersected, at some finite or infinite distance from B₃, by the chord SP₃, which is situated in the same plane with it, namely, in the plane SQ₂P₃. But if we consider the section of the surface, which is made by this last plane, and observe that the two first sides of the triangle SQ₂P₃ pass, by the construction, through a star or point at infinity conjugate to B₃, and through the point B₃ itself, we shall see that, in virtue of a well-known and elementary principle respecting triangles in conics, the third side P₃S must pass through the point D₃, if D₃ be the pole of the right line B₃C₃, which contains upon it the two conjugate points; this pole being taken with respect to the plane section lately mentioned. If then we denote by D₃E₃ the indefinite right line which is, with respect to the surface, the polar of the fixed line B₃C₃, we see that the chord SP₃ must intersect this reciprocal polar also, besides intersecting the line B₃C₃ itself. Conversely this condition, of intersecting these two fixed polars, is sufficient to enable us to draw the chord SP₃ when the point S has been determined, by drawing from the assumed point P the chord PS parallel to a fixed right line. We may then substitute, for a system of two guide-stars and one guide-point, the system of one guide-star and two guide-lines; these lines being (as has been seen) a pair of reciprocal polars, with respect to the given surface.

6. If, then, it be required to inscribe a polygon PP₁P₂...P₂n with any odd number 2n + 1 of sides, which shall pass successively through the same number of given points, A₁A₂...A₂n+1, we may begin by assuming a point P upon the given surface, and drawing through the given points 2n + 1 successive chords, which will in general conduct to a final
point $P_{2n+1}$, distinct from the assumed initial point $P$. And then, by processes of which the nature has been already explained, we can find a point $S$ such that the chord $PS$ shall be parallel to a fixed right line, or shall have a direction independent of the assumed and variable position of $P$; and that the chord $SP_{2n+1}$ shall at the same time cross two other fixed right lines, which are reciprocal polars of each other. In order then to find a new point $P$, which shall satisfy the conditions of the proposed problem, or shall be such as to coincide with the point $P_{2n+1}$, deduced from it as above, we see that it is necessary and sufficient to oblige this sought point $P$ to be situated at one or other extremity of a certain chord $PS$, which shall at once be parallel to a fixed line, and shall also cross two fixed polars. It is clear then that we need only draw two planes, containing respectively these two polars, and parallel to the fixed direction; for the right line of intersection of these two planes will be the chord of solution required; or in other words, it will cut the surface in the two (real or imaginary) points, $P$ and $S$, which are adapted, and are alone adapted, to be positions of the first corner of the polygon to be inscribed.

7. But if it be demanded to inscribe in the same surface a polygon $PP_1P_2 \ldots P_{2n-1}$, with an even number $2n$ of sides, passing successively through the same even number of given points, $A_1A_2 \ldots A_{2n}$, the problem then acquires a character totally distinct. For if, after assuming an initial point $P$ upon the surface, we pass, by $2n$ successive chords, drawn through the given points $A_1$, &c., to a final point $P_{2n}$ upon the surface, which will thus be in general distinct from $P$; it will indeed be possible to assign generally two fixed polars, across which, as two given guide-lines, a certain variable chord $SP_{2n}$ is to be drawn, like the chord $SP_{2n+1}$ of (6); but the chord $PS$ will not, in this question, be parallel to a given line, or directed to a given star; it will, on the contrary, by (3) (4) (5), be bisected by a given diameter, which we may call $AB$; or, we prefer to state the result so, it will be now the supplementary chord $NS$ of the same diametral section of the surface ($N$ being still the point of that surface opposite to $P$), which will have a given direction, and not the chord $PS$ itself. In fact, at the end of (4), we reduced the system of $2n$ guide-points to a system of the centre, two stars, and one point; and in (5) we reduced the system of two stars and a point to the system of a star and two polars. In order then to find a point $P$ which shall coincide with the point $P_{2n}$ deduced from it as above, or which shall be adapted to be the first corner of an inscribed polygon of $2n$ sides passing respectively through the $2n$ given points, $A_1 \ldots A_{2n}$, we must endeavour to find a chord $PS$ which shall be at once bisected by the fixed diameter $AB$, and shall also intersect the two fixed polars above mentioned. And conversely, if we can find any such chord $PS$, it will necessarily be at least one chord of solution of the problem; understanding hereby, that if we set out with either extremity, $P$ or $S$, and draw from it $2n$ successive chords $PP_1$, &c., or $SS_1$, &c., through the $2n$ given points $A_1$, &c., we shall be brought back hereby (as the question requires) to the point with which we started. For, in a process which we have proved to admit of being substituted for the process of drawing the $2n$ chords, we shall be brought first from $P$ to $S$, and then back from $S$ to $P$; or else first from $S$ to $P$, and then back from $P$ to $S$: provided that the chord of solution $PS$ has been selected so as to satisfy the conditions above assigned.

