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\( (p, q) \)-TYPE BETA FUNCTIONS OF SECOND KIND

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Abstract. In the present article, we propose the \((p, q)\)-variant of beta function of second kind and establish a relation between the generalized beta and gamma functions using some identities of the post-quantum calculus. As an application, we also propose the \((p, q)\)-Baskakov–Durrmeyer operators, estimate moments and establish some direct results.

1. Introduction

The quantum calculus (q-calculus) in the field of approximation theory was discussed widely in the last two decades. Several generalizations to the \(q\) variants were recently presented in the book [3]. Further there is possibility of extension of \(q\)-calculus to post-quantum calculus, namely the \((p, q)\)-calculus. Actually such extension of quantum calculus can not be obtained directly by substitution of \(q\) by \(q/p\) in \(q\)-calculus. But there is a link between \(q\)-calculus and \((p, q)\)-calculus. The \(q\) calculus may be obtained by substituting \(p = 1\) in \((p, q)\)-calculus. We mentioned some previous results in this direction. Recently, Gupta [8] introduced \((p, q)\) genuine Bernstein–Durrmeyer operators and established some direct results. \((p, q)\) generalization of Szász–Mirakyan operators was defined in [1]. Also authors investigated a Durrmeyer type modifications of the Bernstein operators in [9]. We can also mention other papers as Bernstein operators [10], Bernstein–Stancu operators [11], Bleimann–Butzer–Hahn operators and Szász–Mirakyan–Kantorovich

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operators. Besides this, we also refer to some recent related work on this topic: e.g. [5], [12] and [13].

Some basic notations of \((p, q)\)-calculus are mentioned below:

The \((p, q)\)-numbers are defined as
\[
[n]_{p,q} = \frac{p^n - q^n}{p - q}.
\]

Obviously, it may be seen that \([n]_{p,q} = p^{n-1} [n]_{q/p} \). In The \((p, q)\)-factorial is defined by
\[
[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, \quad n \geq 1, \quad [0]_{p,q}! = 1.
\]

The \((p, q)\)-binomial coefficient is given by
\[
\begin{bmatrix} n \n k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!}, \quad 0 \leq k \leq n.
\]

For details see [15] and [16].

**Definition 1.1.** The \((p, q)\)-power basis is defined below and it also has a link with \(q\)-power basis as
\[
(x \oplus a)^n_{p,q} = (x + a)(px + qa)(p^2x + q^2a) \cdots (p^{n-1}x + q^{n-1}a).
\]
\[
(x \ominus a)^n_{p,q} = (x - a)(px - qa)(p^2x - q^2a) \cdots (p^{n-1}x - q^{n-1}a).
\]

**Definition 1.2.** The \((p, q)\)-derivative of the function \(f\) is defined as
\[
D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0
\]
and \(D_{p,q} f(0) = f'(0)\), provided that \(f\) is differentiable at 0. Note also that for \(p = 1\), the \((p, q)\)-derivative reduces to the \(q\)-derivative. The \((p, q)\)-derivative fulfills the following product rules
\[
D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x)
\]
\[
D_{p,q}(f(x)g(x)) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x).
\]

The following assertions hold true:
\[
D_{p,q}(x \ominus a)^n_{p,q} = [n]_{p,q} \cdot (px \ominus a)^{n-1}_{p,q}, \quad n \geq 1
\]
\[
D_{p,q}(a \ominus x)^n_{p,q} = -[n]_{p,q} \cdot (a \ominus qx)^{n-1}_{p,q}, \quad n \geq 1,
\]
and \(D_{p,q}(x \oplus a)^0_{p,q} = 0\).

**Definition 1.3.** ([14]) Let \(n\) be a nonnegative integer, we define the \((p, q)\)-gamma function as
\[
\Gamma_{p,q} (n + 1) = \frac{(p \ominus q)^n_{p,q}}{(p - q)^n} = [n]_{p,q}!, \quad 0 < q < p.
\]
Proposition 1.4. The formula of \((p, q)\)-integration by part is given by
\[
\int_a^b f(px) D_{p,q} g(x) \, dp,qx = f(b)g(b) - f(a)g(a) - \int_a^b g(qx) D_{p,q} f(x) \, dp,qx \tag{1.1}
\]

In the present paper, we propose the \((p, q)\)-Baskakov–Durrmeyer operators and estimate some approximation properties, which include asymptotic formula and convergence in terms of modulus of continuity.

