SQUARE INEQUALITY AND STRONG ORDER RELATION

TSUYOSHI ANDO

This paper is dedicated to the memory of late Professor Takayuki Furuta

Communicated by M. S. Moslehian

Abstract. It is well-known that for Hilbert space linear operators 0 \leq A and 0 \leq C, inequality C \leq A does not imply C^2 \leq A^2. We introduce a strong order relation 0 \leq B \ll A, which guarantees that C^2 \leq B^{1/2}AB^{1/2} for all 0 \leq C \leq B, and that C^2 \leq A^2 when B commutes with A. Connections of this approach with the arithmetic-geometric mean inequality of Bhatia–Kittaneh as well as the Kantorovich constant of A are mentioned.

1. Introduction and theorem

Let B(H) denote the space of bounded linear operators on a Hilbert space H. Throughout the paper, a capital letter means an operator in B(H). The order relation A \geq B or equivalently B \leq A for A, B \in B(H) means that both A and B are selfadjoint and A − B is positive (positive semi-definite for matrices). Therefore A \geq 0 means that A is positive. Further, A > 0 means that A \geq 0 and A is invertible, or equivalently A \geq \mu I for some \mu > 0, where I is the identity operator in B(H).

It is well-known that 0 \leq C \leq A does not imply C^2 \leq A^2 in general. We look for a condition on A and B, which guarantees that

0 \leq C \leq B \implies C^2 \leq A^2.
Let us introduce a strong order relation $B \ll A$ for $0 \leq A, B$ as

\[ B \ll A \iff PBP \leq A \text{ for all projection } P. \quad (1.1) \]

**Theorem 1.1.** If $0 \leq C \leq B \ll A$, then $C^2 \leq B^{1/2}AB^{1/2}$ and $C^2 \leq A^2$ whenever $AB = BA$.

**Proof.** Inequality $0 \leq C \leq B$ is characterized by the relation

\[ C = B^{1/2}DB^{1/2} \text{ for some } 0 \leq D \leq I. \quad (1.2) \]

Since each $0 \leq D \leq I$ can be approximated in norm by convex combinations of projections, and since the map $D \mapsto DBD$ is convex in the sense that

\[ \{\lambda D_1 + (1 - \lambda)D_2\}B\{\lambda D_1 + (1 - \lambda)D_2\} \leq \lambda\lambda\lambda_{1}BD_{1} + (1 - \lambda)D_{2}BD_{2} \text{ for all } 0 \leq \lambda \leq 1 \]

we can see from (1.1) and (1.2) that

\[ C^2 = B^{1/2} \cdot (DBD) \cdot B^{1/2} \leq B^{1/2}AB^{1/2}. \]

Further, $B^{1/2}AB^{1/2} \leq A^2$ when $AB = BA$. \qed

**2. Strong order relation**

It is immediate from definition (1.1) that

\[ 0 \leq C \leq B \ll A \implies C \ll A, \quad (2.1) \]

and

\[ 0 \leq B_j \ll A_j \quad (j = 1, 2) \implies \alpha_1 B_1 + \alpha_2 B_2 \ll \alpha_1 A_1 + \alpha_2 A_2 \quad \text{for all } \alpha_1, \alpha_2 \geq 0. \]

The following assertion can be verified easily

\[ 0 \leq A \ll A \iff A = \alpha I \text{ for some } \alpha \geq 0. \]

A little non-trivial fact is that since the square-root map $0 \leq X \mapsto X^{1/2}$ is order-preserving (see [4, p.127])

\[ 0 \leq B \ll A \implies B^{1/2} \ll A^{1/2}. \]

This can be seen as follows: Since

\[ PB^{1/2}P \leq (PBP)^{1/2} \]

for all $B \geq 0$ and all projections $P$, we can conclude that

\[ 0 \leq B \ll A \implies PB^{1/2}P \leq (PBP)^{1/2} \leq A^{1/2} \implies B^{1/2} \ll A^{1/2}. \]

To see further properties of the strong order relation, given a projection $P$, let us consider two maps from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{G})$ with $\mathcal{G} = \text{ran}(P)$, the range space of $P$. First define $(\mathcal{P})X$ by

\[ (\mathcal{P})X := PXP \text{ for all } X \in \mathcal{B}(\mathcal{H}) \]
and second $[P]X$ by

$$[P]X := PXP - (PXPD) \cdot (PDP^{-})^{-1} \cdot (PDP),$$

where $P := I - P$.

