ATTRACTING DYNAMICS OF EXPONENTIAL MAPS

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Abstract. We give a complete classification of hyperbolic components in the space of iterated exponential maps $z \mapsto \lambda \exp(z)$, and we describe a preferred parametrization of those components. More precisely, we associate to every hyperbolic component of period $n$ a finite symbolic sequence of length $n - 1$, we show that every such sequence is realized by a hyperbolic component, and the hyperbolic component specified by any such sequence is unique. This leads to a complete classification of all exponential maps with attracting dynamics, which is a fundamental step in the understanding of exponential parameter space.

1. Introduction

This paper is part of the program to describe the dynamics of exponential maps $\lambda \exp$ and the structure of parameter space, in the spirit of the well-developed body of knowledge about polynomial dynamics. The polynomial theory was pioneered by Douady and Hubbard [DH1] who systematically investigated the Mandelbrot set as the simplest non-trivial example of a holomorphic parameter space. Since then, there has been a lot of further work in this field, much of it based on methods and results developed initially for the Mandelbrot set. Among transcendental entire maps, the exponential family $\lambda \exp$ stands out as the simplest family. It is expected that a good understanding of this space of maps will guide the way for further progress on a study of more classes of entire maps.

For holomorphic spaces of rational maps, such as the space $z \mapsto z^2 + c$ of quadratic polynomials, it is known from the work of Mañé, Sad and Sullivan [MSS] that the set of structurally stable maps is open and dense: a map is structurally stable if it has a neighborhood in which all maps are topologically (and even quasiconformally) conjugate. The complementary locus is called the bifurcation locus within this space of maps; it is closed and nowhere dense. An investigation of a space of rational maps thus starts with a description of connected components in the space of structurally stable maps, called stable components. In most spaces of rational maps (those in which the bifurcation locus is non-empty and not every map has an indifferent orbit), all known stable components consist of hyperbolic maps: there is a uniformly expanding metric in a neighborhood of the Julia set, which is equivalent to the fact that all critical points converge to attracting or

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superattracting cycles \([M2]\). It is conjectured that stable components are always hyperbolic; unfortunately, this has not yet been confirmed for any space.

One reason why quadratic polynomials are the simplest non-trivial rational maps is because they have only one critical point (except for the fixed point \(\infty\)) and of lowest multiplicity; critical points determine the dynamics of rational maps to a large extent. For the space of quadratic polynomials, the bifurcation locus is the boundary of the Mandelbrot set. Douady and Hubbard [DH1] have developed a complete conjectural description of the topology of the Mandelbrot set and its boundary, and they showed that it is a true description if and only if the Mandelbrot set is locally connected. This would imply that every stable component was hyperbolic. They also provided a complete classification of hyperbolic components as part of the complete description of the topology. The importance of the study of the space of quadratic polynomials stems not only from the fact that they are the simplest class of rational maps, but also because renormalization theory [DH2] shows that quadratic polynomials have universal properties; as a consequence, every non-trivial bifurcation locus in spaces of rational maps contains infinitely many homeomorphic copies of the Mandelbrot set [Mc3] (or of Multibrot sets, which are the analogues for maps \(z^d + c\) with \(d > 2\)).

It will be a long way to establish analogous results for transcendental maps, or even to find out which results have analogues and in which sense. The fact that structurally stable maps are dense in many spaces of transcendental entire maps has been established by Eremenko and Lyubich [EL2]; this includes the space of exponential maps. The decisive role of critical points (or critical values) for rational maps is, for transcendental maps, assumed by either critical values or asymptotic values, which are jointly known as singular values. Exponential maps \(\lambda \exp\) have only one singular value of the simplest kind: they have no critical values, \(0\) is the only asymptotic value, and every \(\lambda \exp\) is a universal cover \(\mathbb{C} \to \mathbb{C} \setminus \{0\}\). This makes them good candidates for prototypes of transcendental maps. Probably for this reason, the space of exponential maps has been studied more than any other space of transcendental maps. An exponential map will be called \textit{hyperbolic} if it has an attracting orbit (which necessarily attracts the singular orbit, so there can be at most one attracting orbit). A structurally stable component is called hyperbolic if it consists of hyperbolic maps (this is a slight abuse of notation: hyperbolic dynamics in a strict sense would require a uniformly expanding metric in a neighborhood of the Julia set, but the Julia set is never compact for transcendental maps).

The description of the exponential parameter space was begun in the 1980's by Baker and Rippon [BR], by Eremenko and Lyubich [EL1], [EL2], [EL3], and by Devaney, Goldberg and Hubbard [DGH]. These papers discuss certain fundamental properties of hyperbolic components and of bifurcations (in the case of [EL1], [EL2], [EL3] as an example of a study of more general entire maps), but a description of the global structure of parameter space was in terms of pictures and
conjectures. Eremenko and Lyubich conjectured that every structurally stable exponential map is hyperbolic, so the union of the hyperbolic components would be dense in $\lambda$-space.

For a hyperbolic exponential map, the Julia set has measure zero [EL1], [EL3] but Hausdorff dimension two [Mc2]. There are some results on the topology, in particular if the attracting orbit has period one [AO]. It seems possible to give a more complete description of the topology for exponential maps with attracting orbits of arbitrary periods.

In this paper, we give a complete description of the space of hyperbolic exponential maps. This was part of Chapter III of the author’s habilitation thesis [S1] (of May, 1999) which developed a description of the exponential parameter space in analogy to Douady and Hubbard’s Orsay Notes [DH1] about the Mandelbrot set. An earlier version of this paper was circulated as [S3].

The bifurcation locus of exponential maps is not locally connected. It would be interesting to have a topological criterion (analogous to local connectivity of the Mandelbrot set) which would imply the validity of the conjecture of Eremenko and Lyubich. It seems possible that this could be done in terms of fibers as discussed in [S2]. There are two more conjectures in [EL2] which have now been established [S1], [RS]; they are explained at the end of Section 7.

The set of parameters $\lambda$ for which there is a (necessarily unique) attracting periodic orbit is clearly open; connected components where this happens are hyperbolic components. The period of the attracting orbit is constant throughout the component. Our object is to classify hyperbolic components in the $\lambda$ parameter plane, where $\lambda$ ranges over $\mathbb{C} \setminus \{0\}$. It is known [BR], [EL2], [DGH] that all exponential maps with attracting orbits of period 1 are contained in a single hyperbolic component which is bounded in $\mathbb{C}$; it contains a neighborhood of 0. For period 2, there is a unique hyperbolic component which is contained in a left half plane and unbounded to the left; see also Section 2. All other hyperbolic components have period 3 or more, and are unbounded to the right. Every hyperbolic component is simply connected, except that the period 1 component is punctured at 0.

Here is our main result; it is illustrated in Figure 1.

**Theorem 1.1** (Classification of hyperbolic components). For every period $n \geq 3$, there are countably many hyperbolic components in the space of exponential maps $\lambda \exp$. Each of them is characterized by a sequence

$$s_1, s_2, \ldots, s_{n-1}$$

(its “intermediate external address”), where $s_1, s_2, \ldots, s_{n-2} \in \mathbb{Z}$ with $s_1 = 0$, and $s_{n-1} \in (\mathbb{Z} + \frac{1}{2})$. Conversely, every such sequence is realized by a unique hyperbolic component of period $n$. These hyperbolic components have a natural vertical order in which they stretch out to $+\infty$ along bounded imaginary parts, and this order is the same as the lexicographic order of the corresponding sequences $s_1, s_2, \ldots, s_{n-1}$. 
Figure 1. The space of parameters $\lambda$ for exponential maps $\lambda \exp$, with hyperbolic components indicated in white. Various hyperbolic components are labeled by their intermediate external addresses, or briefly by their periods (in parentheses). The picture has kindly been contributed by Jack Milnor: for every pixel, an approximate test is performed whether or not the corresponding map $\lambda \exp$ has an attracting orbit (with $\lambda$ at the center of the pixel); in addition, the boundaries of hyperbolic components have been emphasized in order to show their shapes more clearly.

In particular, between any pair of consecutive hyperbolic components of period $n$, there are infinitely many hyperbolic components of period $n+1$, ordered like $\mathbb{Z}$.

The numbers $s_1, s_2, \ldots, s_{n-1}$ characterizing any hyperbolic component of period $n$ have a dynamic meaning as follows. Let $\lambda$ be any parameter in the given period $n$ hyperbolic component, and let

$$U_1 \cong U_2 \cong \cdots \cong U_n \rightarrow U_1$$

be the unique cycle of periodic Fatou components for $\lambda \exp$, where $0 \in U_1$, where $\lambda \in U_2$, and where $U_n$ contains a left half plane. Here $\lambda \exp: U_n \rightarrow U_1 \setminus \{0\}$ is a universal cover, and all other $\lambda \exp: U_k \rightarrow U_{k+1}$ are conformal isomorphisms.
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Figure 2. The same space as in Figure 1, drawn differently by a special purpose program of Günter Rottenfußer: this program traces out the boundaries of hyperbolic components, which is possible with arbitrary precision for any given hyperbolic component. Unlike for the Mandelbrot set, in the exponential case it is impossible to test whether the singular orbit “escapes to $\infty$”; instead, in pixel images it is usually tested whether the singular orbit survives some fixed number $N$ of iterations without producing numbers too large to store. This is quite different from the existence of an attracting orbit for the given value of $\lambda$, and logically independent. This picture confirms that pixel test pictures like in Figure 1 are approximately correct.

Define the horizontal lines

$$L(s_k) := \{ z \in \mathbb{C} : \text{Im}(z) = 2\pi s_k - c \};$$

here $c = \text{Im}(\log(\lambda))$, choosing the branch with $|c| < \pi$. Then for $k = 1, \ldots, n-1$, the component $U_k$ contains a curve which is asymptotic to $L(s_k)$ as $\text{Re}(z) \to +\infty$ and which maps to $L(s_{n-1})$ under $(\lambda \exp)^{\circ(n-1-k)}$. Thus $s_k$ specifies precisely which branch of $(\lambda \exp)^{-1}$ carries $U_{k+1}$ to $U_k$. 
Similarly, in the \( \lambda \) parameter plane, if \( n > 3 \) then the points in the hyperbolic component are asymptotic to the line

\[
\text{Im}(\lambda) = 2\pi s_2 \quad \text{as} \quad \text{Re}(\lambda) \to +\infty,
\]

while for \( n = 3 \) they form a neighborhood of this line near \(+\infty\).

Furthermore, if \( H_1 \) and \( H_2 \) are hyperbolic components of any periods greater than 2, then \( H_1 \) lies above \( H_2 \) if and only if its symbol sequence is greater, using lexicographic ordering.

Our results are easily translated into \( \kappa = \log(\lambda) \)-space: then Theorem 1.1 holds for \( n \geq 2 \), and the condition \( s_1 = 0 \) is lifted. However, \( \kappa \)-space is not a true parameter space: every exponential map is represented countably often; adding a constant integer to every intermediate external address in \( \kappa \)-space yields the same map with a different branch of \( \kappa \).

In Section 2, we review necessary properties about exponential maps and state results from earlier papers. In particular, we introduce dynamic rays. Then, in Section 3, we give a combinatorial coding to every hyperbolic component in terms of “intermediate external addresses”, and we show that each intermediate external address is realized by at least one hyperbolic component. The at least one is strengthened to exactly one in Section 4 using a variant of spider theory. This finishes the classification of hyperbolic components. The second main result (in Section 7) constructs, for every hyperbolic component, a preferred parametrization (which even extends to the boundary). The main difficulty is in breaking the symmetry and fixing an origin of the parametrization, which is accomplished using a “dynamic root” of every periodic Fatou component: this is a boundary point which is fixed under the first return map of the component and which is the landing point of at least two periodic dynamic rays. Existence and uniqueness of dynamic roots is shown in Section 6, while Section 5 provides the necessary combinatorial properties of periodic dynamic rays landing at a common point: whenever three or more periodic rays land at a common point, then the dynamics permutes all these rays transitively, and two rays in this orbit are singled out as “characteristic rays”. We conclude this paper with a discussion of further results on hyperbolic components, in particular their boundary properties.

Some notation. We write our exponential maps as \( z \mapsto E_\lambda(z) := \lambda e^z = \exp(z + \kappa) \) with \( \lambda = \exp(\kappa) \), where \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( \kappa \in \mathbb{C} \); usually we will use the branch \( |\text{Im}(\kappa)| \leq \pi \). We will often need \( F(t) = e^t - 1 \), in particular for \( t \in \mathbb{R} \).

