ACCOUNT OF THE ICOSIAN CALCULUS

By

William Rowan Hamilton

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Sir William Rowan Hamilton read a Paper on a new System of Roots of Unity, and of operations therewith connected: to which system of symbols and operations, in consequence of the geometrical character of some of their leading interpretations, he is disposed to give the name of the ‘ICOSIAN CALCULUS.’

This Calculus agrees with that of the Quaternions, in three important respects: namely, 1st, that its three chief symbols, i, k, λ, are (as above suggested) roots of unity, as i, j, k are certain fourth roots thereof: 2nd, that these new roots obey the associative law of multiplication; and 3rd, that they are not subject to the commutative law, or that their places as factors must not in general be altered in a product. And it differs from the Quaternion Calculus, 1st, by involving roots with different exponents; and 2nd by not requiring so far as yet appears) the distributive property of multiplication. In fact, + and −, in these new calculations, enter only as connecting exponents, and not as connecting terms: indeed, no terms, or in other words, no polynomials, nor even binomes, have hitherto presented themselves, in these late researches of the author. As regards the exponents of the new roots, it may be mentioned that in the principal system—for the new Calculus involves a family of systems—there are adopted the equations,

\[ 1 = i^2 = k^3 = λ^5, \quad λ = ik; \]  

so that we deal, in it, with a new square root, cube root, and fifth root, of positive unity; the latter root being the product of the two former, when taken in an order assigned, but not in the opposite order. From these simple assumptions (A), a long train of consistent calculations opens itself out, for every result of which there is found a corresponding geometrical interpretation, in the theory of two of the celebrated solids of antiquity, alluded to with interest by Plato in the Timaeus; namely, the Icosaedron, and the Dodecaedron: whereof the angles may now be unequal. By making \( λ^4 = 1 \), the author obtains other symbolical results, which are interpreted by the Octahedron and the Hexahedron. The Pyramid is, in this theory, almost too simple to be interesting: but it is dealt with by the assumption, \( λ^3 = 1 \), the other equations (A) being untouched. As one fundamental result of those equations (A), which may serve as a slight specimen of the rest, it is found that if we make \( ik^2 = μ \), we shall have

\[ μ^5 = 1, \quad μ = λiλ, \quad λ = μμμ; \]

so that this new fifth root \( μ \) has relations of perfect reciprocity with the former fifth root \( λ \). But there exist more general results, including this, and others, on which Sir W. R. H. hopes to be allowed to make a future communication to the Academy: as also on some applications of the principles already stated, or alluded to, which appear to be in some degree interesting.