The map $(P)$ is defined for all $X$ while $[P]$ is defined only when $PDP$ is invertible in $B(G)$ where $G$ is the ortho-complement of $G$, or equivalently $G = \text{ran}(P)$.

See [1] for more details about the map $[P]$. Sometimes we will abuse $(P)X$ and $[P]X$ as if they are operators in $B(H)$.

It is obvious that, with $I_G$ the identity operator in $B(G)$,

$$\mu I \leq A \leq \lambda I \implies \mu I_G \leq (P)A \leq \lambda I_G.$$  \hspace{1cm} (2.2)

A significant result is the following.

**Theorem 2.1.** For all $A > 0$ and all projections $P$,

$$([P]A)^{-1} = (P)(A^{-1}) \quad \text{and} \quad 0 \leq [P]A \leq A.$$  

*Proof. Along the orthogonal decomposition $H = G \oplus G^{-}$, write

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{11} = PAP$, $A_{12} = PAP^{-}$, $A_{21} = P^{-}AP$ and $A_{22} = P^{-}AP^{-}$.

Everything in the assertion comes from the following decomposition:

$$A = \begin{bmatrix} I_G & A_{12}A_{22}^{-1} \\ 0 & I_G^{-} \end{bmatrix} \cdot \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{bmatrix} \cdot \begin{bmatrix} I_G & 0 \\ A_{22}^{-1}A_{21} & I_G^{-} \end{bmatrix}$$

and the fact that both block operator matrices

$$\begin{bmatrix} I_G & A_{12}A_{22}^{-1} \\ 0 & I_G^{-} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I_G & A_{22}^{-1}A_{21} & 0 \\ 0 & I_G^{-} \end{bmatrix}$$

are invertible with respective inverses

$$\begin{bmatrix} I_G & A_{12}A_{22}^{-1} \\ 0 & I_G^{-} \end{bmatrix}^{-1} = \begin{bmatrix} I_G & -A_{12}A_{22}^{-1} \\ 0 & I_G^{-} \end{bmatrix}$$

and

$$\begin{bmatrix} I_G & 0 \\ A_{22}^{-1}A_{21} & I_G^{-} \end{bmatrix}^{-1} = \begin{bmatrix} I_G & 0 \\ -A_{22}^{-1}A_{21} & I_G^{-} \end{bmatrix}.$$  

In fact

$$A^{-1} = \begin{bmatrix} I_G & 0 \\ -A_{22}^{-1}A_{21} & I_G^{-} \end{bmatrix} \cdot \begin{bmatrix} (P)A^{-1} & 0 \\ 0 & (P^{-})A^{-1} \end{bmatrix} \cdot \begin{bmatrix} I_G & -A_{12}A_{22}^{-1} \\ 0 & I_G^{-} \end{bmatrix}^{-1},$$

and

$$A \geq \begin{bmatrix} I_G & A_{12}A_{22}^{-1} \\ 0 & I_G^{-} \end{bmatrix} \cdot \begin{bmatrix} (P)A & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} I_G & 0 \\ A_{22}^{-1}A_{21} & I_G^{-} \end{bmatrix} = [P]A.$$
Corresponding to (2.2) we have
\[ \mu I \leq A \leq \lambda I \implies \mu I_g \leq [P]A \leq \lambda I_g. \] (2.3)

**Corollary 2.2.** For \( A, B > 0 \),
\[ B \ll A \iff (P)B \leq [P]A \text{ for all projection } P \iff A^{-1} \ll B^{-1}. \]

3. Examples

Given \( A \) with \( \mu I \leq A \leq \lambda I \) for some \( 0 < \mu < \lambda \), we try to find reasonable \( 0 \leq B \) of the form
\[ B = \alpha I - \beta A^{-1} \] with \( \alpha, \beta \geq 0 \) or \( = aA + b \) with \( a \geq 0 \) and real \( b \) for which \( B \ll A \).