Let \( \mathcal{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \mathcal{D}^* := \mathcal{D} \setminus \{0\} \). We write that a curve or sequence in \( \mathbb{C} \) converges to \(+\infty\) or to \(-\infty\) to indicate that the real parts converge to \( \pm\infty \), while the imaginary parts are bounded.

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2. Exponential dynamics

In this section, we will review known properties of exponential dynamics. The map $E_\lambda = \lambda \exp$ has no critical points or critical values, and a unique omitted value 0 also known as singular value. The singular value plays an equally decisive role for exponential dynamics as the critical values do for polynomial dynamics.

Points $z$ with $E_\lambda^{\circ k}(z) \to \infty$ as $k \to \infty$ are known as escaping points; they are completely classified [SZ2]: if the singular value itself does not escape, then the escaping points are on disjoint curves called dynamic rays (or hairs) labeled by external addresses $s = s_1 s_2 s_3 \ldots$, which are infinite sequences over $\mathbb{Z}$ (there is a well-understood exception if the singular value does escape; that case does not matter for our purposes). The dynamic ray at external address $s$ is an injective curve $g_s: [t_s, \infty[ \to \mathbb{C}$ with $\text{Re}(g_s(t)) \to +\infty$ as $t \to \infty$, while $\text{Im}(g_s(t))$ is bounded. The quantity $t_s \geq 0$ depends on $s$ in a well-understood way; we only need bounded sequences $s$, and those have $t_s = 0$. We say that a ray $g_s$ lands at a point $w \in \mathbb{C}$ if $\lim_{t \uparrow t_s} g_s(t)$ exists and is equal to $w$. A ray tail is an unbounded subcurve of a ray: it is a curve $g_s([\tau, \infty[)$ for $\tau > t_s$.

Every point on a dynamic ray is an escaping point, and every escaping point is on such a ray, or the unique limit point of such a ray. We have the dynamic relation

$$E_\lambda(g_s(t)) = g_{\sigma(s)}(F(t))$$

where $\sigma$ is the shift map on external addresses, dropping the first entry. The meaning of the external address of a ray is the following: the set $E_\lambda^{-1}(\mathbb{R}^-)$ is a countable union of horizontal lines, spaced at distance $2\pi i \mathbb{Z}$, and $\mathbb{C} \setminus E_\lambda^{-1}(\mathbb{R}^-)$ are horizontal strips, labeled by $\mathbb{Z}$ so that the strip with label 0 contains the singular value 0 (perhaps on its boundary). Then at least for sufficiently large $t > t_s$, the external address $s$ of $g_s$ is the sequence $s_1 s_2 s_3 \ldots$ of strips visited by the orbit of $g_s(t)$. Not all possible sequences are allowed; the set of allowed sequences is completely understood: it consists of sequences satisfying a certain exponential growth condition [SZ2], and in particular it contains all bounded sequences.

If an exponential map has an attracting periodic point, then the singular value is in a periodic Fatou component which we call the characteristic Fatou component. All periodic orbits, except the unique attracting one, are repelling. We will need a construction and results from [SZ3, Section 4.3]: let $n \geq 2$ be the period of the attracting orbit, let $U_1, U_2, \ldots, U_n = U_0$ be the cycle of periodic Fatou components, labeled cyclically modulo $n$ so that $U_1$ is the characteristic Fatou component, and let $a_1, a_2, \ldots, a_n$ be the attracting periodic orbit labeled so
that \( a_k \in U_k \) for all \( k \). Let \( V_{n+1} \) be a closed neighborhood of \( a_1 \) corresponding to a disk in linearizing coordinates, large enough so as to contain the singular value in its interior. For \( k = 0, 1, \ldots, n \), let
\[
V_k := \{ z \in U_k : E_\lambda^{o(n+1-k)}(z) \in V_{n+1} \}
\]
and \( V := V_0 \cup V_1 \cup \cdots \cup V_{n+1} \). Then \( V_n \) contains a left half plane, and for \( k \in \{ n-1, n-2, \ldots, 1 \} \), \( V_k \) contains a neighborhood of a curve towards \(+\infty\) with \( V_1 \supset V_{n+1} \), while \( V_0 \) contains neighborhoods of infinitely many such curves towards \(+\infty\), spaced equally at integer translates of \( 2\pi i \). The construction assures that \( E_\lambda(V) \subset V \) and that all \( V_k \) are connected and simply connected.

Let \( R := C \setminus V_0 \); it consists of countably many connected components which we will call “regions” \( R_u \): let \( R_0 \) be the region containing the singular value and \( R_u := R_0 + 2\pi i u \), for \( u \in \mathbb{Z} \). Then \( R = \bigcup_u R_u \). Any orbit \( (z_k) \) within the Julia set then has an associated \textit{itinerary} \( u_1, u_2, u_3, \ldots \) such that \( z_k \in R_{u_k} \), for all \( k \). We should emphasize that this itinerary is different from the external address used for example in the construction of dynamic rays: the external address is constructed using inverse images of the negative real axis, which is dynamically not a natural concept. The itineraries as defined here are dynamically natural; compare [SZ3, Sections 4 and 5] for a discussion of the differences. (Note the different fonts for external addresses \( s_1 s_2 \ldots \) and itineraries \( u_1 u_2 \ldots \).

For exponential dynamics with an attracting periodic orbit, every periodic dynamic ray lands at a repelling periodic point [SZ3, Theorem 3.1]; every repelling periodic point is the landing point of a finite positive number of periodic rays [SZ3, Theorem 5.4]; and a periodic ray lands at a periodic point if and only if ray and point have identical itineraries [SZ3, Proposition 4.5]. In particular, different periodic points have different itineraries, while different periodic rays have the same itinerary if and only if they land together.

The following results are known from [EL2], [EL3], [BR]: in \( \lambda \)-space, there is a unique hyperbolic component \( H_1 \) of period 1 which is a bounded neighborhood of the puncture \( \lambda = 0 \) of parameter space; it comes with a conformal isomorphism \( D^* \to H_1 \), \( \mu \to \exp^{-\mu} \) (so that \( \lambda \exp \) has a fixed point with multiplier \( \mu \) if and only if \( \lambda = \mu \exp^{-\mu} \)). All \( \lambda \in \mathbb{C}^* \) with \( |\lambda| < 1/e \) have \( \lambda \in H_1 \). There is a unique period 2 component \( H_2 \) which “almost” occupies a left half plane (in the sense that for every \( \theta \in [\pi/2, 3\pi/2] \), there is an \( R > 0 \) such that for all \( r > R \), the parameter \( \lambda = r \exp(i\theta) \in H_2 \)). Every hyperbolic component \( H \) of period \( n \geq 2 \) is simply connected, and the multiplier map \( \mu : H \to D^* \) is a universal cover.

**Lemma 2.1** (Strong attraction only at far parameters). For every period \( n \geq 2 \) and \( r < 1 \), there is an \( R > 0 \) such that any parameter \( \lambda \) for which there is an attracting orbit of multiplier \( |\mu| \leq r \) has \( |\lambda| > R \).

For every hyperbolic component \( H \) of period \( n \geq 2 \) and \( \lambda \in H \), there is a unique homotopy class of curves \( \gamma([0,\infty[) \to H \cup \{\infty\} \) with \( \gamma(0) = \lambda \) and \( \gamma(\infty) = \infty \) such that \( \mu(\gamma(t)) \to 0 \) as \( t \to \infty \) (homotopy with fixed endpoints).
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Proof. If \( a_1, a_2, \ldots, a_n \) is any periodic orbit of period \( n \) under \( \lambda \exp \), then \( (E^\lambda_a(a_m))' = \prod_k a_k \). If the orbit is attracting, then there is some \(|a_k| < 1\). Hence if \(|\lambda| \leq R\), then all \(|a_k| < \xi\) for some \( \xi \) depending on \( R \) and \( n \). Now \( \text{Re}(a_k) \geq -\xi \) implies that \(|a_{k+1}| \geq |\lambda| \exp(-\xi) \geq \exp(-\xi)/e \) (if \(|\lambda| < 1/e\), then \( \lambda \exp \) has an attracting fixed point). Hence we have a lower bound for all \(|a_k|\) and hence for \(|\mu|\).

It follows that any curve \( \gamma'([0, \infty[) \to \mathbb{D}^* \) with \( \lim_{t \to -\infty} \gamma'(t) = 0 \) lifts under any branch of the inverse multiplier map to a curve \( \gamma([0, \infty[) \to H \) with \( \lim_{t \to -\infty} \gamma(t) = \infty \). Conversely, any two curves \( \gamma_1, \gamma_2: ([0, \infty[) \to H \) which limit at \( \infty \) and with \( \mu(\gamma_1(t)) \to 0 \) as \( t \to -\infty \) project under \( \mu \) to two curves \( \gamma_1', \gamma_2': [0, \infty[ \to \mathbb{D}^* \), and these are homotopic in \( \mathbb{D}^* \); hence \( \gamma_1 \) and \( \gamma_2 \) are also homotopic in \( H \) (different branches of \( \mu^{-1} \) can be compensated by loops around 0 in \( \mathbb{D}^* \) and do not matter). \( \Box \)

Remark. This lemma does not show that any two curves \( \gamma_{1,2}: ([0, \infty[) \to H \cup \{\infty\} \) with \( \gamma_i(\infty) = \infty \) are homotopic if the condition \( \mu(\gamma_{1,2}(t)) \to 0 \) is dropped: it would even be conceivable that, for some \( \vartheta \in \mathbb{R} \), a branch of \( \mu^{-1}([0, e^{\vartheta}]\) (an internal parameter ray, Definition 4.5) tends to \( \infty \) as \(|\mu| \to 1\).

It was conjectured by Eremenko and Lyubich [EL2] that this does not happen. The proof of this is not easy: see [S1, Section V] and [RS].

3. Classification of hyperbolic components

Hyperbolic components of Multibrot sets have the helpful property that they have a unique “center” in which the dynamics is postcritically finite [Mc1], [M1], [ES]. If an exponential map has an attracting orbit, it can never be postsingularly finite (with multiplier 0); the center of hyperbolic components “is at \( \infty \)” (in the sense of Lemma 2.1). Fairly enough, it turns out that hyperbolic components of exponential maps have a different feature unknown to the polynomial case: since they stretch out to \( +\infty \) like parameter rays, they can be described by a slight generalization of external addresses: we need finite sequences of integers, followed by a half-integer.

Definition 3.1 (Intermediate external address). An intermediate external address of period \( n \geq 2 \) is a finite sequence \( s_1 s_2 \ldots s_{n-2} s_{n-1} \) with \( s_k \in \mathbb{Z} \) for \( k \leq n-2 \) and \( s_{n-1} \in (\mathbb{Z} + \frac{1}{2}) \).

The lexicographic order on external addresses (infinite sequences over \( \mathbb{Z} \)) extends naturally to intermediate external addresses such as \( s = s_1 s_2 \ldots s_{n-1} \). Intermediate external addresses of period \( n \) (which consist of \( n-1 \) numbers) label hyperbolic components of period \( n \).

As usual, we start with a dynamic consideration.

Definition 3.2 (Attracting dynamic ray). Consider an exponential map \( E_\lambda \) with an attracting orbit of period \( n \geq 2 \) and let \( s = s_1 s_2 \ldots s_{n-1} \) be an intermediate external address of period \( n \). As always, let \( a_1 \) be the attracting periodic
point in the characteristic Fatou component $U_1$. We say $E_\lambda$ has an *attracting dynamic ray at external address* $\underline{s}$ if there is a curve $\gamma: [0, \infty] \to U_1$ such that the following hold:

- $\gamma(0) = a_1$;
- $\lim_{t \to \infty} E_\lambda^{0k} \gamma(t) = +\infty$ for $k = 0, 1, \ldots, n - 2$;
- $\lim_{t \to \infty} E_\lambda^{0(n-1)} \gamma(t) = -\infty$;
- every dynamic ray at an external address $s' < \underline{s}$ is below $\gamma$;
- every dynamic ray at an external address $s' > \underline{s}$ is above $\gamma$.

**Remark.** Since $\gamma$ is in a Fatou component, it must be disjoint from every dynamic ray. Both $\gamma$ and any given dynamic ray tend to $+\infty$ at bounded imaginary parts, so the ray must be above or below $\gamma$ in the following sense: for real $\xi$ sufficiently large, $\gamma$ cuts the half plane $\{ z \in \mathbb{C} : \Re(z) > \xi \}$ into two unbounded parts, one above and one below $\gamma$, and every dynamic ray must tend to $+\infty$ within one of these two parts.

