AN APPLICATION OF THE TOPOLOGICAL
RIGIDITY OF THE SINE FAMILY

Gaofei Zhang

Nanjing University, Department of Mathematics
Hankou Road, No. 22, Nanjing, 210093, P. R. China; zhanggf@hotmail.com

Abstract. By using a result of Domínguez and Sienra on the topological rigidity of the Sine
family, we give a different proof of a result in [8] which says that, for any bounded type irrational
number $0 < \theta < 1$, the boundary of the Siegel disk of $e^{2\pi i \theta} \sin(z)$ is a quasi-circle passing through
exactly two critical points $\pi/2$ and $-\pi/2$.

1. Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be two continuous maps. We say that $f$ and $g$ are
topologically equivalent to each other if there are two homeomorphisms $\theta_1, \theta_2 : \mathbb{C} \to \mathbb{C}$ such that $f = \theta_1^{-1} \circ g \circ \theta_2$. The following lemma on the topological rigidity of the
Sine family was proved by Domínguez and Sienra (Lemma 1, [3]).

Lemma 1.1. Let $f$ be an entire function. If $f(z)$ is topologically equivalent to
$\sin(z)$, then $f(z) = a + b \sin(cz + d)$ where $a, b, c, d \in \mathbb{C}$, and $b, c \neq 0$.

The main purpose of this paper is to use this lemma to give a new but simpler
proof of the following result, which was previously proved in [8].

Theorem. Let $0 < \theta < 1$ be a bounded type irrational number. Then the
boundary of the Siegel disk of $f_\theta(z) = e^{2\pi i \theta} \sin(z)$ is a quasi-circle which passes through
exactly two critical points $\pi/2$ and $-\pi/2$.

Here is the idea of the new proof. Following the idea of Cheritat [2], we first
construct a Ghys-like model $G_\theta(z)$ for the map $e^{2\pi i \theta} \sin(z)$. Next we do a stan-
dard quasi-conformal surgery on the model map $G_\theta(z)$ and get an entire function
$g_\theta(z)$. We then derive the theorem from Lemma 1.1 and the fact that $g_\theta$ and $f_\theta$ are
topologically equivalent to each other.

It is interesting to contrast the proof here with the one in [8]. The proof in
[8] is based on a non-symmetric model map $\tilde{f}_\theta(z)$. One of the most important
characteristics of this model map is its periodicity which plays a key role in the
proof there. In this paper, we use the symmetric model map $G_\theta$, which does not
have the periodicity. A priori, the resulted entire map $g_\theta$ may not be periodic either.

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But because of the topological rigidity of the Sine family, the map \( g_\theta \) turns out to be equal to \( f_\theta \), and this implies the theorem.

I would like to mention that by using trans-quasiconformal surgery introduced in [7] and the techniques in this paper, the theorem was recently extended to David type Siegel disks of the Sine family[9].

2. A Ghys-like model

We use the idea of Cheritat in the following construction (see [2]). Let \( \Delta \) be the unit disk and \( T(z) = \sin(z) \). It follows that the map \( T(z) \) has exactly two critical values 1 and \(-1\). Let \( D \) be the component of \( T^{-1}(\Delta) \) which contains the origin. The following lemma is obvious and we leave the proof to the reader.

**Lemma 2.1.** \( D \) is a Jordan domain which is symmetric about the origin. Moreover, \( \partial D \) passes through exactly two critical points \( \pi/2 \) and \(-\pi/2 \), and the map \( T|_{\partial D} : \partial D \to \partial \Delta \) is a homeomorphism.

For \( k \in \mathbb{Z} \), let \( D_k = \{ z + k\pi \mid z \in D \} \). It follows that \( D_0 = D \). Note that for any two distinct integers \( k \) and \( j \), if \( \partial D_k \cap \partial D_j \neq \emptyset \), then any point in \( \partial D_k \cap \partial D_j \) must be a critical point of \( \sin(z) \). This, together with Lemma 2.1, implies

**Lemma 2.2.** For any \( k \in \mathbb{Z} \), \( \partial D_k \cap \partial D_{k+1} = \{ k\pi + \pi/2 \} \). For any two distinct integers \( k \) and \( j \) with \( |k - j| \neq 1 \), \( \partial D_k \cap \partial D_j = \emptyset \). In particular, if \( \Lambda \subset \mathbb{Z} \) contains infinitely many elements but \( \Lambda \neq \mathbb{Z} \), then \( \bigcup_{k \in \Lambda} \partial D_k \) is disconnected.