2. \((p, q)\)-beta Function of Second Kind

Let \(m, n \in \mathbb{N}\), we define \((p, q)\)-beta function of second kind as
\[
B_{p,q}(m, n) = \int_0^\infty \frac{x^{m-1}}{(1 \oplus px)^{m+n}} \, dp,qx
\]

Theorem 2.1. Let \(m, n \in \mathbb{N}\). We have the following relation between \((p, q)\)-beta and \((p, q)\)-gamma function:
\[
B_{p,q}(m, n) = q^{[2-m(m-1)]/2} p^{-m(m+1)/2} \frac{\Gamma_{p,q}(m) \Gamma_{p,q}(n)}{\Gamma_{p,q}(m+n)}.
\]

Proof. We know that
\[
D_{p,q} \frac{1}{(1 \oplus x)^n_{p,q}} = - \frac{p[n]_{p,q}}{(1 \oplus px)^{n+1}_{p,q}}
\]

If we choose \(f(x) = x^m\) and \(g(x) = -1_{p[m+n]_{p,q}(1 \oplus x)^{m+n}}\) and use (1.1) we have
\[
B_{p,q}(m+1, n) = \int_0^\infty \frac{x^m}{(1 \oplus px)^{m+n+1}} \, dp,qx
\]

\[
= - \frac{p^{-m}}{p[m+n]_{p,q}} \int_0^\infty (px)^m \, dp,qx \frac{1}{(1 \oplus x)^{m+n}_{p,q}}
\]

\[
= \frac{p^{-m}}{p[m+n]_{p,q}} \int_0^\infty D_{p,q} x^m \, dp,qx \frac{1}{(1 \oplus qx)^{m+n}_{p,q}}
\]

\[
= \frac{p^{-m}[m]_{p,q}}{p[m+n]_{p,q}} \int_0^\infty x^{m-1} \, dp,qx
\]

\[
= \frac{p^{-m-1}[m]_{p,q}}{q^{m-1}[m+n]_{p,q}} \int_0^\infty (qx)^{m-1} \, dp,qx
\]

\[
= \frac{p^{-1}[m]_{p,q}}{(pq)^m [m+n]_{p,q}} \int_0^\infty x^{m-1} \, dp,qx
\]

\[
= \frac{p^{-1}[m]_{p,q}}{(pq)^m [m+n]_{p,q}} B_{p,q}(m, n),
\]

\[
B_{p,q}(1, n) = \int_0^\infty \frac{1}{(1 \oplus px)^{n+1}_{p,q}} \, dp,qx = - \frac{1}{p[n]_{p,q}} \int_0^\infty D_{p,q} \frac{1}{p[n]_{p,q}} \, dp,qx = \frac{1}{p[n]_{p,q}}
\]
and

\[ B_{p,q}(m,n) = \frac{p^{-1}[m-1]_{p,q}}{(pq)^{m-1}[m+n-1]_{p,q}} B_{p,q}(m-1,n) \]

\[ = \frac{p^{-1}[m-1]_{p,q}}{(pq)^{m-1}[m+n-1]_{p,q}} \frac{p^{-1}[m-2]_{p,q}}{(pq)^{m-2}[m+n-2]_{p,q}} \cdots \frac{p^{-1}}{p [n+1]_{p,q}} B_{p,q}(1,n) \]

\[ = \frac{p^{-1}[m-1]_{p,q}}{(pq)^{m-1}[m+n-1]_{p,q}} \frac{p^{-1}[m-2]_{p,q}}{(pq)^{m-2}[m+n-2]_{p,q}} \cdots \frac{p^{-1}}{p [n+1]_{p,q}} \frac{q}{pq^{n-1}[n]_{p,q}} \]

\[ = \frac{qp^{-m} \Gamma_{p,q}(m) \Gamma_{p,q}(n)}{(pq)^{(m-1)/2} \Gamma_{p,q}(m+n)} \]

\[ \square \]

3. (\(p, q\))-Baskakov–Durrmeyer Operators and Moments

The \((p, q)\)-analogue of Baskakov operators for \(x \in [0, \infty)\) and \(0 < q < p \leq 1\) is defined as

\[ B_{n,p,q}(f; x) = \sum_{k=0}^{n} b_{n,k}^{p,q}(x) f \left( \frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}} \right) , \quad (3.1) \]

where

\[ b_{n,k}^{p,q}(x) = \left[ \begin{array}{c} n + k - 1 \\ k \end{array} \right]_{p,q} p^{k+n(n-1)/2} q^{k(k-1)/2} x^{k} (1 + x)^{n+k}. \]

In case \(p = 1\), we get the \(q\)-Baskakov operators \([2, 3]\). If \(p = q = 1\), we get at once the well known Baskakov operators.