**Lemma 3.3** (Attracting dynamics has external addresses). For every exponential map $E_\lambda$ with an attracting orbit of period $n \geq 2$, there is a unique intermediate external address $\underline{s} = s_1 s_2 \ldots s_{n-1}$ of period $n$ such that there is an attracting dynamic ray at external address $\underline{s}$. Every exponential map from the same hyperbolic component has an attracting dynamic ray at the same external address $\underline{s}$.

**Proof.** The periodic Fatou component $U_0 = U_n$ contains a left half plane. By simple connectivity, it contains a unique homotopy class of curves connecting the attracting periodic point $a_n$ to $-\infty$ eventually within a left half plane, even eventually along $\mathbb{R}^-$. This homotopy class of curves can be pulled back $n-1$ steps to a preferred homotopy class of curves within $U_1$ connecting $a_1$ to $+\infty$. Choose one such curve $\gamma \subset U_1$. This curve avoids dynamic rays, and it is easy to check that the supremum of external addresses of dynamic rays below $\gamma$ has well-defined $n-1$ initial entries in $\mathbb{Z}$: the curve $\gamma$ and its first $n-2$ iterates tend to $+\infty$ (with bounded imaginary parts), so the first $n-1$ entries in the supremum are just the labels of the strips containing the iterates of $\gamma$ (with respect to inverse images of $\mathbb{R}^-$ used in the construction of external addresses). Similarly, the infimum of external addresses of dynamic rays above $\gamma$ supplies $n-1$ well-defined first entries which differ from the lower external addresses only in the last entry and only by one. It is easy to confirm that the external address does not depend on the choice of $\gamma$ or on the parameter chosen from the hyperbolic component of $E_\lambda$ (the intermediate external address is a discrete object which depends continuously on the parameter and on $\gamma$). $\square$

We thus have a combinatorial coding for every hyperbolic component (the unique component of period 1 is coded by the empty sequence), and our goal is
to show that each coding is realized by exactly one hyperbolic component. This will be done in Theorem 3.5 (existence) and Corollary 4.4 (uniqueness).

First we need a lemma to prove the existence of an attracting orbit.

**Lemma 3.4 (Singular orbit in horizontal strip).** Suppose that for some parameter \( \lambda \) there is a real number \( h > 3 \) such \( \Re(\lambda) > h \) and the initial segment \( z_1 = 0, z_2 = \lambda, \ldots, z_n \) of the singular orbit has the property that \( |\Im(z_k)| < h \) for \( 1 \leq k \leq n \). Suppose moreover that \( z_n \) is the first point on the singular orbit with \( \Re(z_n) < 0 \). Then the map \( E_\lambda \) has an attracting periodic orbit of exact period \( n \), and the attracting basin contains the left half plane \( \Re(z) \leq \Re(z_n) + 1 \). As \( \Re(\lambda) \to \infty \) with fixed height \( h \) of the strip, the multiplier tends to 0.

The proof needs a couple of unpleasant calculations, but its idea is very simple: the geometry of the strip containing the singular orbit assures that absolute values of orbit points are dominated by the real parts, and the real parts grow exponentially. Once the orbit reaches a point with negative real part, its absolute value dominates the remaining orbit by far, and its image is extremely close to 0. The contraction coming from the exponential map at this point is far greater than the expansion along the previous orbit, starting at the singular value 0. Therefore, any sufficiently small disk around 0 will map after \( n \) steps to a much smaller (almost-) disk close to the origin. In order to map this disk into itself, its size has to be chosen so that it is neither too large (or we would lose control in the estimates) nor too small (or it would not contain the images after \( n \) steps). It turns out that things work if we choose the disk so that its image at \( z_n \) has radius 1.

**Proof.** The points \( z_2, \ldots, z_{n-1} \) of the orbit are contained within the strip \( S := \{ z \in \mathbb{C} : |\Im(z)| < h \} \) at positive real parts. We show that they all have real parts greater than \( h \). Indeed, this is true for \( z_2 = \lambda \) by assumption, and for the others it follows by induction using \( |\lambda| > h > 1: |z_k| = |\lambda| \exp(\Re(z_{k-1})) > he^h \), so \( \Re(z_k) > h \) for \( k = 2, \ldots, n-1 \) and \( \Re(z_n) < -h \).

Now we show for \( m \leq n \)

\[
(1) \quad \prod_{k=2}^{m} (|z_k| + 1) < (|z_m| + 1)^2.
\]

Indeed, for \( m = 1 \) the empty product on the left equals 1, while \( \Re(z_1) > 3 \) by assumption. For the inductive step, we only need to prove \( (|z_m| + 1)^2 < |z_{m+1}| + 1 \).

We will use the inequality \( (\sqrt{2} x + 1)^2 < 3 \exp(x) \) for all \( x \geq 2 \) and estimate for \( \Re(z_m) > 0 \) as follows:

\[
(|z_m| + 1)^2 < (\sqrt{2} \Re(z_m) + 1)^2 < 3 \exp(\Re(z_m)) < |\lambda| \exp(|z_m|) < |z_{m+1}| + 1.
\]

Our next claim is about \( z_n \), the first point with negative real part:

\[
(2) \quad e|\lambda| \exp(\Re(z_n)) < (|z_n| + 1)^{-2}.
\]
Indeed, we have $|z_n| = |\lambda| \exp(\text{Re}(z_{n-1})) > he^h > 3 \exp(3) > 60$, thus $|\text{Re}(z_n)| > he^h/\sqrt{2} > 40$ and $|z_n| + 1 < \sqrt{2}|\text{Re}(z_n)|$. Using the inequality $2ex^3 < e^x$ for all $x \geq 8$, it follows
$$e|\lambda|(|z_n| + 1)^2 < 2e|\lambda||\text{Re}(z_n)|^2 \leq 2e|\text{Re}(z_n)|^3 < \exp(|\text{Re}(z_n)|).$$
Since the real part of $z_n$ is negative, the claim follows.

Now we can start the actual proof of the lemma. Let $D_n$ be the open disk of radius 1 around the point $z_n$. Pulling back by the dynamics, we obtain open neighborhoods $D_{n-1}$ around $z_{n-1}, \ldots, D_1$ around $z_1 = 0$. These pull-backs are contracting at every step: the derivative of $E_\lambda$ at a point $z'_k \in D_k$ is equal to $E_\lambda(z'_k)$, and its absolute value is bounded above by $|z_{k+1}| + 1$ (using the inductive fact that $\text{dist}(z_k, \partial D_k) \leq 1$). The inverse map is thus contracting with contraction factor at most $1/(|z_{k+1}| + 1)$, and the domain $D_1$ contains a disk around the origin with radius at least $\varrho = \prod_{k=2}^n (|z_k| + 1)^{-1} > (|z_n| + 1)^{-2}$ by equation (1) above (since we have a bound on the contraction for every point on the disks, the claim follows without invoking Koebe’s theorems).

On the other hand, all the points in $D_n$ are contained in the left half plane $\text{Re}(z) \leq \text{Re}(z_n) + 1$. For every point $z$ in this left half plane, $|E_\lambda(z)| \leq |\lambda| \exp(\text{Re}(z_n) + 1) = e|\lambda| \exp(\text{Re}(z_n)) < \varrho$ by equation (2) above. It follows that $E_\lambda(D_n) \subset D_1$ and hence $E_\lambda^n(D_1) \subset D_1$. This is a proper inclusion, so there is an attracting orbit of period at most $n$. The period clearly cannot be smaller than $n$. If within the same strip with imaginary parts bounded by $h$, $\text{Re}(\lambda)$ becomes large, the size of the image of $D_n$ within $D_1$ gets much smaller compared to the size of $D_1$, and the multiplier tends to 0. \( \square \)

Now we come to the existence theorem. We restrict to periods $n \geq 3$ because the hyperbolic components of periods 1 and 2 are completely classified: the unique component of period 1 in $\lambda$-space is coded by the empty external address, and the component of period 2 is coded by the address $\underline{\omega} = 0$ of length 1.

**Theorem 3.5 (Existence of hyperbolic components).** For every $n \geq 3$ and every intermediate external address $\underline{s} = s_1s_2\ldots s_{n-1}$ of period $n$ with $s_1 = 0$, there is a hyperbolic component in $\lambda$-space in which every exponential map has an attracting dynamic ray at external address $\underline{s}$.

This hyperbolic component contains an analytic curve tending to $+\infty$ with imaginary parts converging to $2\pi s_2$ such that along this curve the multipliers of the attracting orbit tend to 0.

**Proof.** Let $s^+_n := s_{n-1} + \frac{1}{2} \in \mathbb{Z}$ and define two periodic external addresses of period $n - 1$ via $s^- := s_1s_2\ldots s_{n-1}$ and $s^+ := s_1s_2\ldots s^+_n$. Let $A := 1 + \max_k \{|s_k|\}$.

In [SZ2, Proposition 3.4] (or [SZ1, Theorem 2.3]), the existence of dynamic rays $g_{s^+}, g_{s^-}$ was shown: these are curves $g_{s^\pm} : [0, \infty[ \to \mathbb{C}$ satisfying
\begin{equation}
(3) \quad g_{s^\pm}(t) = t - \kappa + 2\pi is_1 + r_{s^\pm}(t) \quad \text{with} \quad |r_{s^\pm}(t)| < 2e^{-t}(|\kappa| + 2 + 2\pi AC')
\end{equation}
for $t \geq 1 + 2 \log(|\kappa| + 3)$, where $C' < 2.5$ is a universal constant. The same statement with the same bound holds also for all $\sigma^k(\xi^\pm)$ (replacing $s_1$ by the appropriate entry, of course). In particular, if we let

$$\tau := 1 + 2 \log(|\kappa| + 3 + 2\pi AC'),$$

then $\tau \geq \max\{1 + 2 \log(|\kappa| + 3), \log 2 + \log(|\kappa| + 2 + 2\pi AC')\}$ so that (3) holds and

$$|g_{\xi^\pm}(t) - (t - \kappa + 2\pi is_1)| = |r_{\xi^\pm}(t)| < e^{-(t-\tau)}$$

for $t \geq \tau$. After $n - 2$ iterations, the ray $g_{\xi^\pm}(|\tau, \infty[)$ map to

$$E_\lambda^{(n-2)}(g_{\xi^\pm}(|\tau, \infty[)) = g_{\sigma^{n-2}(\xi^\pm)}([F^{(n-2)}(\tau), \infty[)$$

with

$$g_{\sigma^{n-2}(\xi^\pm)}(t) = t - \kappa + 2\pi is_{n-1}^\pm + r_{\xi^\pm}(t)$$

with $|r_{\xi^\pm}(t)| < e^{-(t-\tau)} \leq 1$. Define a curve

$$\gamma'_\kappa: [F^{(n-2)}(\tau), \infty[ \to C \text{ via } \gamma'_\kappa(t) = t - \kappa + 2\pi is_{n-1};$$

it has the property that $E_\lambda(\gamma'_\kappa) \subset R^-$. The construction assures that the two ray tails $g_{\sigma^{n-2}(\xi^\pm)}([F^{(n-2)}(\tau), \infty[)$ are above respectively below $\gamma'_\kappa([F^{(n-2)}(\tau), \infty[)$ (asymptotically by $i\pi$), and all three curves are disjoint. Moreover, every dynamic ray $g_{\xi'}$ with $\xi' < s_{n-1}$ (that is, $s'_1 < s_{n-1}$) is eventually below $\gamma'$, and if $\xi' > s_{n-1}$ (that is, $s'_1 > s_{n-1}$), then the ray $g_{\xi'}$ is eventually above $\gamma'$.

Pulling back $n - 2$ times along equal branches of $E_\lambda^{(n-1)}$ on $C \setminus R_0^-$, it follows that there is a curve $\gamma_\kappa: [|\tau, \infty| \to C$ between the two rays $g_{\xi^\pm}([\tau, \infty[)$ with

$$E_\lambda^{(n-2)}(\gamma_\kappa(t)) = \gamma'_\kappa(F^{(n-2)}(\tau)) = F^{(n-2)}(\tau) - \kappa + 2\pi is_{n-1}.$$

Just like dynamic rays, this curve inherits the bound for $t \geq \tau$

$$\gamma_\kappa(t) = t - \kappa + 2\pi is_1 + r_\kappa(t) \quad \text{with} \quad |r_\kappa(t)| < 2e^{-(1+\ell)}(\tau + 2 + 2\pi AC')e^{-(t-\tau)}$$

(it follows from the construction of $\gamma_\kappa$ using $n - 2$ pull-backs that $\gamma_\kappa$ satisfies asymptotically the same bounds (3) as the two rays $g_{\xi^\pm}$ which surround $\gamma_\kappa$; this can also be verified in the proof of (3) in [SZ1], [SZ2]).