**Theorem 3.1.** Let \( \mu I \leq A \leq \lambda I \) with \( 0 < \mu < \lambda \) and \( \alpha, \beta \geq 0 \). Then validity of
\[ 0 \leq \alpha - \frac{\beta}{t} \leq t \text{ for all } t \in [\mu, \lambda] \] implies that \( \alpha I - \beta A^{-1} \ll A \).

**Proof.** Given a projection \( P \), let \( X := [P]A \). Since by (2.3) \( \mu I_g \leq X \leq \lambda I_g \) with \( G = \text{ran}(P) \), the assumption implies
\[ 0 \leq \alpha I_g - \beta X^{-1} \leq X. \]
Since \( X^{-1} = (P)(A^{-1}) \) by Theorem 2.1, considering \( X \) and \( X^{-1} \) as operators in \( B(H) \) we have
\[ P(\alpha I - \beta A^{-1})P \leq [P]A \leq A, \]
which is just the assertion. \( \Box \)

Suppose that \( \mu I \leq A \leq \lambda I \) with \( 0 < \mu < \lambda \) and that for \( \alpha, \beta \geq 0 \)
\[ 0 \leq \alpha - \frac{\beta}{t} \leq t \text{ for all } t \in [\mu, \lambda], \]
or equivalently
\[ \alpha \mu \leq \beta \text{ and } h(t) := t^2 - \alpha t + \beta \geq 0 \text{ for all } t \in [\mu, \lambda]. \] (3.1)
In this case, define a function \( f_{\alpha,\beta}(t) \) by
\[ f_{\alpha,\beta}(t) := \alpha - \frac{\beta}{t} \text{ for } t \in [\mu, \lambda]. \] (3.2)

Next determine \( a \geq 0 \) and real \( b \) by the relations
\[ a \mu + b = \alpha - \frac{\beta}{\mu} \text{ and } a \lambda + b = \alpha - \frac{\beta}{\lambda}, \] (3.3)
and define an affine function \( g_{\alpha,\beta}(t) \) by
\[ g_{\alpha,\beta}(t) := at + b \text{ for } t \in [\mu, \lambda]. \] (3.4)

**Corollary 3.2.** Suppose that (3.1) is satisfied and that \( f_{\alpha,\beta}(t) \) and \( g_{\alpha,\beta}(t) \) are defined according to (3.2) and (3.3) respectively. Then
\[ 0 \leq g_{\alpha,\beta}(A) \leq f_{\alpha,\beta}(A) \ll A, \text{ so that } g_{\alpha,\beta}(A) \ll A. \]
Proof. Since \( f_{\alpha,\beta}(t) \) is concave by (3.2) and \( g_{\alpha,\beta}(t) \) is affine by (3.4), and by (3.3)
\[
g_{\alpha,\beta}(\mu) = f_{\alpha,\beta}(\mu) \quad \text{and} \quad g_{\alpha,\beta}(\lambda) = f_{\alpha,\beta}(\lambda)
\]
we can conclude that \( g_{\alpha,\beta}(t) \leq f_{\alpha,\beta}(t) \) on \([\mu, \lambda]\). Then via functional calculus and by Theorem 3.1 and implication (2.1)

\[
0 \leq g_{\alpha,\beta}(A) \leq f_{\alpha,\beta}(A) \ll A, \quad \text{so that} \quad g_{\alpha,\beta}(A) \ll A.
\]

\[
\square
\]

In the remaining part of this section, under the assumption on a pair \((\alpha, \beta)\) as in Corollary 3.2, we will investigate when the extremal cases as \( f_{\alpha,\beta}(\mu) = \mu \) or \( f_{\alpha,\beta}(\lambda) = \lambda \) occur.