Remark 3.1. Starting with the following relations between \((p, q)\)-calculus and \(q\)-calculus:

\[ \left[ \begin{array}{c} n + k - 1 \\ k \end{array} \right]_{p,q} = p^{k(n-1)/2} q^{(k-1)/2} \left[ \begin{array}{c} n + k - 1 \\ k \end{array} \right]_{q/p} \]

and

\[ (x \oplus a)_{p,q}^{n} = p^{n(n-1)/2} (x + a)^{n}_{q/p} \]

and using moments of \(q\)-Baskakov operators (see [2, 3]), it can easily be verified by simple computation that

\[ B_{n,p,q}(1; x) = 1, B_{n,p,q}(t; x) = x, B_{n,p,q}(t^2; x) = x^2 + \frac{p^{n-1}x}{[n]_{p,q}} \left( 1 + \frac{p}{q} x \right). \]
Definition 3.2. Using \((p, q)\)-beta function of second kind, we propose below for \(x \in [0, \infty), 0 < q < p \leq 1\) the \((p, q)\) analogue of Baskakov–Durrmeyer operators

\[
D_{n}^{p,q}(f; x) = [n - 1]_{p,q} \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)q^{\frac{k(k+1)-2}{2}}p^{(k+1)(k+2)/2} \int_{0}^{\infty} \left[ \frac{n+k-1}{k} \right]_{p,q} t^{k} \frac{t^{k}}{(1 \oplus pt)_{p,q}} f(p^{k}t)d_{p,q}t \tag{3.2}
\]

where \(b_{n,k}^{p,q}(x)\) is as defined in (3.1).

Lemma 3.3. For \(x \in [0, \infty), 0 < q < p \leq 1\), we have

1. \(D_{n}^{p,q}(1; x) = 1\)
2. \(D_{n}^{p,q}(t; x) = \frac{1}{q^{2}[n-2]_{p,q}} + \frac{[2]_{p,q}}{p^{2}q^{2}[n-2]_{p,q}} x + \frac{1}{p^{n}} x\)
3. \(D_{n}^{p,q}(t^{2}; x) = \frac{[2]_{p,q}}{q^{4}[n-2]_{p,q}[n-3]_{p,q}} x^{2} + \frac{[3]_{p,q}}{p^{4}q^{4}[n-2]_{p,q}[n-3]_{p,q}} x^{2} + \frac{[n+2][3]_{p,q} + q[2]_{p,q}[3]_{p,q}}{p^{2}[n+2]_{p,q}[n-2]_{p,q}[n-3]_{p,q}} x^{2} + \frac{1}{p^{n+2}} x^{2}\).

Proof. Using (3.2) and Remark 3.1, we have

\[
D_{n}^{p,q}(1; x) = [n - 1]_{p,q} \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)q^{\frac{k(k+1)-2}{2}}p^{(k+1)(k+2)/2} \times \int_{0}^{\infty} \left[ \frac{n+k-1}{k} \right]_{p,q} t^{k} \frac{t^{k}}{(1 \oplus pt)_{p,q}} d_{p,q}t
\]

\[
= [n - 1]_{p,q} \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)q^{\frac{k(k+1)-2}{2}}p^{(k+1)(k+2)/2} \times \left[ \frac{n+k-1}{k} \right]_{p,q} B_{p,q}(k + 1, n - 1)
\]

\[
= B_{n,p,q}(1; x) = 1.
\]
Next using the identity \([k + 1]_{p,q} = q^k + p[k]_{p,q}\) and applying Remark 3.1, we have