The curve $\gamma_\kappa$ clearly satisfies the second and third conditions for attracting dynamic rays; the last two are asymptotically satisfied in the sense that for every $\xi'$, the ray $g_{\xi'}$ is above or below $\gamma_\kappa$ (as needed) for sufficiently large $t$ depending on $\xi'$. The first condition requires an attracting orbit, which not every exponential map has.
For any $R \geq 0$, set $\kappa_+^R := R + 2\pi is_1 \pm i\pi = R \pm i\pi$ and $I_R := [\kappa_-^R, \kappa_+^R]$ (a vertical interval of length $2\pi$); then $|\kappa| \leq R + \pi$ for all $\kappa \in I_R$. Set $t_R := 1 + 2\log(R + \pi + 3 + 2\pi AC') > \tau$. For $t \geq t_R$, the bound (5) implies $\text{Im}(\gamma_{\kappa_+^R}(t)) < 0$ and $\text{Im}(\gamma_{\kappa_-^R}(t)) > 0$. For all $\kappa \in I_R$, we have

$$\Re(\gamma_{\kappa}(t_R)) < t_R - R + 1.$$ 

Now fix $R$ large enough so that $t_R - R + 1 < 0$; this is possible since $t_R = O(\log R)$. As $\kappa$ moves from $\kappa_-^R$ to $\kappa_+^R$, by continuity there must be an intermediate value $\kappa^*$ where $\gamma_{\kappa^*}(t^*) = 0$ for some $t^* > t_R > \tau$. The point of this construction is, of course, that $\kappa^*$ has an attracting periodic orbit of period $n$ with the required properties, at least when $R$ is sufficiently large.

Indeed, with $\lambda^* := \exp(\kappa^*)$, the first $n$ postsingular points $0$, $E_{\lambda^*}(0)$, \ldots, $E_{\lambda^*}^{(n-1)}(0)$ are in the strip $|\text{Im}(z)| \leq 2\pi A + \pi + 1$: the $2\pi A$ comes from the bound $|s_k| < A$; the $\pi$ is the bound on $\text{Im}(\kappa)$, and the final 1 is the error bound for the rays. The postsingular orbit $E_{\lambda^*}(0) = \lambda^*, \ldots, E_{\lambda^*}^{(n-2)}(0)$ has positive real parts, while $\Re(E_{\lambda^*}^{(n-1)}(0)) \ll 0$. Now if $R$ is large enough, then Lemma 3.4 shows that there is an attracting orbit of exact period $n$ for $\kappa^*$, and $E_{\lambda^*}^{(n-1)}(\gamma_{\kappa^*}(t^*, \infty[])) \subset U_n$. Hence $\gamma_{\kappa^*}(t^*, \infty[])$ is in the attracting basin with $\gamma_{\kappa^*}(t^*) = 0$, so $\gamma_{\kappa^*}(t^*, \infty[]) \subset U_1$.

Connect $\gamma_{\kappa^*}$ to the attracting periodic point $a_1 \in U_1$ and call this resulting curve $\gamma: [0, \infty[]) \to \mathbb{C}$. It clearly satisfies the first three conditions for attracting dynamic rays. We argued above that the last two conditions were satisfied at least for large $t$. But since the curve $\gamma$ is in the characteristic Fatou component, it is disjoint from all dynamic rays, and it is indeed an attracting dynamic ray at external address $\underline{s} = s_1 s_2 \ldots s_{n-1}$.

By Lemma 3.3, every parameter $\lambda$ in the hyperbolic component of $\lambda^*$ has an attracting dynamic ray at external address $\underline{s}$. This finishes the existence part of the theorem.

To justify the asymptotics, start with large $R$, hence large $t^* > t_R = O(\log R)$, and observe that $\gamma_{\kappa^*}(t^*) = 0$ implies, using (5), that $\kappa^* = t^* + 2\pi is_1 + r_{\kappa^*}(t^*)$ with $|r_{\kappa^*}(t^*)| \to 0$, hence $\text{Im}(\kappa^*) \to 2\pi s_1 = 0$. For $n = 3$, we have $\lambda^* = E_{\lambda^*}^{(n-1)}(\gamma_{\kappa^*}(t^*)) = \gamma_{\lambda^*}'(F(t)) = F(t) - \kappa + 2\pi is_2$, hence $\text{Im}(\lambda^*) \to 2\pi s_2$. For $n > 3$, we argue similarly:

$$\lambda^* = E_{\lambda^*}^{(n-1)}(\gamma_{\kappa^*}(t^*)) = g_{\sigma(\underline{s}_2)}(F(t^*)) + o(1) = F(t^*) - \kappa + 2\pi is_2 + o(1)$$

$$= F(t^*) - t^* + 2\pi (s_2 - s_1) + o(1).$$

This way, we have shown the existence of a map $R \mapsto (\kappa^*, t^*)$ (for sufficiently large $R$) with $\Re(\kappa^*) = R$ and $\gamma_{\kappa^*}(t^*) = 0$. Note that $E_{\lambda^*}^{(n-1)}(t^*) \in \mathbb{R}^-$ for all $t^* = t^*(R)$. It is quite easy to check that $(\partial/\partial \kappa^*)(E_{\lambda^*}^{(n-1)}(t^*)) \to \infty$ as $t^* \to \infty$, so the implicit function theorem shows that the graph of $R \mapsto \kappa^*(R)$ is analytic. It follows from Lemma 3.4 that the attracting multiplier tends to 0 as $R \to \infty.$ \qed
We know from Lemma 2.1 that every hyperbolic component $H$ has a preferred homotopy class of curves $\gamma: [0, \infty) \to H \cup \{\infty\}$ with $\gamma(0) \in H$ and $\gamma(\infty) = \infty$ such that multipliers tend to 0 along this curve. These preferred homotopy classes of curves give a natural vertical order to hyperbolic components, much as the order for dynamic rays: we say that some hyperbolic component is above another hyperbolic component if the corresponding preferred homotopy classes of curves have the appropriate vertical order.

**Corollary 3.6** (Relative position of hyperbolic component). The vertical order of hyperbolic components is the same as the lexicographic order of their intermediate external addresses.

**Proof.** This follows from the previous proof as follows: if $s_0' > s_0''$ are two intermediate external addresses, then there is a periodic external address $s$ between them: $s_0' > s > s_0''$. In the construction in the proof of Theorem 3.5, the curve $\gamma_\kappa$ for $s_0'$ is always above $g_s$, while the corresponding curve for $s_0''$ is always below $g_s$. □

**Remark.** This vertical order can also be expressed in terms of parameter rays [S1], [F]: these are differentiable curves $G_\lambda: (t, \infty) \to \mathbb{C}$ with $G_\lambda(t) \to +\infty$ as $t \to \infty$ such that for $\lambda = G_\lambda(t)$, the singular value escapes with $0 = g_\lambda(t)$; parameter rays are thus disjoint from hyperbolic components. With the notation of the proof of Corollary 3.6, there is a parameter ray $G_s^+$, and the hyperbolic components for $s_0'$ and $s_0''$ are above, respectively below, this ray.

As this paper was being submitted, a manuscript by Devaney, Fagella and Jarque [DFJ] was released which contains the same sufficient condition for the existence of hyperbolic components as in our Theorem 3.5.

4. Uniqueness of the classification

In the following lemma, we construct a curve $\gamma = \gamma_- \cup \{a_1\} \cup \gamma_+$ in the dynamical plane of every exponential map which has an attracting orbit with positive real multiplier. The curve $\gamma_+$ will be used in this section to construct fundamental domains for exponential dynamics, while $\gamma_-$ will be used in Section 7 to construct a “dynamic root”, which helps parametrizing hyperbolic components in a dynamically meaningful fashion.

**Lemma 4.1** (Attracting dynamic ray to boundary fixed point). Let $E_\lambda$ be an exponential map which has an attracting periodic orbit with positive real multiplier and with period $n \geq 2$. Then there is a proper analytic curve $\gamma: \mathbb{R} \to U_1$ which contains the orbit of 0 under $E_\lambda^\circ n$. This curve is unique up to reparametrization and can be written $\gamma = \gamma_- \cup \{a_1\} \cup \gamma_+$ with two disjoint subcurves $\gamma_\pm$ which have the following properties:

- $\gamma_+$ is an attracting dynamic ray and contains the orbit of 0 under $E_\lambda^\circ n$; it connects $a_1$ to $+\infty$;
- $\gamma_-$ starts at $a_1$ and lands at some $w \in \partial U_1$ with $E_\lambda^\circ n(w) = w$. 


Proof. There is a unique open neighborhood $D$ of $a_1$ which corresponds to a round disk in linearizing coordinates around $a_1$ and which contains the singular value 0 on its boundary. Let $\gamma_D \subset D$ be the curve corresponding to a diameter in linearizing coordinates such that 0 sends $D_1$ as a round disk in linearizing coordinates around $E$. Let $\gamma_D$ into itself. Any proper analytic curve which contains the orbit of 0 under $E_1^n$ must be an extension of $\gamma_D$ because the orbit of 0 accumulates at $a_1$, so $\gamma$ is unique if it exists.

The point $a_1$ cuts $\gamma_D$ into two radii; let $\gamma'_+$ be the one which ends at 0 and $\gamma'_-$ the other one (then $\gamma_D = \gamma'_+ \cup \{a_1\} \cup \gamma'_-$). The set $E_\lambda^o(-n)(\gamma'_+) \cap U_1$ consists of countably many curves; let $\gamma_+$ be the one which extends $\gamma'_+$: it starts at $a_1$ and lands at $0$, running through 0. Since $\gamma'_+$ is differentiable at 0, every branch of $E_\lambda^{-1}(\gamma'_+)$ has bounded imaginary parts, and $\gamma_+$ approaches $+\infty$ along bounded imaginary parts. Then $\gamma_+$ is an attracting dynamic ray: it satisfies the first three conditions of Definition 3.2, and by Lemma 3.3, the intermediate external address of $\gamma_+$ is uniquely determined by the hyperbolic component containing $E\lambda$.

There is a unique curve $\gamma_- \subset U_1 \setminus \{a_1\}$ which extends $\gamma'_-$ and which satisfies $E_\lambda^o(\gamma_-) = \gamma_-$: such a curve can be constructed by an infinite sequence of pull-backs, starting at $\gamma'_-$ and always choosing the branch which extends $\gamma'_-$. Then $\gamma_-$ is analytic and $E_\lambda^o: \gamma_- \to \gamma_-$ is a homeomorphism.

The curve $\gamma_-$ can be parametrized (in many ways) as $\gamma_-: \mathbb{R} \to U_1$ so that $E_\lambda^o(\gamma_-(t)) = \gamma_-(t + 1)$. We have $\lim_{t \to +\infty} \gamma_-(t) = a_1$. We want to show that $\lim_{t \to -\infty} \gamma_-(t)$ exists in $\partial U_1$. We will use a modification of the known standard proofs for the landing of external dynamic rays of polynomials. Let

$$U' := U_1 \setminus \bigcup_{k \geq 0} E_\lambda^{okn}(0) \quad \text{and} \quad U'' := E_\lambda^{(on)}(U') \cap U_1.$$ 

Then both domains are open and $E_\lambda^{on}: U'' \to U'$ is a holomorphic covering map, hence a local isometry with respect to the unique normalized hyperbolic metrics of $U''$ and $U'$, and the inclusion $U'' \hookrightarrow U'$ is a contraction. Therefore, the hyperbolic distance in $U'$ between any $\gamma_-(t - 1)$ and $\gamma_-(t)$ is less than between $\gamma_-(t)$ and $\gamma_-(t + 1)$. By continuity, there is an $s > 0$ such that the hyperbolic distance in $U'$ between $\gamma_-(t)$ and $E_\lambda^o(\gamma_-(t))$ is at most $s$, for all $t < 0$. But since $\gamma_-(t) \to \partial U_1$ as $t \to -\infty$ (points $\gamma_-(t)$ for large negative $t$ need longer and longer to iterate near $a_1$), and the density of the hyperbolic metric tends to $\infty$ near $\partial U_1$, it follows that $|\gamma_-(t) - E_\lambda^o(\gamma_-(t))| \to 0$ as $t \to -\infty$. Therefore, any limit point of $\gamma_-$ is either $\infty$ or a fixed point of $E_\lambda^o$. Since the limit set is non-empty and connected, while the set of fixed points is discrete, it follows that $\gamma_-$ lands at a well-defined boundary point of $U_1$ which is either $\infty$ or fixed under $E_\lambda^o$. If the landing point is a fixed point of $E_\lambda^o$, then the curve of the claim is $\gamma := \gamma_- \cup \{a_1\} \cup \gamma_+$. All that remains is to show that the curve $\gamma_-$ does not land at $\infty$.