8. To inscribe then, for example, a gauche chiliagon in an ellipsoid, $PP_1 \ldots P_{999}$, or $SS_1 \ldots S_{999}$, under the condition that its thousand successive sides shall pass successively
through a thousand given points \(A_1A_2 \ldots A_{1000}\), we are conducted to seek to inscribe, in the same given ellipsoid, a chord \(PS\), which shall be at once bisected by a given diameter \(AB\), and also crossed by a given chord \(CD\) and by the polar of that given chord. Now in general when any two proposed right lines intersect each other, their respective polars also intersect, namely, in the pole of the plane of the two lines proposed. Since then the sought chord \(PS\) intersects the polar of the given chord \(CD\), it follows that the polar of the same sought chord \(PS\) must intersect the given chord \(CD\) itself. We may therefore reduce the problem to this form: to find a chord \(PS\) of the ellipsoid which shall be bisected by a given diameter \(AB\), and shall also be such that while it intersects a given chord \(CD\) in some point \(E\), its polar intersects the prolongation of that given chord, in some other point \(F\).

9. The two sought points \(E\), \(F\), as being situated upon two polars, are of course conjugate relatively to the surface; they are therefore also conjugate relatively to the chord \(CD\), or, in other words, they cut that given chord harmonically. The four diametral planes \(ABC\), \(ABE\), \(ABD\), \(ABF\), compose therefore an harmonic pencil; the second being, in this pencil, harmonically conjugate to the fourth; and being at the same time, on account of the polars, conjugate to it also with respect to the surface, as one diametral plane to another. When the ellipsoid becomes a sphere, the conjugate planes \(ABE\), \(ABF\) become rectangular; and consequently the sought plane \(ABE\) bisects the angle between the two given planes \(ABC\) and \(ABD\). This solves at once the problem for the sphere; for if, conversely, we thus bisect the given dihedral angle \(CABD\) by a plane \(ABE\), cutting the chord \(CD\) in \(E\), and if we take the harmonic conjugate \(F\) on the same given chord prolonged, and draw from \(E\) to \(F\) lines meeting ordinately the given diameter \(AB\), these two right lines will be situated in two rectangular or conjugate diametral planes, and will satisfy all the other conditions requisite for their being polars of each other; but each intersects the given chord \(CD\), or that chord prolonged, and therefore each intersects also, by (8), the polar of that chord; each therefore satisfies all the transformed conditions of the problem, and gives a chord of solution, real or imaginary. More fully, the ordinate \(EE'\) to the diameter \(AB\), drawn from the internal point of harmonic section \(E\) of the chord \(CD\), gives, when prolonged both ways to meet the surface, the chord of real solution, \(PS\); and the other ordinate \(FF'\) to the same diameter \(AB\), which is drawn from the external point of section \(F\) of the same chord \(CD\), and which is itself wholly external to the surface, is the chord of imaginary solution. But because when we return from the sphere to the ellipsoid, or other surface of the second order, the condition of bisection of the given dihedral angle \(CABD\) is no longer fulfilled by the sought plane \(ABE\), a slight generalization of the foregoing process becomes necessary, and can easily be accomplished as follows.