\[
D_n^{p,q}(t; x)
= [n - 1]_{p,q} \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)q^{k(k+1)/2}p(n+1)(k+2)/2 \int_0^\infty \left[ n + k - 1 \atop k \right]_{p,q} \frac{t^{k+1}p^k}{(1 \pm pt)^{k+n}} d_{p,q}t
= [n - 1]_{p,q} \sum_{k=0}^{\infty} q^{k(k+1)/2}p(n+1)(k+2)/2b_{n,k}^{p,q}(x) \left[ n + k - 1 \atop k \right]_{p,q} p^kB_{p,q}(k+2, n - 2)
= \sum_{k=0}^{\infty} p^{-k}q^{-k-1}b_{n,k}^{p,q}(x) \left[ k + 1 \atop n - 2 \right]_{p,q}
= \frac{1}{[n - 2]_{p,q}p^2} \sum_{k=0}^{\infty} q^{-k-1}b_{n,k}^{p,q}(x)(q^k + p[k]_{p,q})
= \frac{1}{[n - 2]_{p,q}q^2}B_{n,p,q} \left[ k + 1 \atop n - 2 \right]_{p,q} + \frac{[n]_{p,q}x}{p^2q^2[n - 2]_{p,q}}
\]

Further using the identity \([k + 2]_{p,q} = q^{k+1} + pq^k + p^2[k]_{p,q}\) and by Remark 3.1, we get

\[
D_n^{p,q}(t^2; x) = [n - 1]_{p,q} \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)q^{k(k+1)/2}p(n+1)(k+2)/2 \times \left[ n + k - 1 \atop k \right]_{p,q} \frac{t^{k+2}p^{2k}}{(1 \pm pt)^{k+n}} d_{p,q}t
= [n - 1]_{p,q} \sum_{k=0}^{\infty} q^{k(k+1)/2}p(n+1)(k+2)/2b_{n,k}^{p,q}(x)
\times \left[ n + k - 1 \atop k \right]_{p,q} p^{2k}B_{p,q}(k + 3, n - 3)
= \sum_{k=0}^{\infty} q^{-(2k+3)}p^{-5}b_{n,k}^{p,q}(x) \left[ k + 2 \atop n - 2 \right]_{p,q} \frac{k + 1}{p^2} \left[ n - 3 \atop p,q \right]_{p,q}
= \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x)p^{-5}q^{-(2k+3)} \left( p^3[k]_{p,q}^2 + q^k(p[2]_{p,q} + p^2)[k]_{p,q} + q^{2k}[2]_{p,q} \right) \frac{1}{[n - 2]_{p,q}[n - 3]_{p,q}}
= \frac{1}{[n - 2]_{p,q}[n - 3]_{p,q}} \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) \left( \frac{p^{n-1}}{q^{k-1}[k]_{p,q}} \right) \frac{p^{-7-2n}}{q^5} + (p[2]_{p,q} + p)\left( \frac{p^{n-1}}{q^{k-1}[k]_{p,q}} \right) \frac{p^{-3-n}}{q^4} + q^{-3}[2]_{p,q}
\]
We consider the following class of functions:

\[
\begin{align*}
q^{-3}[2]_{p,q} B_{n,p,q}(1; x) + & \frac{p^{-3-n}}{q^4} \frac{([2]_{p,q} + p)[n]_{p,q}}{[n-2]_{p,q}[n-3]_{p,q}} B_{n,p,q}(t; x) \\
+ & \frac{p^{-7-2n}}{2[n]_{p,q}} B_{n,p,q}(t^2; x) \\
= & \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} \frac{([2]_{p,q} + p)[n]_{p,q}}{p^{3+n}q^4[n-2]_{p,q}[n-3]_{p,q}} x \\
+ & \frac{[n]_{p,q}}{p^{7+2n}q^5[n-2]_{p,q}[n-3]_{p,q}} x^2 + \frac{[n]_{p,q}}{p^{8+n}q^5[n-2]_{p,q}[n-3]_{p,q}} x + \\
& \frac{[n]_{p,q}}{p^{7+2n}q^6[n-2]_{p,q}[n-3]_{p,q}} x^2.
\end{align*}
\]

\[
\begin{align*}
= & \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} \frac{([2]_{p,q} + p)[n]_{p,q}}{p^{3+n}q^4[n-2]_{p,q}[n-3]_{p,q}} x \\
+ & \frac{[n]_{p,q}}{p^{7+2n}q^5[n-2]_{p,q}[n-3]_{p,q}} x^2 + \frac{[n]_{p,q}}{p^{8+n}q^5[n-2]_{p,q}[n-3]_{p,q}} x + \\
& \frac{[2]_{p,q}[3]_{p,q}}{p^{12+n}q^5[n-2]_{p,q}[n-3]_{p,q}} x^2 \\
+ & \frac{[n]_{p,q}}{p^{8+n}q^5[n-2]_{p,q}[n-3]_{p,q}} x + \frac{[n]_{p,q}}{p^{7+2n}q^6[n-2]_{p,q}[n-3]_{p,q}} x^2.
\end{align*}
\]