Suppose that $\gamma_-(t) \to \infty$ as $t \to -\infty$. Then also $E_\lambda^o(\gamma_-(t)) = \gamma_-(t - 1) \to \infty$ as $t \to -\infty$. Since $\gamma_- \subset U_1$, it follows that $E_\lambda^o(n-1)(\gamma_-) \subset U_0 = U_n$ and
Then conjugate to the dynamics on the dynamic plane of $k$ into a conformal conjugation. Since the multipliers at the attracting fixed points are the same, we can define the periodic point $a_1$ under iteration of $E_\lambda^{2n}$.

Remark. In fact, it is not difficult to show that $\gamma$ is a hyperbolic geodesic of $U_1$ [S1]: this is easier if the first return dynamics is conjugated to the map $M \circ \exp: H^- \to H^+$, where $H^-$ is the left half plane, $\exp: H^- \to D^*$ is a universal cover and $M: D \to H^-$ is an appropriate conformal isomorphism.

Using the analytic curve $\gamma$ from Lemma 4.1, we construct fundamental domains for $E_\lambda$ as follows (provided the attracting multiplier is positive real): there is a subcurve $\gamma' \subset \gamma$ which connects the singular value 0 to $+\infty$ (in fact, $\gamma' = \gamma_+ \setminus (\gamma'_+ \cup \{0\})$); the full inverse image $E_\lambda^{-1}(\gamma')$ is a countable collection of analytic curves which connect $-\infty$ to $+\infty$ and which differ by translation by $2\pi i \mathbb{Z}$. The connected components of the complement in $C$ of these curves are fundamental domains for $E_\lambda$; each of them is mapped by $E_\lambda$ conformally onto $C \setminus \gamma'$, and each has bounded imaginary parts because $\gamma'$ is differentiable in its endpoint 0.

Theorem 4.2 (Conformal conjugation). Suppose that two exponential maps have attracting orbits of equal period $n \geq 2$ with equal positive real multipliers, and both have attracting dynamic rays at the same intermediate external address $s_1 s_2 \cdots s_{n-1}$. Suppose in addition that for both maps, the fundamental domains as constructed above are such that a single fundamental domain contains both the periodic point $a_0$ and the singular value. Then both maps are conformally conjugate.

Proof. Step 1. Let $E_\lambda$ and $E_{\lambda'}$ be two exponential maps satisfying the assumptions of the theorem. Let $V_1$ be an open round disk with respect to linearizing coordinates of $a_1$, large enough so as to contain 0 in its interior. For $k = 2, 3, \ldots, n+1$, let $V_k \subset U_k$ be the domain $E_\lambda^{(k-1)}(V_1)$, and let $V_0 := E_\lambda^{-1}(V_1)$. Then $V_1 \supset V_{n+1}$ and $V_0 \supset V_n$. Denote the corresponding sets for $E_{\lambda'}$ as $U'_k$ and $V'_k$, where the size of $V'_1$ is chosen so that the dynamics on it is conformally conjugate to the dynamics on $V_1$, respecting the singular value.

Step 2. We will construct a quasiconformal homeomorphism $\varphi: C \to C$ from the dynamic plane of $E_\lambda$ to the dynamic plane of $E_{\lambda'}$ which will eventually turn into a conformal conjugation. Since the multipliers at the attracting fixed points are the same, we can define $\varphi|_{V_k}: V_k \to V'_k$ as conformal isomorphisms for $k =$
Note that (7) implies 
\[ \varphi(\gamma_k) = \varphi(\gamma_k) \]
This fixes the homeomorphism \( \varphi \), and it becomes unique by fixing the value at a single point. Since \( \varphi(\gamma_k) \) connects \( V_k \) to \( V_k' \) for every \( k \), and we require that \( \varphi(\gamma_k) \) be homotopic to the analogous curve \( \gamma_k' \) relative to \( V_1' \cup \cdots \cup V_n' \). This fixes the homeomorphism \( \varphi : C \to C \) uniquely up to homotopy, and it is well known that \( \varphi \) may be chosen so as to be quasiconformal. Note that we do not claim that the extension of \( \varphi \) away from the \( V_k \) respects the dynamics.

Step 3. Our goal is to promote \( \varphi \) to a conformal conjugation via a sequence of quasiconformal maps \( \varphi_j \) with \( \varphi_0 := \varphi \) and
\[ \varphi_j \circ E_\lambda = E_{\lambda'} \circ \varphi_{j+1}. \]
We will show that \( \varphi_j \to \text{id} \) uniformly on compact sets, which implies that \( E_\lambda = E_{\lambda'} \). The construction is inspired by the theory of spiders [HS].

By induction over \( j \), we will assure that \( \varphi_j|_{V_k} = \varphi_0|_{V_k} \) for all \( j \) and all \( k \), and in particular \( \varphi_j(0) = 0 \) and \( \varphi_j(\lambda) = \lambda' \); moreover, \( \varphi_j(\gamma_k) \) is homotopic to \( \varphi_0(\gamma_k) \) and hence to \( \gamma_k' \), relative to \( V_1' \cup \cdots \cup V_n' \), for all \( k \) and \( j \).

To construct \( \varphi_{j+1} \) from \( \varphi_j \), note that both \( \varphi_j \circ E_\lambda \) and \( E_{\lambda'} \) are universal covers from \( C \) to \( C' \), so there is a homeomorphism \( \varphi_{j+1} : C \to C \) which satisfies (7), and it becomes unique by fixing the value at a single point. Since \( \varphi_j \circ E_\lambda(0) = \lambda' = E_{\lambda'}(0) \), we can and will require \( \varphi_{j+1}(0) = 0 \) to fix \( \varphi_{j+1} \) everywhere uniquely. Note that (7) implies \( \varphi_{j+1}(z + 2\pi i) = \varphi_{j+1}(z) + 2\pi i \) for all \( z \). By (7) and (6), we have
\[ (E_{\lambda'} \circ \varphi_{j+1})|_{V_k} = (\varphi_j \circ E_\lambda)|_{V_k} = (\varphi_0 \circ E_\lambda)|_{V_k} = (E_{\lambda'} \circ \varphi_0)|_{V_k} \subset V_{k+1}. \]
To justify \( \varphi_{j+1}|_{V_k} = \varphi_0|_{V_k} \), all that remains to show is that \( \varphi_{j+1} \) maps \( V_k \) onto \( V_k' \) and not onto another branch of \( (E_\lambda')^{-1}(V_{k+1}) \). It is here that the assumption about the attracting dynamic rays comes in.

Step 3a: the case \( k = 1, 2, \ldots, n-1 \). Since \( (E_{\lambda'} \circ \varphi_{j+1})(\gamma_k) = (\varphi_j \circ E_\lambda)(\gamma_k) = \varphi_j(\gamma_k) \) is homotopic to \( \gamma_k' = E_{\lambda'}(\gamma_k) \) relative \( V_1' \cup \cdots \cup V_n' \), it follows that \( \varphi_{j+1}(\gamma_k) \) is homotopic to a \( 2\pi i \mathbb{Z} \)-translate of \( \gamma_k \) relative \( E_{\lambda'}^{-1}(V_1' \cup \cdots \cup V_n') \).
Now \( \varphi_{j+1}(0) = 0 \) implies \( \varphi_{j+1}|_{V_1} = \varphi_0|_{V_1} \), and since \( \gamma_1 \) is attached to \( V_1 \), it follows that \( \varphi_{j+1}(\gamma_1) \) is homotopic to \( \gamma'_1 \).

For \( k = 2, 3, \ldots, n - 1 \), the number of \( 2\pi i \mathbb{Z} \)-translates of \( \gamma_k \) between \( \gamma_k \) and \( \gamma_1 \) is coded in the intermediate external address, and it equals the number of \( 2\pi i \mathbb{Z} \)-translates of \( \gamma_k' \) between \( \gamma_k' \) and \( \gamma_1' \). This quantity must be respected by \( \varphi_{j+1} \) in the sense that the number of translates of \( \gamma_k' \) between \( \gamma_1' \) and \( \varphi_{j+1}(\gamma_k) \) is the same as between \( \gamma_k \) and \( \gamma_1 \), so \( \varphi_{j+1}(\gamma_k) \) is homotopic to \( \gamma_k' \) and \( \varphi_{j+1}|_{V_k} = \varphi_0|_{V_k} \). For \( k = 2 \), this implies that \( \varphi_{j+1}(\lambda) = \lambda' \). (For later use in Corollary 4.4, we note that this reasoning is unchanged if a constant integer is added to all \( s_{k'} \) : only the differences \( s_{k'} - s_1 \) matter).

Step 3b: the case \( k = 0 \). Since \( V_0 \) is invariant under translation by \( 2\pi i \mathbb{Z} \) and the curve \( \gamma_n \) runs towards \( -\infty \), not \( +\infty \), a different argument is needed to show that \( \varphi_{j+1}|_{V_0} = \varphi_0|_{V_0} \). This is built into the construction: since \( \varphi_{j+1}(0) = 0 \), it follows that \( \varphi_{j+1} \) maps the \( E_\lambda \)-fundamental domain containing 0 to the \( E_{\lambda'} \)-fundamental domain containing 0 (up to homotopies of the boundary curve). Since \( a_0 \) and \( a'_0 \) are in the same fundamental domains of \( E_\lambda \) respectively of \( E_{\lambda'} \) as 0, it follows \( \varphi_{j+1}(a_0) = a'_0 = \varphi_0(a_0) \); now \( \varphi_{j+1}|_{V_0} = \varphi_0|_{V_0} \) follows from the covering property.

Finally, \( \varphi_0(\gamma_n) \) connects \( a'_0 \) to \( -\infty \) in the complement of \( V_1' \cup \cdots \cup V_n' \) and their associated curves \( \gamma_{k'}' \); since there is only one homotopy class of such curves, it follows that \( \varphi_{j+1}(\gamma_n) \) is homotopic to \( \varphi_0(\gamma_n) \) rel \( V_1' \cup \cdots \cup V_n' \).

Step 4. We have a sequence \( (\varphi_j) \) of homeomorphisms \( \mathbb{C} \to \mathbb{C} \); it follows from (7) that all are quasiconformal with the same maximal dilatation as \( \varphi_0 \). All \( \varphi_j \) coincide on \( \bigcup_k V_k \), and all fix the points 0, \( 2\pi i \), and \( \infty \), and quasiconformal homeomorphisms with these properties form a compact space. On the domain \( E^{(j-1)}_\lambda(\bigcup_k V_k) \) with \( j \geq 1 \), the map \( \varphi_j \) is conformal and coincides with all \( \varphi_{j'} \) for \( j' \geq j \). Now \( \bigcup_j E^{(j-1)}_\lambda(\bigcup_k V_k) \) fills up the entire Fatou set, while the Julia set has measure zero by [EL1], [EL3]. Therefore, the support of the bounded quasiconformal dilatation of the \( \varphi_j \) converges to zero, so the \( \varphi_j \) converge uniformly on compact subsets of \( \mathbb{C} \) to an automorphism of \( \mathbb{C} \) fixing 0, \( 2\pi i \) and \( \infty \), hence to the identity. Finally, \( \varphi_j(\lambda) = \lambda' \) implies \( E_\lambda = E_{\lambda'} \). This is what we claimed. \( \diamond \)

In order to conclude that every hyperbolic component is uniquely described by its intermediate external address, we first state a lemma with routine proof.

**Lemma 4.3** (Same fundamental domain). Every hyperbolic component of exponential maps with period \( n \geq 2 \) contains a map for which the attracting orbit has positive real multiplier, and for which a single fundamental domain contains both the periodic point \( a_0 \) and the singular value.