10. Conceive, as before, that on the diameter \(AB\) the ordinate \(EE'\) is let fall from the internal point of section \(E\), and likewise the ordinates \(CC'\) and \(DD'\) from \(C\) and \(D\); and draw also, parallel to that diameter, the right lines \(CC''\), \(DD''\), \(EE''\), from the same three points \(C\), \(D\), \(E\), so as to terminate on the diametral plane through \(O\) which is conjugate to the same diameter; in such a manner that \(OC''\), \(OB''\), \(OE''\) shall be parallel and equal to the ordinates \(C'\), \(D'\), \(E'\); and that the segments \(CE\), \(ED\) of the chord \(CD\) shall be proportional to the segments \(C''E''\), \(D''E''\) of the base \(C''D''\) of the triangle \(C''D''O\), which is situated in the diametral plane, and has the centre \(O\) for its vertex. For the case of the sphere, the vertical angle \(C''OD''\) of this triangle is, by (9), bisected by the line \(OE''\); wherefore the sides \(OC''\), \(OD''\),
or their equals, the ordinates \( c', d' \), are, in this case, proportional to the segments \( c''e'', e''d'' \) of the base, or to the segments \( ce, ed \) of the chord: while the squares of the ordinates are, for the same case of the sphere, equal to the rectangles \( ac'b, ad'b \), under the segments of the diameter \( ab \). Hence, for the sphere, the squares of the segments of the given chord are proportional to the rectangles under the segments of the given diameter, these latter segments being found by letting fall ordinates from the ends of the chord; or, in symbols, we have the proportion,

\[
CF^2 : DF^2 :: CE^2 : ED^2 :: AC'B : AD'B.
\]

But, by the general principles of geometrical deformation, the property, thus stated, cannot be peculiar to the sphere. It must extend, without any further modification, to the ellipsoid; and it gives at once, for that surface, the two points of harmonic section, \( E \) and \( F \), of the given chord \( cd \), through which points the two sought chords of real and imaginary solution are to pass; these chords of solution are therefore completely determined, since they are to be also ordinates, as before, to the given diameter \( ab \). The problem of inscription for the ellipsoid is therefore fully resolved; not only when, as in (6), the number of sides of the polygon is odd, but also in the more difficult case (7), when the number of sides is even.

11. If the given surface be a hyperboloid of two sheets, one of the two fixed polars will still intersect that surface, and the fixed chord \( cd \) may still be considered as real. If the given diameter \( ab \) be also real, the proportion in (10) still holds good, without any modification from imaginaries, and determines still a real point \( E \), with its harmonic conjugate \( F \), through one of other of which two points still passes a chord of real solution, while through the other point of section still is drawn a chord of imaginary solution, reciprocally polar to the former. But if the diameter \( ab \) be imaginary, or in other words, if it fail to meet the proposed hyperboloid at all, we are led to consider, instead of it, an ideal diameter \( a'b' \), having the same real direction, but terminating, in a well-known way, on a certain supplementary surface; in such a manner that while \( A \) and \( B \) are now imaginary points, the points \( a' \) and \( b' \) are real, although not really situated on the given surface; and that

\[
OA^2 = OB^2 = -OA'^2 = -OB'^2.
\]

The points \( c' \) and \( d' \) are still real, and so are the rectangles \( ac'b \) and \( ad'b \), although \( A \) and \( B \) are imaginary; for we may write,

\[
AC'B = OA^2 - OC'^2, \quad AD'B = OA^2 - OD'^2,
\]

and the proportion in (10) becomes now,

\[
CF^2 : DF^2 :: CE^2 : ED^2 :: OC'^2 + OA'^2 : OD'^2 + OA'^2.
\]

It gives therefore still a real point of section \( E \), and a real conjugate point \( F \); and through these two points of section of \( cd \) we can still draw two real right lines, which shall still ordinately cross the real direction of \( ab \), and shall still be two reciprocal polars, satisfying all the transformed conditions of the question, and coinciding with two chords of real and imaginary solution. For the double-sheeted hyperboloid, therefore, as well as for the ellipsoid,
the problem of inscribing a *gauche chiliagon*, or other *even-sided polygon*, whose sides shall pass successively, and in order, through the same given number of points, is solved by a system of *two polar chords*, which we have assigned geometrical processes to determine; and the solutions are still, in general, *four* in number; two of them being still real, and two imaginary.

