A THEOREM CONCERNING
POLYGONIC SYNGRAPHY

By

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A Theorem concerning Polygonic Syngraphy.

By Sir William R. Hamilton.

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Professor Sir William Rowan Hamilton exhibited the following Theorem, to which he had been conducted by that theory of geometrical syngraphy of which he had lately submitted to the Academy a verbal and hitherto unreported sketch, and on which he hopes to return in a future communication.

Theorem. Let \( A_1, A_2, \ldots, A_n \) be any \( n \) points (in number odd or even) assumed at pleasure on the \( n \) successive sides of a closed polygon \( BB_1B_2 \ldots B_n-1 \) (plane or gauche), inscribed in any given surface of the second order. Take any three points, \( P, Q, R \), on that surface, as initial points, and draw from each a system of \( n \) successive chords, passing in order through the \( n \) assumed points \( (A) \), and terminating in three other superficial and final points, \( P', Q', R' \). Then there will be (in general) another inscribed and closed polygon, \( CC_1C_2 \ldots C_{n-1} \), of which the \( n \) sides shall pass successively, in the same order, through the same \( n \) points \( (A) \); and of which the initial point \( C \) shall also be connected with the point \( B \) of the former polygon, by the relations

\[
\frac{ael}{be} \frac{\beta \gamma}{\alpha \epsilon \lambda} = \frac{a' e' l'}{b' e'} \frac{\beta' \gamma'}{\alpha' \epsilon' \lambda'}, \quad \frac{bfm}{ca} \frac{\gamma \alpha}{\beta \zeta \mu} = \frac{b' f' m'}{c' a'} \frac{\gamma' \alpha'}{\beta' \zeta' \mu'}, \quad \frac{egn}{ab} \frac{\alpha \beta}{\gamma \eta \nu} = \frac{e' g' n'}{a'b'} \frac{\alpha' \beta'}{\gamma' \eta' \nu'};
\]

where

\[
a = QR, \quad b = RP, \quad c = PQ,
\]
\[
e = BP, \quad f = BQ, \quad g = BR,
\]
\[
l = CP, \quad m = CQ, \quad n = CR,
\]
\[
a' = Q'R', \quad b' = R'P', \quad c' = P'Q',
\]
\[
e' = BP', \quad f' = BQ', \quad g' = BR',
\]
\[
l' = CP', \quad m' = CQ', \quad n' = CR';
\]

while \( \alpha \beta \gamma \epsilon \zeta \eta \lambda \mu \nu \), and \( \alpha' \beta' \gamma' \epsilon' \zeta' \eta' \lambda' \mu' \nu' \), denote the semidiameters of the surface, respectively parallel to the chords \( abcefglmn, a'b'c'e'f'g'l'm'n' \).

As a very particular case of this theorem, we may suppose that \( PQ'R'PQ'R' \) is a plane hexagon in a conic, and \( BC \) its Pascal’s line.