Let \( \phi : \hat{\mathbb{C}} - \bar{\Delta} \to \hat{\mathbb{C}} - \bar{\Delta} \) be the Riemann map such that \( \phi(\infty) = \infty \) and \( \phi(1) = \pi/2 \). Since \( \Delta \) and \( D \) are both symmetric about the origin, we have

**Lemma 2.3.** \( \phi \) is odd.

For \( z \in \mathbb{C} \), let \( z^* \) denote the symmetric image of \( z \) about the unit circle. Define

\[
G(z) = \begin{cases} 
T \circ \phi(z) & \text{for } z \in \mathbb{C} - \Delta, \\
(T \circ \phi(z^*))^* & \text{for } z \in \Delta - \{0\}.
\end{cases}
\]

From the construction of \( G(z) \), we have

**Lemma 2.4.** \( G(z) \) is holomorphic in \( \mathbb{C} - \{0\} \) and is symmetric about the unit circle. Moreover, \( G(z) \) is odd.

By Lemma 2.1, we see that \( G|_{\partial \Delta} : \partial \Delta \to \partial \Delta \) is a critical circle homeomorphism. By Proposition 11.1.9 of [5], we get

**Lemma 2.5.** There exists a unique \( t \in [0,1) \) such that \( e^{2\pi i t} G|_{\partial \Delta} : \partial \Delta \to \partial \Delta \) is a critical circle homeomorphism of rotation number \( \theta \).

Let \( t \in [0,1) \) be the number given in Lemma 2.5. Let us denote \( e^{2\pi i t} G(z) \) by \( G_\theta(z) \). By Herman–Swiatek’s quasi-symmetric linearization theorem on critical circle mappings [6], it follows that there is a quasi-symmetric homeomorphism \( h : \partial \Delta \to \partial \Delta \) such that \( h(1) = 1 \) and \( G_\theta|_{\partial \Delta} = h^{-1} \circ R_\theta \circ h \), where \( R_\theta \) is the rigid rotation given by \( \theta \).
Lemma 2.6. $G_\theta$ and $h$ are both odd.

Proof. The assertion that $G_\theta$ is odd follows from that $G(z)$ is odd (Lemma 2.4). Now let us prove that $h$ is odd. First let us show that $h(-1) = -1$. Let $U(N)$ be the number of the points in $\{G_\theta^k(1) \mid k = 1, \cdots, N\}$ which lie in the upper half circle. Let $L(N)$ be the number of the points in $\{G_\theta^k(-1) \mid k = 1, \cdots, N\}$ which lie in the lower half circle. Note that $G_\theta$ is odd, it follows that $U(N) = L(N)$. Since the angle length of the image of the upper half circle under $h$ is equal to the limit of $2\pi U(N)/N$ as $N \to \infty$, and the angle length of the image of the lower half circle under $h$ is equal to the limit of $2\pi L(N)/N$ as $N \to \infty$, it follows that the angle length of the images of the upper half circle and the lower half circle under $h$ are equal to each other. This implies that $h(-1) = -1$.

To show that $h$ is odd, let $t(z) = -h(-z)$. We have $t(1) = 1 = h(1)$. Since $t^{-1} \circ R_\theta \circ t(z) = -G_\theta|\partial\Delta(-z) = G_\theta|\partial\Delta(z)$, it follows that $t = h$. This proves Lemma 2.6. \qed

Let $H : \overline{\Delta} \to \overline{\Delta}$ be the Douady–Earle extension of $h$. We refer the reader to [4] for the definition and properties of Douady–Earle extension. It follows that $H$ is odd also. In particular, $H(0) = 0$. Define

\begin{equation}
\tilde{G}_\theta(z) = \begin{cases} 
G_\theta(z) & \text{for } z \in \mathbb{C} - \Delta, \\
H^{-1} \circ R_\theta \circ H(z) & \text{for } z \in \Delta.
\end{cases}
\end{equation}

For $k \in \mathbb{Z}$, let $\Delta_k = \phi^{-1}(D_k)$. Note that $\Delta_0 = \Delta$.