**Proposition 3.3.** If \( f_{\alpha,\beta}(\mu) = \mu \), then \( \mu \leq \alpha \leq 2\mu \) and \( \beta = (\alpha - \mu)\mu \). Conversely if \( \mu \leq \alpha \leq 2\mu \), then the pair \((\alpha, \beta)\) with \( \beta := (\alpha - \mu)\mu \) satisfies condition (3.1) and \( f_{\alpha,\beta}(\mu) = \mu \).

Proof. Since the assumption \( \mu = f_{\alpha,\beta}(\mu) = \alpha - \frac{\beta}{\mu} \) implies \( \beta = (\alpha - \mu)\mu \), so that \( \alpha \geq \mu \). Since by (3.1)
\[
h(t) = (t - \mu)\{t - (\alpha - \mu)\} \geq 0 \quad \text{for all} \quad t \in [\mu, \lambda]
\]
we have \( \alpha - \mu \leq \mu \), that is, \( \alpha \leq 2\mu \).

Conversely, suppose that \( \mu \leq \alpha \leq 2\mu \). Define \( \beta := (\alpha - \mu)\mu \). Clearly \( \beta \geq \alpha \mu \) and \( f_{\alpha,\beta}(\mu) = \mu \). Since \( \alpha - \mu \leq \mu \), we have \( h(t) \geq 0 \) on \([\mu, \lambda]\), so that (3.1) is satisfied.

We notice the following concrete examples.

(i) When \( \alpha = 2\mu \) and \( \beta = \mu^2 \),
\[
f_{\alpha,\beta}(t) = \mu(2 - \frac{\mu}{t}) \quad \text{and} \quad g_{\alpha,\beta}(t) = \frac{\mu}{\lambda}\{t + (\lambda - \mu)\}.
\]

(ii) When \( \alpha = \mu \) and \( \beta = 0 \), \( f_{\alpha,\beta}(t) = g_{\alpha,\beta}(t) = \mu \).

**Proposition 3.4.** The requirement \( f_{\alpha,\beta}(\lambda) = \lambda \) is possible only when \( \lambda \leq 2\mu \) or equivalently \( 2\lambda \leq \frac{\lambda^2}{\lambda - \mu} \) and
\[
2\lambda \leq \alpha \leq \frac{\lambda^2}{\lambda - \mu} \quad \text{and} \quad \beta = \lambda(\alpha - \lambda).
\]

\(3.5\)

Conversely when \( \lambda \leq 2\mu \), any pair \((\alpha, \beta)\) with (3.5) satisfies condition (3.1) and \( f_{\alpha,\beta}(\lambda) = \lambda \).

Proof. The requirement \( f_{\alpha,\beta}(\lambda) = \lambda \) implies \( \beta = \lambda(\alpha - \lambda) \). On the other hand, condition (3.1)
\[
(t - \lambda)\{t - (\alpha - \lambda)\} \geq 0 \quad \text{for all} \quad t \in [\mu, \lambda]
\]
implies \( \alpha - \lambda \geq \lambda \), whence \( \alpha \geq 2\lambda \). Again, since by (3.1) \( \alpha \mu \geq \beta = \lambda(\alpha - \lambda) \), we have \( \alpha \leq \frac{\lambda^2}{\lambda - \mu} \), so that
\[
2\lambda \leq \alpha \leq \frac{\lambda^2}{\lambda - \mu}.
\]
The proof of the converse direction is similar.  \(\square\)
We notice the following concrete examples.