**Proof.** Let \( H \) be the hyperbolic component and let \( \mu: H \to \mathbb{D}^* \) be the multiplier map. Since it is a universal covering, we may find a map \( E_{\lambda_0} \in H \) with
\[ \mu(\lambda_0) > 0. \] There is a curve in \( H \) starting at \( \lambda_0 \) so that the \( \mu \)-image of this curve describes a circle in \( D^* \) around the origin in positive orientation; let \( \lambda_1 \in H \) be the endpoint of the curve.

For \( E_{\lambda_0} \), recall the analytic curve \( \gamma_+ \) from Lemma 4.1: it is an extension of a diameter in linearizing coordinates around \( a_1 \). Let \( \gamma_+^1 \) be the subcurve between 0 and \( E_{\lambda_0}^{\infty}(0) \). As the parameter \( \lambda \) is deformed from \( \lambda_0 \) to \( \lambda_1 \), the curve \( \gamma_+^1 \) extends to a homotopy \( \gamma_+^1(\lambda) \) of curves within \( U_1(\lambda) \) connecting 0 to \( E_{\lambda_1}^{\infty}(0) \) in the complement of the respective singular orbits. In particular, we obtain a curve \( \gamma_+^1(\lambda_1) \) connecting 0 to \( E_{\lambda_1}^{\infty}(0) \) within \( U_1(\lambda_1) \) in the complement of the singular orbit, and this curve is unique up to homotopy. This is not a closed curve, but the beginning and endpoint are on the same radius with respect to linearizing coordinates of \( a_1 \), so it makes sense to say that \( \gamma_+^1(\lambda_1) \) winds once around \( a_1 \).

For the map \( E_{\lambda_1} \), Lemma 4.1 provides another curve which connects 0 to \( E_{\lambda_1}^{\infty}(0) \) (part of a linearizing radius), and this has winding number zero. Call this curve \( \tilde{\gamma}_+^0(\lambda_1) \).

For \( k = 1, 2, \ldots, n \), pull back both curves under \( E_{\lambda_1}^{(n-k)} \), choosing the branch ending at \( E_{\lambda_1}^{(n-k)}(0) \) (as in Lemma 4.1): this yields two curves \( \gamma_+^0(\lambda_1) \) and \( \tilde{\gamma}_+^0(\lambda_1) \) connecting 0 to \( +\infty \); their winding numbers around \( a_1 \) differ by 1.

All the countably many \( E_{\lambda_1}^{-k} \)-preimages of \( \gamma_+^0(\lambda_1) \) bound the fundamental domains of \( E_{\lambda_1} \) as defined before Theorem 4.2, while the \( E_{\lambda_1}^{-k} \)-preimages of \( \tilde{\gamma}_+^0(\lambda_1) \) are deformations of the fundamental domain boundaries of \( E_{\lambda_0} \): their difference shows that between \( E_{\lambda_0} \) and \( E_{\lambda_1} \), the periodic point \( a_0 \) has jumped one fundamental domain up.

A finite repetition of this process (forward or backward) can bring \( a_0 \) into the fundamental domain which contains 0. \( \blacklozenge \)

**Corollary 4.4** (Uniqueness of classification). The intermediate external addresses associated to hyperbolic components associate a bijection between hyperbolic components and intermediate external addresses with \( s_1 = 0 \) of any given period.

**Proof.** By Lemma 3.3, every hyperbolic component is associated to a unique intermediate external address \( s_1, s_2, \ldots, s_{n-1} \); this defines a “classification map” from hyperbolic components to intermediate external addresses. Every component contains a parameter as described in Lemma 4.3, say with multiplier \( \mu = \frac{1}{2} \). By Theorem 4.2 two such exponential maps are identical if their external addresses \( s_1 s_2 \ldots s_{n-1} \) and \( s'_1 s'_2 \ldots s'_{n-1} \) satisfy \( s_k - s_1 = s'_k - s'_1 \) for all \( k \), so the classification map is injective. Since every address with \( s_1 = 0 \) is realized (Theorem 3.5), only addresses with \( s_1 = 0 \) are realized, and the classification map is a bijection. \( \blacklozenge \)

**Remark.** In the proof that the intermediate external address uniquely describes hyperbolic components, we singled out attracting exponential maps for which the same fundamental domain contains the periodic point \( a_0 \) and the singular value 0. This was convenient for the proof, but such maps are dynamically
not special enough so that the corresponding locus in parameter space would stand out. We do not believe that these maps have any real significance other than that they are helpful in the proof. There is, however, a dynamically significant locus within every hyperbolic component, called “central internal ray”: this ray has significance both dynamically and in parameter space and allows us to give a preferred parametrization of hyperbolic components. This will be discussed in Section 7, but some preparations are more conveniently done here.

Recall that the multiplier map \( \mu: H \to \mathbb{D}^* \) is a universal covering map for every hyperbolic component \( H \) of period \( n \geq 2 \), hence there is a conformal isomorphism \( \Phi: H \to H \) with \( \mu = \exp \circ \Phi \), and \( \Phi \) is unique up to translation by \( 2\pi i \mathbb{Z} \) in the range (throughout this paper, \( H \) denotes the left half plane). For our preferred parametrization of the component, we have to specify a choice of this integer translation, which is a combinatorial problem.

**Definition 4.5** (Internal rays of hyperbolic components). An internal ray at angle \( \vartheta \in \mathbb{R}/\mathbb{Z} \) of a hyperbolic component \( H \) is any branch of \( \mu^{-1}(\{0, e^{2\pi i \vartheta}\}) \), where \( \mu: H \to \mathbb{D}^* \) is the multiplier map. In other words, an internal ray is a connected component of the locus in \( H \) where \( \text{arg}(\mu) \) is constant.

For every fixed angle \( \vartheta \in \mathbb{R}/\mathbb{Z} \), there are countably many parameter rays with the same angle (except for the unique period 1 component in \( \lambda \) space), and on each of them the map \( |\mu| \) induces a homeomorphism onto \( ]0, 1[ \).

The internal parameter rays at angle 0 are the loci where the multiplier is real and positive, and they can be distinguished dynamically by the following generalization of Lemma 4.3. Note that the conformal isomorphism \( \Phi: H \to H \) induces a vertical order of parameter rays induced by imaginary parts within \( H \); this order does not depend on the ambiguity in the definition of \( \Phi \) (but since \( H \) is a left half plane, for hyperbolic components with unbounded positive real parts this order is “upside down” when parameter rays are ordered with respect to imaginary parts in their approach to \( +\infty \)).

**Lemma 4.6** (Internal parameter rays at angle 0). Every exponential map which has an attracting orbit of period \( n \geq 2 \) with real positive multiplier has an associated integer-valued index \( \Delta' \) which specifies how many fundamental domains the periodic point \( a_0 \) is above the singular value (or below, if the index is negative). The index \( \Delta' \) is constant for exponential maps on the same internal parameter ray. For every hyperbolic component, the index \( \Delta' \) induces an order preserving bijection between \( \mathbb{Z} \) and parameter rays at angle 0. In particular, for every conformal isomorphism \( \Phi: H \to H \) with \( \mu = \exp \circ \Phi \), there is an integer \( m = m(H, \Phi) \) depending only on \( H \) and \( \Phi \) such that every \( \lambda \in H \) with \( \mu(\lambda) > 0 \) has \( \text{Im}(\Phi(\lambda)) = 2\pi(\Delta' - m) \).

**Proof.** Both \( a_0 \) and the singular value are in some fundamental domain, and the latter are ordered like \( \mathbb{Z} \); therefore, there is a well-defined integer \( \Delta' \) which
specifies how many fundamental domains \( a_0 \) is above 0. By continuity, the index is constant along internal parameter rays at angle 0.

Moving one parameter ray up within the hyperbolic component (i.e. moving to a parameter ray on which \( \text{Im}(\Phi) \) is greater by \( 2\pi \) for any given \( \Phi \)) can be achieved by a deformation as in the proof of Lemma 4.3, and this increases the index \( \Delta' \) by one. \( \square \)

5. Characteristic rays and permutations

In this section, we investigate periodic points at which at least two periodic dynamic rays land, and show that the first return map of the periodic points permutes their rays transitively. This property is well known from quadratic polynomials; it depends on the fact that there is a single singularity, not on the degree of the map.

**Definition 5.1** (Essential orbit, characteristic point and rays). A periodic orbit will be called *essential* if at least two dynamic rays land at each of its points. Suppose that a point \( z \) on an essential orbit is the landing point of two dynamic rays which separate the singular value from all the other points on the orbit of \( z \); then the point \( z \) will be called the *characteristic point* of its orbit. The *characteristic rays* of the orbit will be the two dynamic rays landing at the characteristic point which separate the singular value from all the other rays landing at the same orbit.

The following result describes the combinatorics of dynamic rays landing together. The statement is the same as for polynomials, but the usual proof (using “widths of sectors”) does not apply without modification. Still, essential ideas are borrowed from Milnor [M3].

**Lemma 5.2** (Permutation of dynamic rays). Every essential periodic orbit has exactly one characteristic point and exactly two characteristic rays at every point. If more than two dynamic rays land at any periodic point, then the first return map of the periodic point permutes these rays cyclically.

**Proof.** Let \( z_1, z_2, \ldots, z_n = z_0 \) be a periodic orbit of period \( n \), labeled in the order of the dynamics, and let \( r \geq 2 \) be the number of dynamic rays at each of these points. This number is constant along the orbit. The \( r \) rays landing at any given point \( z_k \) cut the complex plane into \( r \) open connected components which will be called the “sectors” at \( z_k \). These dynamic rays connect \( z_k \) to \( +\infty \), so exactly one of the sectors at \( z_k \) contains a left half plane.

Consider any sector which does not contain a left half plane. Let \( m \) be the position of the first difference in the external addresses of the two rays bounding the sector (with \( m = 1 \) if the first entries are different); this will be called the *singular index* of the sector. For the sector which does contain a left half plane, we let the singular index be \( m := 0 \). Clearly, any sector at \( z_k \) with index \( m \geq 1 \)
maps homeomorphically onto a sector at $z_{k+1}$ with index $m - 1$ (for $m = 1$, it follows from the fact that the sector must contain a horizontal line segment which stretches infinitely to the right and maps onto an infinite segment of $\mathbb{R}^-$). It follows that for each sector, the index equals the number of iterations it takes for this sector to map over a left half plane.

If the index of a sector $S$ at some $z_k$ equals 0 so that the sector contains a left half plane, then $S$ will not map forward homeomorphically. If $g_z$ and $g_{z'}$ are the two dynamic rays bounding $S$, then $E_\lambda(g_z) = g_{\sigma(z)}$ and $E_\lambda(g_{z'}) = g_{\sigma(z')}$. bound a sector $S'$ at $z_{k+1}$ containing the singular value; we call $S'$ the image sector of $S$. If the image sector also contains a left half plane, then the index remains 0; otherwise, the rays at $z_{k+1}$ separate the singular value from a left half plane, and the index is strictly greater than 0. Each $z_k$ has exactly one sector with index 0 (the unique sector containing a left half plane).

Every sector at every point $z_k$ is periodic (in the sense just defined, not as a subset of $\mathbb{C}$), and so is the sequence of the indices. The index sequence of each sector must of course contain at least one index 0, and it cannot be the constant sequence: if all the entries of one cycle of sectors were equal to 0, then all the other cycles of sectors could never have index 0, a contradiction.

Among the $nr$ sectors based at the $n$ points $z_1 \cdots z_n$, there is at least one with largest index. Let $S$ be such a sector and let $z_1$ be the periodic point at which $S$ is based (possibly after cyclic relabeling). If $S$ contains a periodic point $z_k \neq z_1$, then at least one of the sectors at $z_k$ has index at least as large as $S$; after replacing $S$ with such a sector, we may assume that $S$ contains no periodic point $z_k$. Let $S_0$ be the unique sector (at $z_0$) with image sector $S$. Then $S_0$ must contain a left half plane (or the index of $S_0$ would exceed that of $S$), hence $S$ contains the singular value. This makes $z_1$ the characteristic point of its orbit, and the rays bounding $S$ are the characteristic rays. This shows the first statement.

Let $\alpha_1 > \alpha_2 > \cdots > \alpha_r$ be the indices of the sectors at $z_1$; no two of them can be equal because otherwise the corresponding sectors would map forward homeomorphically until they contained a left half plane at the same time. Of course, $\alpha_r = 0$ is the sector containing a left half plane.