Lemma 2.7. $\tilde{G}_\theta$ is odd. The critical set of $\tilde{G}_\theta$ is contained in the set $\tilde{G}_\theta^{-1}(\partial\Delta) = \bigcup_{k \in \mathbb{Z}} \partial\Delta_k$, and moreover, if $\Lambda \subset \mathbb{Z}$ contains infinitely many elements but $\Lambda \neq \mathbb{Z}$, then the set $\bigcup_{k \in \Lambda} \partial\Delta_k$ is disconnected.

Proof. The first assertion follows from the construction of $\tilde{G}_\theta$. The second one follows from Lemma 2.2. \qed

Let $\nu_0$ be the complex structure in $\Delta$ which is the pull back of the standard complex structure by $H$. Since $H$ is odd, we have

Lemma 2.8. $\nu_0(-z) = \nu_0(z)$.

Now we can define a $\tilde{G}_\theta(z)$-invariant complex structure $\nu$ on the complex plane. The procedure is standard. For $z \in \Delta$, define $\nu = \nu_0$. For $z \notin \Delta$, there are two cases. In the first case, there is some integer $m \geq 1$ such that $\tilde{G}_\theta^m(z) \in \Delta$. In this case, define $\nu(z)$ to be the pull back of $\nu_0(\tilde{G}_\theta^m(z))$ by $\tilde{G}_\theta^m$. In the second case, the forward orbit of $z$ does not intersect the unit disk. In this case, define $\nu(z) = 0$. Since $\tilde{G}_\theta$ is odd, by Lemma 2.8, we have

Lemma 2.9. $\nu(-z) = \nu(z)$.
Now by Ahlfors–Bers’s theorem [1], there is a unique quasi-conformal homeomorphism $\psi$ of the Riemann sphere which solves the Beltrami equation given by $\nu$, and which fixes 0 and the infinity, and maps 1 to $\pi/2$.

Lemma 2.10. $\psi$ is odd.

Proof. Let $r(z) = -\psi(-z)$. Let $\nu_r$ and $\nu_\psi$ denote the dilations of $r$ and $\psi$, respectively. By Lemma 2.9, it follows that $\nu_r(z) = \nu_\psi(z)$. Since $r(0) = \psi(0) = 0$ and $r(\infty) = \psi(\infty) = \infty$, we get that $r(z) = a\psi(z)$ for some constant $a$. This implies that $\theta(z) = \psi(-z)) = a^2\psi(z)$. It follows that $a^2 = 1$. We then have $a = 1$ or $a = -1$. If $a = -1$, we get $\psi(-z) = \psi(z)$ for all $z$. This is impossible since $\psi(z)$ is a homeomorphism. Therefore $a = 1$. The Lemma follows.

Let $g_\theta(z) = \psi \circ \tilde{G}_\theta \circ \psi^{-1}(z)$ and let $\Omega = \psi(\Delta)$. By the construction, we get

Lemma 2.11. $g_\theta(z)$ is an odd entire function which has a Siegel disk $\Omega$ centered at the origin with rotation number $\theta$. Moreover, $\Omega$ is symmetric about the origin, and $\partial \Omega$ is a quasi-circle passing through exactly two critical points $\pi/2$ and $-\pi/2$.

For $k \in \mathbb{Z}$, let $\Omega_k = \psi(\Delta_k)$. It follows that $\Omega = \Omega_0$ and each $\Omega_k$ is a component of $g_\theta^{-1}(\Omega_0)$. By Lemma 2.7, we get

Lemma 2.12. The critical set of $g_\theta$ is contained in the set $g_\theta^{-1}(\partial \Omega_0) = \bigcup_{k \in \mathbb{Z}} \partial \Omega_k$. Moreover, if $\Lambda \subset \mathbb{Z}$ contains infinitely many elements but $\Lambda \neq \mathbb{Z}$, then the set $\bigcup_{k \in \Lambda} \partial \Omega_k$ is disconnected.

3. Topological equivalence

Lemma 3.1. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ and $g: \mathbb{C} \rightarrow \mathbb{C}$ be two continuous maps such that $f = g$ on the outside of the unit disk. If in addition, $f: \Delta \rightarrow \tilde{\Delta}$ and $g: \Delta \rightarrow \tilde{\Delta}$ are both homeomorphisms, then $f$ and $g$ are topologically equivalent to each other.