12. If the given surface be a hyperboloid of *one sheet*, then not only may the diameter $AB$ be real or imaginary, but also the chord $CD$ may or may not cease to be real; for the two fixed polars will now either *both meet* the surface, or else *both fail* to meet it in any two real points. When $AB$ and $CD$ are both real, the proportion in (10), being put under the form

$$\frac{CF^2}{DF^2} :: \frac{CE^2}{ED^2} :: \frac{OA^2 - OC'^2}{OA^2 - OD'^2},$$

shews that the point of section $E$ and its conjugate $F$ will be real, if the points $C'$ and $D'$ fall *both* on the diameter $AB$ itself, or *both* on that diameter prolonged; that is, if the extremities $C$ and $D$ lie *both within* or *both without* the interval between the two parallel tangent planes to the surface which are drawn at the points $A$ and $B$: under these conditions therefore there will still be *two real right lines*, which may still be called the *two chords of solution*; but because these lines will still be two reciprocal polars, they will now (like the two fixed polars above mentioned) either *both meet* the hyperboloid, or else *both fail* to meet it; and consequently there will now be either *four real*, or else *four imaginary* solutions. If $AB$ and $CD$ be still both real, but if the chord $CD$ have one extremity within and the other extremity without the interval between the two parallel tangent planes, the proportion above written will assign a *negative ratio* for the squares of the segments of $CD$; the points of section $E$ and $F$, and the *two polar chords* of solution, become therefore, in this case, *themselves imaginary*; and of course, by still stronger reason, the four solutions of the problem become then imaginary likewise. If $CD$ be real, but $AB$ imaginary, the proportion in (11) conducts to two real points of section, and consequently to two real chords, which may, however, correspond, as above, either to four real or to four imaginary solutions of the problem. And, finally, it will be found that the same conclusion holds good also in the remaining case, namely, when the chord $CD$ becomes imaginary, whether the diameter $AB$ be real or not; that is, when the two fixed polars do not meet, in any real points, the single-sheeted hyperboloid.

13. Although the case last mentioned may still be treated by a modification of the proportion assigned in (10), which was deduced from considerations relative to the sphere, yet in order to put the subject in a clearer (or at least in another) point of view, we may now resume the problem for the ellipsoid as follows, without making any use of the spherical deformation. It was required to find two lines, reciprocally polar to each other, and ordinatey crossing a given diameter $AB$ of the ellipsoid, which should also cut a given chord $CD$ of the same surface, internally in some point $E$, and externally in some other point $F$. Bisect $CD$ in $G$, and conceive $EF$ to be bisected in $H$; and besides the four old ordinates to the diameter $AB$, namely $CC'$, $DD'$, $EE'$, and $FF'$, let there be now supposed to be drawn, as two new ordinates to the same diameter, the lines $GG'$ and $HH'$. Then $G'$ will bisect $CD'$, and $H'$ will bisect $E'F'$; while the centre $O$ of the ellipsoid will still bisect $AB$. And because the points $E'$ and $F'$ are
harmonic conjugates, not only with respect to the points \(A\) and \(B\), but also with respect to the points \(C'\) and \(D'\), we shall have the following equalities:

\[
H'F'^2 = H'E'^2 = H'\alpha' . \ H'B' = H'C' . \ H'D' = H'O^2 - OA^2 = H'G'^2 - C'G'^2.
\]

Hence,

\[
OH'^2 - C'\dot{H}'^2 = OA^2 - C'G'^2,
\]

that is,

\[
OH' = \frac{OA^2 + OC'^2 - C'G'^2}{2OG'} = \frac{OA^2 + OC' . OD'}{OC' + OD'}.
\]

Now each of these two last expressions for \(OH'\) remains real, and assigns a real and determinate position for the point \(H'\), even when the points \(C', D'\), or the points \(A, B\), or when both these pairs of points at once become imaginary; for the points \(O\) and \(G'\) are still in all cases real, and so are the squares of \(OA\) and \(C'G'\), the rectangle under \(OC'\) and \(OD'\), and the sum \(OC' + OD'\). Thus \(H'\) can always be found, as a real point, and hence we have a real value for the square of \(H'E'\), or \(H'F'\), which will enable us to assign the points \(E'\) and \(F'\) themselves, or else to pronounce that they are imaginary.