Consider any cycle $C$ of sectors and let $\alpha > 0$ be the largest index within its period. Since indices are always decreasing unless they are equal to 0, the index $\alpha$ must occur for a sector containing the singular value but not a left half plane. Let $z_k$ be the periodic point at which this sector is based. If $z_k = z_1$, then the sector with index $\alpha$ is the sector at $z_1$ containing the singular value. Hence all sectors for which the largest index is realized at $z_1$ are on the same orbit. This is true even if the sequence of indices contains several maxima and one of them is realized at $z_1$.

If, however, $z_k \neq z_1$, then $\alpha \leq \alpha_{r-1}$ because the point $z_1$ is within the sector at $z_k$ with index $\alpha$, and so are all the sectors at $z_1$ with indices $\alpha_1 > \cdots > \alpha_{r-1}$. The cycle $C$ of sectors must map through $z_1$, but the only sector at $z_1$ it can map
through is the sector containing a left half plane (the maximum $\alpha$ is not assumed at $z_1$).

It follows that there are at most two cycles of sectors: their representatives at $z_1$ must include either the sector containing the singular value or the sector containing a left half plane, or both. Suppose that not all sectors are on the same orbit. Then the sector at $z_1$ containing a left half plane is fixed under the first return map of $z_1$ and has period $n$, and all the other $r-1$ sectors at $z_1$ are on the same orbit, so they are permuted transitively by the first return map of the dynamics and have period $(r-1)n$. But all sectors must have equal periods because all dynamic rays have equal periods, and this is possible only if $r = 2$. \[\]

6. Dynamic roots

For an understanding of the dynamics, the most important rays are those which land together. We will now show that such are associated to attracting Fatou components.

We will need the partition constructed in [SZ3, Section 4.3] and reviewed in Section 2. Let $u_1,u_2,\ldots,u_{n-1}$ be the first $n-1$ entries of the itinerary of the singular value (such that the singular value 0 is in the region labeled $u_1$, etc.). By definition of the labels, $u_1 = 0$. The $n$th entry is not defined.

For a dynamic ray $g_s$ with bounded external address $s$, it may happen that there are two curves within $U_1$ which connect the singular value to $+\infty$ such that they separate $g_s$ from $U_0$. In this case, we will say that the ray $g_s$ is surrounded by $U_1$, and rays $g_{s'}$ with bounded external addresses $s'$ sufficiently close to $s$ will also be surrounded by $U_1$, so this is an open property in the sequence space $S$. The Fatou component $U_1$ contains infinitely many non-homotopic curves connecting the singular value to $+\infty$, namely pull-backs of curves connecting $E^{o(n-1)}(0)$ to $\infty$ within the Fatou component $U_n$ containing a left half plane. Hence each ray which is not surrounded by $U_1$ is either above or below $U_1$ in the sense that the ray approaches $+\infty$ above or below all such curves within $U_1$. Similarly, we will say that rays are above or below $U_1$ (but not $U_0$). Since every $U_i$ (for $i = 1,2,\ldots,n-1$) is disjoint from its $2\pi i \mathbb{Z}$-translates, and $U_{n-1}$ contains a preimage of an unbounded part of $\mathbb{R}^-$, each $U_i \neq U_0$ has bounded imaginary parts.

**Lemma 6.1** (Periodic external address above $U_1$). Let $U_1$ be the characteristic Fatou component of an exponential map with attracting orbit of period $n \geq 2$. Then

$$g^+ := \inf \{ g \in S : g_s \text{ is above } U_1 \} \quad \text{and} \quad g^- := \sup \{ g \in S : g_s \text{ is below } U_1 \}$$

define two periodic external addresses of period $n$ with equal itineraries.
Proof. It may not be clear a priori that $s^+$ and $s^-$ are well-defined sequences in $\mathbb{Z}^N$, but their first entries are certainly integers $s^+_1 \geq s^-_1$. If $s^+_1 = s^-_1$, then iteration gives
\[ \sigma(s^+) = \inf\{ \underline{s} \in S : g_{\underline{s}} \text{ is above } U_2 \} \quad \text{(and similarly for } \sigma(s^-)) \]
so that $s^+_2 \geq s^-_2$ are well-defined. Repeating this argument shows that
\[ \sigma^k(s^+) = \inf\{ \underline{s} \in S : g_{\underline{s}} \text{ is above } U_{k+1} \} \]
where $k$ is the first integer such that $s^+_{k+1} > s^-_{k+1}$. Now $U_{k+1}$ surrounds or contains a preimage of an unbounded part of $\mathbb{R}^-$, hence a preimage of a left half plane, and the next step is different:
\[ \sigma^{k+1}(s^+) = \inf\{ \underline{s} \in S : g_{\underline{s}} \text{ is below } U_1, \text{ and } U_{k+2} \text{ does not separate } g_{\underline{s}} \text{ from } U_1 \} \]
(the problem is that the lower part of $U_{k+1}$ maps above $U_1$, while the upper part maps below $U_1$, and we are interested only in the image of the upper part; another way of saying this is that $\sigma^{k+1}(s^+)$ is the infimum of sequences which are above those parts of $U_{k+2}$ that are below $U_1$). Note that the infimum still has finite first entry because some part of $U_{k+2}$ is below $U_1$. We can continue to iterate this:
\[ \sigma^{k+m}(s^+) = \inf\{ \underline{s} \in S : g_{\underline{s}} \text{ is below } U_m, \text{ and } U_{k+m+1} \text{ does not separate } g_{\underline{s}} \text{ from } U_m \} \]
where $m$ is the first integer such that $U_m$ surrounds or contains a preimage of an unbounded part of $\mathbb{R}^-$, or there is a preimage of an unbounded part of $\mathbb{R}^-$ below $U_m$ which separates $U_m$ from $U_{k+m+1}$. If that happens, we get
\[ \sigma^{k+m+1}(s^+) = \inf\{ \underline{s} \in S : g_{\underline{s}} \text{ is above } U_{k+m+2} \} \]
and we are back to the initial situation. Repeating these arguments for a total of $n - 2$ iterations, we see that either
\[ (8) \quad \sigma^{n-2}(s^+) = \inf\{ \underline{s} \in S : g_{\underline{s}} \text{ is above } U_{n-1} \}, \]
or there is a $k' \leq n$ such that (8) is false, but
\[ (9) \quad \sigma^{n-2}(s^+) = \inf\{ s \in S : g_s \text{ is below } U_{k'}, \text{ and } U_{n-1} \text{ does not separate } g_s \text{ from } U_{k'} \}. \]
In the case of (8), we get
\[ \sigma^{n-1}(s^+) = \inf\{ s \in S : g_s \text{ is in the same region } R_u \text{ as } U_1, \text{ and } U_n = U_0 \text{ does not separate } g_s \text{ from } U_1 \} \]
Similarly, in the case of (9), we get
\[ \sigma^n(s^+) = \inf \{ s \in S : g^s \text{ is in the same region } R_{U_k + 1}, \text{ and } U_n \text{ does not separate } g^s \text{ from } U_k + 1 \} \]
and again \( \sigma^n(s^+) = \inf \{ s \in S : g^s \text{ is above } U_1 \} \).

Thus in every case \( \sigma^n(s^+) = s^+ \), and this is a periodic sequence over \( \mathbb{Z} \) with period \( n \). The same applies to \( s^- \), and both have itineraries of period (dividing) \( n \).

The first \( n - 1 \) entries in the itinerary of \( g_{s^+} \) are \( u_1 u_2 \cdots u_{n-1} \), which are the same as in the itinerary of \( U_1 \) or of the singular value. The \( n \)th entry in the itinerary of \( U_1 \) is not defined (because the image component \( U_n \) extends through all fundamental domains), but we saw above that the \( n \)th entry in the itinerary of \( g_{s^+} \) is either \( u_1 \) or \( u_{k+1} \), and \( g_{s^-} \) has the same itinerary.

**Theorem 6.2** (Two rays at boundary fixed point). Every periodic Fatou component with attracting dynamics of period \( n \geq 2 \) has a unique point on its boundary which is fixed by the first return map of the component and which is the landing point of at least two periodic dynamic rays. The characteristic point of this periodic orbit is on the boundary of the characteristic Fatou component.

**Proof.** Existence follows from Lemma 6.1: the periodic dynamic rays \( g_{s^+} \) and \( g_{s^-} \) have identical itineraries, so by the results of [SZ3] mentioned in Section 2, they land at a common periodic point \( z \), say. In order to prove that \( z \in \partial U_1 \), let \( l \) be the hyperbolic distance of \( z \) to \( \partial U_1 \) in the hyperbolic domain \( \mathbb{C} \setminus \bigcup_{k \geq 0} E^\infty_\Lambda(k)(0) \).

Assume that \( l > 0 \). The hyperbolic distance between the unique periodic inverse image of \( z \) and \( U_0 = U_n \) is then less than \( l \). We take \( n - 1 \) further pull-back steps along the periodic orbit of \( z \); since the itinerary of \( z \) in those steps is the same as that of the singular orbit, the branches for the pull-back of \( z \) are those mapping \( U_n \) to \( U_{n-1}, U_{n-2}, \ldots, U_1 \), and hyperbolic distances are decreased in every step. After \( n \) steps, \( z \) is mapped back to itself and its hyperbolic distance to \( U_1 \) has decreased. This contradiction shows that \( l = 0 \) and \( z \in \partial U_1 \).

Let \( z_1 \) be the characteristic point of the orbit of \( z \) (cf. Definition 5.1 and Lemma 5.2). The two characteristic rays separate the singular value from the orbit of \( z \) (except \( z_1 \) itself) and from a left half plane. If \( z_1 \neq z \), then the characteristic rays would separate \( z \in \partial U_1 \) from a left half plane, and this is a contradiction. This proves the existence claim for the periodic Fatou component \( U_1 \), and for the others it follows easily.

Now we show uniqueness. The point \( z_1 \), together with the two characteristic rays landing at it, cut \( \mathbb{C} \) into two open parts; let \( V \) be the one containing the
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singular value. Suppose that there is another periodic point \( z'_1 \in \partial U_1 \) which is fixed by the first return map of \( U_1 \) and which is the landing point of two periodic dynamic rays. We have \( z'_1 \in V \) because \( V \) contains \( \overline{U_1 \setminus \{z_1\}} \). The two characteristic dynamic rays landing at \( z'_1 \) are then contained in \( V \) as well. But by symmetry between \( z_1 \) and \( z'_1 \), it also follows that the two characteristic rays landing at \( z'_1 \) bound an open sector \( V' \) which contains \( z_1 \) and all its rays, and this is a contradiction. \( \Box \)

**Remark.** The same statement holds also for parabolic dynamics; the proof requires only the same modifications as in [SZ3, Section 4.3].

**Definition 6.3** (Dynamic root). In any exponential dynamics with attracting orbit of period \( n \geq 2 \), the unique point of the characteristic Fatou component which is fixed under the first return map of the component and which is the landing point of at least two dynamic rays (as described in Theorem 6.2) will be called the *dynamic root* of the characteristic Fatou component.

**Lemma 6.4** (Rays at dynamic root). *In attracting exponential dynamics, the two characteristic rays of the dynamic root of the characteristic Fatou component separate this Fatou component from all other periodic Fatou components.*

**Proof.** Let \( U_1 \) be the characteristic Fatou component and let \( z_1 \) be its dynamic root. The characteristic dynamic rays at \( z_1 \) separate the singular value and thus \( U_1 \) from all other points on the orbit of \( z_1 \). Every periodic Fatou component \( U_i \) has a unique point \( z_i \) on its boundary (Theorem 6.2). If \( z_i \neq z_1 \), then \( U_i \) is separated from the singular value by the characteristic ray pair. Let the periods of the attracting orbit and of \( z_1 \) be \( n \) and \( k \), respectively. The number of different periodic Fatou components with \( z_1 \) on its boundary is exactly \( n/k \) and the first return map of \( z_1 \) must permute these \( k \) components cyclically. Hence the gaps between cyclically adjacent periodic Fatou components at \( z_1 \) are also permuted cyclically, and at least one of them must contain a periodic dynamic ray landing at \( z_1 \); hence all gaps do, and all periodic Fatou components at \( z_1 \) are separated by periodic dynamic rays landing at \( z_1 \). (Conversely, it follows from Lemma 5.2 that all the rays landing at \( z_1 \) are separated by periodic Fatou components provided at least two periodic Fatou components have \( z_1 \) as their common dynamic root.) \( \Box \)

7. Parametrization of hyperbolic components

**Theorem 7.1** (Parametrization of hyperbolic components). *For every hyperbolic component \( H \) of period \( n \geq 2 \), there is a unique conformal isomorphism \( \Phi_H: H \to H \) with \( \mu = \exp \circ \Phi_H \) such that if \( \mu(\lambda) > 0 \), then \( \Im(\Phi_H(\lambda)) = 2\pi \Delta \), where the integer \( \Delta \) specifies how many fundamental domains the periodic point \( a_0 \in U_0 \) is above the dynamic root \( z_0 \) of \( U_0 \) (or below if \( \Delta < 0 \)).