\[
\begin{align*}
= & \frac{[2]_{p,q}}{q^3[n-2]_{p,q}[n-3]_{p,q}} + \left( \frac{p^3 q (q + 2p) + 1}{p^3 q^4[n-2]_{p,q}[n-3]_{p,q}} + \frac{p^3 q (q + 2p) + 1}{p^{10+n}q^3[n-3]_{p,q}} \right) x \\
+ & \frac{q^2 + pq + p^2}{p^9+nq^2[n-2]_{p,q}[n-3]_{p,q}} x^2 + \frac{[3]_{p,q}}{p^{10+n}q^3[n-3]_{p,q}} x^2 \\
+ & \frac{(p^{n+2}[3]_{p,q} + q[2]_{p,q}[3]_{p,q})}{p^{12+n}q^6[n-2]_{p,q}[n-3]_{p,q}} x^2 + \frac{1}{p^{7+2n}} x^2.
\end{align*}
\]

4. Weighted approximation

We consider the following class of functions:
Let \( H_x \) be the set of all functions \( f \) defined on \([0, \infty)\) satisfying the condition \(|f(x)| \leq M_f (1 + x^2)\), where \( M_f \) is a constant depending only on \( f \). By \( C_x \), we denote the subspace of all continuous functions belonging to \( H_x \). Also, let \( C_x^n \) be the subspace of all functions \( f \in C_x \), for which \( \lim_{x \to \infty} \frac{f(x)}{1 + x^2} \) is finite. The norm on \( C_x^n \) is \( \|f\|_x = \sup_{x \in [0, \infty)} |f(x)| \).

Now we shall discuss the weighted approximation theorem, where the approximation formula holds true on the interval \([0, \infty)\).

**Theorem 4.1.** Let \( p = p_n \) and \( q = q_n \) satisfies \( 0 < q_n < p_n \leq 1 \) and for \( n \) sufficiently large \( p_n \to 1, q_n \to 1 \) and \( q_n^2 \to 1 \) and \( p_n^2 \to 1 \). For each \( f \in C_x^n \), we have

\[
\lim_{n \to \infty} \|D_n^{p_n,q_n} (f) - f\|_x = 0.
\]

**Proof.** Using the Theorem in [7] we see that it is sufficient to verify the following three conditions

\[
\lim_{n \to \infty} \|D_n^{p_n,q_n} (t^\nu, x) - x^\nu\|_x = 0, \quad \nu = 0, 1, 2. \tag{4.1}
\]

Since \( D_n^{p_n,q_n} (1, x) = 1 \) the first condition of (4.1) is fulfilled for \( \nu = 0 \).

We can write for \( n > 3 \)

\[
\|D_n^{p_n,q_n} (t, x) - x\|_x \leq \frac{1}{q_n p_n^2 |n - 2| p_n q_n} + \left( \frac{[2]_{p_n,q_n}}{p_n q_n^2 |n - 2| p_n q_n} + \frac{1}{p_n^q} - 1 \right) \sup_{x \in [0, \infty)} \frac{x}{1 + x^2}
\]

and

\[
\|D_n^{p_n,q_n} (t^2, x) - x^2\|_x \leq \left( \frac{q_n^2 + p_n q_n + p_n^2}{p_n^q q_n^2 |n - 2| p_n q_n} + \frac{[3]_{p_n,q_n}}{p_n^q q_n^3 |n - 3| p_n q_n} + \frac{p_n^q [2]_{p_n,q_n} + q_n [2]_{p_n,q_n} [3]_{p_n,q_n}}{p_n^q [2]_{p_n,q_n} + q_n [2]_{p_n,q_n} [3]_{p_n,q_n}} \right) \times \sup_{x \in [0, \infty)} \frac{x}{1 + x^2}
\]

\[
+ \left( \frac{1}{p_n^{q+2n} - 1} \sup_{x \in [0, \infty)} \frac{x^2}{1 + x^2} \right)
\]

\[
+ \left( \frac{\left( p_n^q q_n (q_n + 2p_n) + 1 \right) [3]_{p_n,q_n}}{p_n^q q_n^3 [n - 2]_{p_n,q_n} [n - 3]_{p_n,q_n}} + \frac{p_n^q q_n (q_n + 2p_n) + 1}{p_n^q q_n^3 [n - 2]_{p_n,q_n} [n - 3]_{p_n,q_n}} \right) \sup_{x \in [0, \infty)} \frac{x}{1 + x^2}
\]

\[
+ \frac{[2]_{p_n,q_n}}{q_n^3 [n - 2]_{p_n,q_n} [n - 3]_{p_n,q_n}}
\]

which implies that

\[
\lim_{n \to \infty} \|D_n^{p_n,q_n} (t, x) - x\|_x = 0
\]
and
\[
\lim_{n \to \infty} \| D_n^{p_n,q_n} (t^2, x) - x^2 \|_{x^2} = 0.
\]
Thus the proof is completed. \(\square\)