(iii) Let $0 < \lambda \leq 2\mu$. When $\alpha := \frac{\lambda^2}{\lambda - \mu}$ and $\beta := \frac{\lambda^2 \mu}{\lambda - \mu}$,

\[
f_{\alpha, \beta}(t) = \frac{\lambda^2}{\lambda - \mu} \left\{ 1 - \frac{\mu}{t} \right\} \quad \text{and} \quad g_{\alpha, \beta}(t) = \frac{\lambda}{\lambda - \mu} \{ t - \mu \}.
\]

(iv) Let $0 < \lambda \leq 2\mu$. When $\alpha := 2\lambda$ and $\beta := \lambda^2$,

\[
f_{\alpha, \beta}(t) = \lambda \left\{ 2 - \frac{\lambda}{t} \right\} \quad \text{and} \quad g_{\alpha, \beta}(t) = \frac{\lambda}{\mu} \left\{ t - (\lambda - \mu) \right\}.
\]

4. Connection with known results

Bhatia and Kittaneh [3] established a remarkable matrix arithmetic-geometric mean inequality. It says that for any $n \times n$ matrices $A, C \geq 0$ and any unitarily invariant norm $\| \cdot \|$ (see [2, p.91] for definition)

\[
\| AC \| \leq \| \left( \frac{A + C}{2} \right)^2 \|.
\]

Taking the operator norm, this inequality is extended to the case of Hilbert space operators. Taking $A^{-1}$ in place of $A$, this theorem for the operator norm says

\[
C + A^{-1} \leq 2 \cdot I \quad \Rightarrow \quad A^{-1} C^2 A^{-1} \leq I \quad \Rightarrow \quad C^2 \leq A^2,
\]

or

\[
0 \leq C \leq 2 \cdot I - A^{-1} \quad \Rightarrow \quad C^2 \leq A^2.
\]

Therefore this corresponds to the case that $\alpha = 2, \beta = 1, \mu = \frac{1}{2}$ and any number $\lambda$ with $\lambda I \geq A$.

Suppose that $0 < A$ has maximum spectrum $\lambda$ and minimum spectrum $\mu$. The numbers $\lambda$ and $\mu$ can be expressed in terms of norms related to $A$. In fact

\[
\lambda = \| A \| \quad \text{and} \quad \mu = \| A^{-1} \|^{-1}. \tag{4.1}
\]

The number

\[
\kappa_A := \frac{(\lambda + \mu)^2}{4\lambda \mu} \tag{4.2}
\]

is called the Kantorvich constant of $A$. Then it is clear from (4.1) and (4.2) that

\[
\kappa_A = \frac{(\| A \| \cdot \| A^{-1} \| + 1)^2}{4\| A \| \cdot \| A^{-1} \|}.
\]

The following fact has been known (see [4, Chapter III] for more detail):

**Theorem 4.1.** For $A > 0$,

\[
0 \leq C \leq A \quad \Rightarrow \quad C^2 \leq \kappa_A \cdot A^2.
\]

Let us show how this can be incorporated into our theory. The following proposition can be checked immediately.
Proposition 4.2. When $\alpha = \frac{4\lambda\mu}{\lambda+\mu}$ and $\beta = \frac{4\lambda^2\mu^2}{(\lambda+\mu)^2}$ the pair $(\alpha, \beta)$ satisfies condition (3.1) and

$$f_{\alpha,\beta}(t) = \frac{4\lambda\mu}{\lambda+\mu} \left( 1 - \frac{\lambda\mu}{(\lambda+\mu)t} \right)$$
and
$$g_{\alpha,\beta}(t) = \frac{4\lambda\mu}{(\lambda+\mu)^2} \cdot t = \kappa_A^{-1} t.$$

Therefore $\kappa_A^{-1} \cdot A \ll A$.

Now Theorem 4.1 is deduced from Proposition 4.2 and Theorem 1.1 as follows:

$$0 \leq C \leq A \implies \kappa_A^{-1} C \leq \kappa_A^{-1} A \ll A \implies \kappa_A^{-2} C^2 \leq \kappa_A^{-1} A^2 \implies C^2 \leq \kappa_A \cdot A^2.$$

Notice that the above argument shows that

$$0 \leq C \leq \kappa_A^{-1/2} \cdot A \implies C^2 \leq A^2.$$

References


Hokkaido University (Emeritus), Sapporo 060, Japan
E-mail address: ando@es.hokudai.ac.jp