Proof. Define $\theta_2(z) = z$ for $z \notin \Delta$ and $\theta_2(z) = g^{-1} \circ f(z)$ for $z \in \Delta$. It follows that $\theta_2: \mathbb{C} \rightarrow \mathbb{C}$ is a homeomorphism. Let $\theta_1 = \text{id}$. Then $f = \theta_1^{-1} \circ g \circ \theta_2$. The Lemma follows.

Let $\phi: \tilde{C} - \overline{\Delta} \rightarrow \tilde{C} - \overline{D}$ be map in the definition of $G(z)$. Let $\eta: \tilde{C} \rightarrow \tilde{C}$ be a homeomorphic extension of $\phi$. As before let $T(z) = \sin(z)$. It follows that $T(z)$ is topologically equivalent to $T \circ \eta$. Let $t \in [0, 1)$ be the number in Lemma 2.5. Let $S(z) = e^{2\pi it}T \circ \eta(z)$. It follows that $S(z)$ is topologically equivalent to $T(z)$. By Lemma 3.1, we have

Lemma 3.2. $S(z)$ is topologically equivalent to $\tilde{G}_\theta(z)$.

Lemma 3.3. $g_\theta(z)$ is topologically equivalent to $T(z)$.

Proof. By the construction of $g_\theta$, it follows that $g_\theta$ is topologically equivalent to $\tilde{G}_\theta$. The Lemma then follows from Lemma 3.2.

Proof of the Theorem. By Lemma 1.1, it follows that $g_\theta(z) = a + b\sin(cz + d)$ where $a, b, c, d \in \mathbb{C}$ and $b, c \neq 0$. Since $g_\theta(-z) = -g_\theta(z)$ by Lemma 2.11, by
differentiating on both sides of the equation, we get
\[ \cos(cz + d) = \cos(-cz + d) \]
for all \( z \). It follows that
\[ \sin(d) \sin(cz) = 0 \]
for all \( z \). Since \( c \neq 0 \), it follows that \( d = k\pi \) for some integer \( k \). Therefore, we may assume that \( g_\theta(z) = a + b\sin(cz) \). Since \( g_\theta(0) = 0 \), it follows that \( a = 0 \). This implies that \( g_\theta(z) = b\sin(cz) \).

Since \( g'_\theta(\pi/2) = 0 \), it follows that \( c \) is some odd integer. By changing the sign of \( b \), we may assume that \( c \) is positive. Suppose \( c = 2l + 1 \) for some integer \( l \geq 0 \). Recall that \( \Omega_0 = \Omega \) is the Siegel disk of \( g_\theta \) centered at the origin. For \( k \in \mathbb{Z} \), let \( E_k = \{ z + k\pi \mid z \in \Omega \} \). Since \( \Omega_0 \) is symmetric about the origin, it follows that every \( E_k \) is a component of \( g^{-1}_\theta(\Omega_0) \). Since \( \partial \Omega_0 \) passes through \( \pi/2 \) and \( -\pi/2 \) by Lemma 2.11, it follows that for every \( k \in \mathbb{Z} \), \( \pi/2 + k\pi \in \partial E_k \cap \partial E_{k+1} \), and hence that the set \( \bigcup_{k\in\mathbb{Z}} \partial E_k \) is connected. By Lemma 2.12, we get \( g^{-1}_\theta(\partial \Omega_0) = \bigcup_{k\in\mathbb{Z}} \partial E_k \).

By Lemma 2.12 again, it follows that the critical set of \( g_\theta \) is contained in \( \bigcup_{k\in\mathbb{Z}} \partial E_k \). Since \( \partial E_0 = \partial \Omega_0 \) passes through exactly two critical points \( \pi/2 \) and \( -\pi/2 \) of \( g_\theta(z) \) and since \( g'_\theta(z) = (-1)^k g'_\theta(z + k\pi) \), it follows that every critical point of \( g_\theta \) has the form \( \pi/2 + k\pi \) where \( k \in \mathbb{Z} \) is some integer. This implies that \( c = 1 \). It follows that \( b = e^{2\pi i \theta} \) and therefore \( g_\theta(z) = f_\theta(z) \). This completes the proof of the theorem. \( \square \)

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References

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