**Proof.** We already know from Lemma 4.6 the existence of a conformal isomorphism \( \Phi: H \to H \) with \( \mu = \exp \circ \Phi_H \), which is unique up to addition of a constant
in $2\pi i \mathbb{Z}$ in the range, so we only need to justify the combinatorial interpretation of imaginary parts if $\mu(\lambda) > 0$.

Choose any $\lambda \in H$. Let $z_1$ be the dynamic root of the characteristic Fatou component $U_1$ and let $\xi_1 < \xi_2$ be the external addresses of its two characteristic rays. Let $z_0$ be the dynamic root of $U_0$ (with $E_\lambda(z_0) = z_1$) and let $\xi'_1 < \xi'_2$ be the least and greatest external addresses (with respect to lexicographic ordering) of two dynamic rays landing at $z_0$. By continuity, these external addresses are the same for every $\lambda \in H$.

Since the rays $g_{\xi_1}$ and $g_{\xi_2}$ surround 0, they bound the characteristic sector at $z_1$, which is the image of the sector at $z_0$ containing a left half plane. It follows that $\sigma(\xi'_1) = \xi_1$ and $\sigma(\xi'_2) = \xi_2$, and there is an $m \in \mathbb{Z}$ with $\xi'_2 = m \xi_2$ and $\xi'_1 = (m + 1) \xi_1$ (concatenation). By the translation symmetry, it follows that for every $m' \in \mathbb{Z}$, the rays at addresses $m' \xi_2$ and $(m' + 1) \xi_1$ land together at $z_0 + 2\pi i (m' - m)$. Since $\xi_1$ and $\xi_2$ are characteristic external addresses, no $\sigma^k(\xi_1)$ or $\sigma^k(\xi_2)$ is contained in $]\xi_1, \xi_2[$; therefore, no $\sigma^k(\xi_1)$ or $\sigma^k(\xi_2)$ is contained in any $]m' \xi_1, m' \xi_2[$ for $m' \in \mathbb{Z}$. It follows that there is an $m_1 \in \mathbb{Z}$ with

$$m_1 \xi_2 < \xi_1 < \xi_2 < (m_1 + 1) \xi_1;$$

since the ray pair $g_{\xi_1}$ and $g_{\xi_2}$ surround the singular value 0, it follows that the ray pair at addresses $m_1 \xi_2$ and $(m_1 + 1) \xi_1$ surrounds 0 as well. We had seen above that the ray pair at addresses $m \xi_2$ and $(m + 1) \xi_1$ lands at $z_0$.

Now suppose that $\mu(\lambda) > 0$. Then $z_0$ is exactly $m - m_1$ fundamental domains above 0. All we used for this reasoning are the external addresses $\xi_1$, $\xi_2$, $\xi'_1$ and $\xi'_2$ which are the same for any $\lambda \in H$, so the index $m - m_1$ is the same throughout $H$.

Recall the index $\Delta'$ from Lemma 4.6 which specifies, whenever $\mu(\lambda) > 0$, how many fundamental domains $a_0$ are above 0. Therefore, $\Delta := \Delta' - (m - m_1)$ specifies how many fundamental domains $a_0$ are above $z_0$. By Lemma 4.6, every conformal isomorphism $\Phi: H \to \mathbb{H}$ with $\mu = \exp \circ \Phi$ has an integer $m_\Phi \in \mathbb{Z}$ such that for all $\lambda \in H$ with $\mu(\lambda) > 0$, $\Im(\Phi(\lambda)) = 2\pi (\Delta' - m_\Phi)$. Setting $\Phi_H := \Phi + 2\pi i \left( m_\Phi - (m - m_1) \right)$ yields the desired conformal isomorphism. It is clearly unique. \hfill \Box

Having stated this theorem, we should outline why this parametrization is indeed a preferred one, for example over the one from Lemma 4.6 counting fundamental domains between $a_0$ and the singular value. The dynamic root clearly has dynamic significance, and the invariant curve $\gamma^-$ from Lemma 4.1 is part of an analytic curve containing the singular orbit. Both might be linked for any particular exponential map: the following lemma states when this happens. It should come as no surprise that the locus of such maps stands out in parameter space; this is discussed at the end of this section.

**Lemma 7.2 (Invariant curve lands at dynamic root).** For every exponential map which has an attracting orbit of period $n \geq 2$ with positive real multiplier,
the invariant curve $\gamma_-$ from Lemma 4.1 lands at the dynamic root $z_1 \in \partial U_1$ if and only if $\Delta = 0$.

Proof. The curve $\gamma_-$ starts at $a_1$ and lands at a point $w_1 \in \partial U_1$ with $E_\lambda^{o_n}(w_1) = w_1$. Let $\gamma_0 \subset U_0$ be the branch of $E_\lambda^{-1}(\gamma_-)$ starting at $a_0$; it lands at a point $w_0 \in \partial U_0$ with $E_\lambda^{o_n}(w_0) = w_0$. Clearly, $w_1 = E_\lambda(w_0)$, so $w_0$ is the dynamic root of $U_0$ if and only if $w_1$ is the dynamic root of $U_1$.

Recall the analytic curve $\gamma_+ \subset U_1$ from Lemma 4.1: it connects $a_1$ to $+\infty$ such that $\gamma_+ := E_\lambda^{o_n}(\gamma_+) \subset \gamma_+$, and $\gamma_+$ contains the singular orbit within $U_1$. The fundamental domains for $E_\lambda$ are bounded by $E_\lambda^{-1}(\gamma_+ \setminus \gamma_-)$.

Since $\gamma_0$ is disjoint from the boundary of the fundamental domains, it follows that $w_0$ is in the same fundamental domain as $a_0$, so $w_0$ can be the dynamic root $z_0 \in U_0$ only if $a_0$ and $z_0$ are in the same fundamental domain, i.e. only if $\Delta = 0$.

Conversely, we show that $w_0$ is the only boundary point of $U_0$ which is fixed by $E_\lambda^{o_n}$ and which is in the same fundamental domain as $a_0$; this implies that whenever $\Delta = 0$, then $\gamma_0$ lands at $z_0$ and $\gamma_-$ lands at $z_1$.

Suppose there is another point $w'_0 \in \partial U_0$ with $E_\lambda^{o_n}(w'_0) = w'_0$ in the same fundamental domain as $w_0$. There is a curve $\gamma'_0 \subset U_0$ with $E_\lambda^{o_n}(\gamma'_0) \supset \gamma'_0$ which lands at $w'_0$ (take a point $b'_0$ in a linearizing neighborhood of $w'_0$ and pull it back repeatedly so as to obtain a sequence $(b'_k)$ converging to $w'_0$; these can be connected by a curve $\gamma'_0$ as required). Let $(b_k)$ be an analogous sequence of points converging to $w_0$.

Connect $b_0$ to $b'_0$ by a differentiable curve $\Gamma_0 \subset U_0$ which avoids the fundamental domain boundaries and the curve $E_\lambda^{o(n-1)}(\gamma_+)$ which contains the singular orbit within $U_0$. Then $E_\lambda^{-n}(\Gamma_0)$ contains a curve $\Gamma_1$ connecting $b_1$ to $b'_1$ and homotopic to $\Gamma_0$ in $U'_0 := U_0 \setminus (E_\lambda^{-1}(\gamma_+) \cup \{a_0\})$ (the fact that the same branch of $E_\lambda^{-n}$ fixes both $w_0$ and $w'_0$ uses the assumption that both points are within the same fundamental domain and would fail if $w'_0$ was an arbitrary fixed point of $E_\lambda^{-n}$ in $\partial U_0$). This can be repeated, and the hyperbolic lengths of $\Gamma_k$ within $U_k'$ become shorter each time, while $b_k, b'_k \to \partial U_0$, and this implies $w_0 = w'_0$. \qed

For the following definition, recall the conformal isomorphism $\Phi_H: H \to \mathbb{H}$ which maps parameter rays of $H$ to horizontal lines in $\mathbb{H}$.

Definition 7.3 (Height and central internal ray). Given a hyperbolic component $H$ of period $n \geq 2$ with preferred conformal isomorphism $\Phi_H: H \to \mathbb{H}$, we define the height of a parameter ray as the number $h \in \mathbb{R}$ such that $\text{Im}(\Phi_H(\lambda)) = 2\pi h$ for $\lambda$ on this parameter ray.

The central parameter ray of $H$ is the parameter ray at height $h = 0$.

Remark. The angle of a parameter ray is the projection of $h$ to $\mathbb{R}/\mathbb{Z}$. If $\lambda \in H$ has $\mu(\lambda) > 0$, then clearly $h = \Delta$. Note that for components of period $n \geq 3$ in $\lambda$-space (and of period $n \geq 2$ in $\kappa$-space) a ray with larger height is below a ray with smaller height: see the discussion after Definition 4.5. This is
unavoidable if among nearby rays one wants rays at larger heights to have larger angles of their multipliers.

Figure 3. Hyperbolic components of periods 3 and 4, with internal parameter rays at integer heights drawn in. In both components, the central parameter rays are highlighted. Their landing points are the roots of the component (at which the components bifurcate from components of period 1 respectively 2).

**Remark.** We have classified hyperbolic components in terms of intermediate external addresses. A different coding would be in terms of the external addresses of the two characteristic dynamic rays landing at the root of the characteristic Fatou component: clearly, all exponential maps from the same hyperbolic component have the same external addresses at the dynamic rays landing at the root, so there is an algorithm to turn the intermediate external address of the component into the external addresses of the characteristic rays. The converse is much easier: knowing the external addresses of the two characteristic rays, it is not hard to write down the intermediate external address of the attracting dynamic ray and thus of the component. Both algorithms can be found in [RS]. While the coding in terms of intermediate external addresses leads to the easier classification, the characteristic external addresses are more easily related to the structure of the Multibrot sets $\mathcal{M}_d$ and their limiting configurations as $d \to \infty$.

The boundary of hyperbolic components and bifurcations. While this paper completely describes hyperbolic components of exponential maps, their boundary properties are discussed in [S1, Section V] and [RS]. To complete the picture, we mention some results here.

By Theorem 7.1, every hyperbolic component $H$ (except the period 1 component in $\lambda$-space) comes with a preferred conformal isomorphism $\Phi_H:H \to \mathbb{H}$. It extends as a homeomorphism to the closures $\overline{\Phi_H:H} \to \overline{\mathbb{H}}$. This result requires a much better understanding of the exponential parameter space. It implies that every hyperbolic component has connected boundary $\partial H = \overline{\Phi_H^{-1}(i\mathbb{R})}$, which was conjectured by Eremenko and Lyubich [EL2].
Different hyperbolic components may have common boundary points. This happens if and only if one hyperbolic component bifurcates from another, and then both components have a unique boundary point in common. The structure of bifurcations between hyperbolic components in exponential parameter space is cleared up in [RS]. In particular, this proves a third conjecture in [EL2] which states that there are infinitely many “trees” of hyperbolic components such that two components are in the same tree if and only if they can be connected via a finite chain of components so that adjacent components in the chain have common boundary.

Every internal parameter ray of $H$ at height $h$ lands at a well-defined parameter in $\partial H$ with indifferent orbit so that the landing point depends continuously on $h$. The landing point of the central parameter ray ($h = 0$) is called the root of $H$, and it is significant in several ways: the root of $H$ is the only boundary point which may simultaneously be a boundary point of another hyperbolic component $H_0$, and this happens if and only if $H$ bifurcates from $H_0$. Moreover, every boundary point of $H$ is the landing point of one or two external parameter rays (curves consisting of parameters for which the singular orbit converges to $\infty$ under iteration, see [S1, Chapter II] or [F]). Boundary points of $H$ at positive heights are the landing points of parameter rays which come from $+\infty$ below $H$, while boundary points at negative heights are landing points of parameter rays above $H$. The root point of $H$ is the landing point of two parameter rays, one above and one below $H$ [S1, Chapter IV]. This shows once more the significance of our preferred parametrization from Theorem 7.1.

References

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