14. We see at the same time, from the values \(H'O^2 - OA^2\) and \(H'G'^2 - C'G'^2\) above assigned for \(H'E'^2\) or \(H'F'^2\), that these two sought points \(E'\) and \(F'\) must both be real, unless the two fixed points \(A\) and \(C'\) are themselves both real, since \(O, C', H'\), are, all three, real points. But for the ellipsoid, and for the double sheeted hyperboloid, we can in general oblige the points \(C, D,\) and their projections \(C', D'\), to become imaginary, by selecting that one of the two fixed polars which does not actually meet the surface; for these two sorts of surfaces, the two polar chords of solution of the problem of inscription of a gauche polygon with an even number of sides passing through the same number of given points, and therefore found anew to be two real lines, although only one of them will actually intersect the surface, and only two of the four polygons will (as before) be real. And even for the single-sheeted hyperboloid, in order to render the two chords of solution imaginary lines, it is necessary that the two given polars should actually meet the surface; for otherwise the polar lines deduced will still be real. It is necessary also, for the imaginariness of the two lines deduced, that the given diameter \(AB\) should be itself a real diameter, or in other words that it should actually intersect the hyperboloid. But even when the given chord \(CD\) and the given diameter \(AB\) are thus both real, and when the surface is a single-sheeted hyperboloid, it does not follow that the two chords of solution may not be real lines. We shall only have failed to prove their reality by the expressions recently referred to. We must resume, for this case, the reasonings of (12), or some others equivalent to them, and we find, as in that section of this Abstract, for the imaginariness of the two sought polar lines, the condition that one of the two extremities of the given and real chord \(CD\) shall fall within, and that the other extremity of that chord shall fall without the interval between the two real and parallel tangent planes to the single-sheeted hyperboloid, which are drawn at the extremities of the real diameter \(AB\). Sir W. R. Hamilton confesses that the case where all these particular conditions are combined, so as to render imaginary the two polar lines of solution, had not occurred to him when he made to the Royal Irish Academy his communication of June, 1849.
15. It seems to him worth while to notice here that instead of the foregoing metric processes for finding (when they exist) the two lines of solution of the problem, the following graphic process of construction of those lines may always, at least in theory, be substituted, although in practice it will sometimes require modification for imaginaries. In the diametral plane $ABC$ draw a chord $K'D'L'$, which shall be bisected at the known point $D'$ by the given diameter $AB$; and join $CK, CL$. These joining lines will cut that diameter in the two sought points $E', F'$; which being in this manner found, the two sought lines of solution $EE', FF'$, are constructed without any difficulty. For the sphere, the ellipsoid, and the hyperboloid of two sheets, although not always for the single-sheeted hyperboloid, this simple and graphic process can actually be applied, without any such modification from imaginaries as was above alluded to. The consideration of non-central surfaces does not enter into the object of the present communication; nor has it been thought necessary to consider in it any limiting or exceptional cases, such as those where certain positions or directions become indeterminate, by some peculiar combinations of the data, while yet they are in general definitely assignable, by the processes already explained.

16. Sir William Rowan Hamilton is unwilling to add to the length of this communication by any historical references; in regard to which, indeed, he does not consider himself prepared to furnish anything important, as supplementary to what seems to be pretty generally known, by those who feel an interest in such matters. He has however taken some pains to inquire, from a few geometrical friends, whether it is likely that he has been anticipated in his results respecting the inscription of gauche polygons in surfaces of the second order; and he has not hitherto been able to learn that any such anticipation is thought to exist. Of course he knows that he must, consciously and unconsciously, be in many ways indebted to his scientific contemporaries, for their instructions and suggestions on these and on other subjects; and also to his acquaintance, imperfect as it may be, with what has been done in earlier times. But he conceives that he only does justice to the yet infant Method of Quaternions (communicated to the Royal Irish Academy for the first time in 1843), when he states that he considers himself to owe, to that new method of geometrical research, not merely the results stated to the Academy in the summer of 1849, respecting these inscriptions of gauche polygons, and several other connected although hitherto unpublished results, which to him appear remarkable, but also the suggestion of the mode of geometrical investigation which has been employed in the present Abstract. No doubt the principles used in it have all been very elementary, and perhaps their combination would have cost no serious trouble to any experienced geometer who had chosen to attack the problem. But to his own mind the whole foregoing investigation presents itself as being (what in fact in his case it was) a mere translation of the quaternion analysis into ordinary geometrical language, on this particular subject. And he will not complicate the present Abstract by giving, on this occasion, any account of those other theorems respecting polygons in surfaces, to which the Calculus of Quaternions has conducted him, but of which he has not seen how to translate the proofs (for it is easy to translate the results) into the usual language of geometry.