We give the following theorem to approximate all functions in \(C_{x^2} [0, \infty)\).

**Theorem 4.2.** Let \(p = p_n\) and \(q = q_n\) satisfies \(0 < q_n < p_n \leq 1\) and for \(n\) sufficiently large \(p_n \to 1\), \(q_n \to 1\) and \(q_n^* \to 1\) and \(p_n^* \to 1\). For each \(f \in C_{x^2} [0, \infty)\), we have
\[
\lim_{n \to \infty} \sup_{x \in [0, \infty)} \frac{|D_n^{p_n,q_n} (f, x) - f (x)|}{(1 + x^2)^{1+\alpha}} = 0.
\]

**Proof.** For any fixed \(x_0 > 0\),
\[
\sup_{x \in [0, \infty)} \frac{|D_n^{p_n,q_n} (f, x) - f (x)|}{(1 + x^2)^{1+\alpha}}
\leq \sup_{x \leq x_0} \frac{|D_n^{p_n,q_n} (f, x) - f (x)|}{(1 + x^2)^{1+\alpha}} + \sup_{x \geq x_0} \frac{|D_n^{p_n,q_n} (f, x) - f (x)|}{(1 + x^2)^{1+\alpha}}
\leq \| D_n^{p_n,q_n} (f) - f \|_{C[0, x_0]} + \| f \|_{x^2} \sup_{x \geq x_0} \frac{|D_n^{p_n,q_n} (1 + t^2, x)|}{(1 + x^2)^{1+\alpha}}
+ \sup_{x \geq x_0} \frac{|f (x)|}{(1 + x^2)^{1+\alpha}}.
\]

The first term of the above inequality tends to zero from well known Korovkin’s theorem. By Lemma 3.3 for any fixed \(x_0 > 0\) it is easily seen that \(\sup_{x \geq x_0} \frac{|D_n^{p_n,q_n} (1 + t^2, x)|}{(1 + x^2)^{1+\alpha}}\) tends to zero as \(n \to \infty\). We can choose \(x_0 > 0\) so large that the last part of above inequality can be made small enough. \(\square\)

**Remark 4.3.** For \(q \in (0, 1)\) and \(p \in (q, 1]\) it is seen that \(\lim_{n \to \infty} [n]_{p,q} = 1/(q - p)\). In order to consider convergence of \((p, q)\) Baskakov operators we assume \(p = (p_n)\) and \(q = (q_n)\) such that \(0 < q_n < p_n \leq 1\) and for \(n\) sufficiently large \(p_n \to 1\), \(q_n \to 1\) and \(p_n^* \to 1\) and \(q_n^* \to 1\).

5. Quantitative Approximation

Let \(C_B [0, \infty)\) denote the space of all real valued continuous and bounded functions on \([0, \infty)\). In this space we consider the norm
\[
\| f \|_{C_B} = \sup_{x \in [0, \infty)} | f (x) |.
\]

Now we give the first and second order modulus of continuity of function \(f \in C_B\) (see [4], [6]). The first modulus of continuity is defined as
\[
\omega_1 (f; \delta) = \sup_{x,a,v \geq 0 \atop |u-v| \leq \delta} | f (x + u) - f (x + v)|.
\]
and the second order modulus of continuity is defined

$$\omega_2(f; \delta) = \sup_{x,u,v \geq 0 \atop |u-v| \leq \delta} |f(x+2u) - 2f(x+u+v)f(x+2v)|, \quad \delta \geq 0.$$  

We will use the Steklov mean function for \(f \in C_B\)

$$f_h(x) = \frac{4}{h^2} \int_0^h \int_0^h [2f(x+u+v) - f(x+2(u+v))] \, dudv. \quad (5.1)$$

Since \(f_h \in C_B\) we can write

$$f_h(x) - f(x) = \frac{4}{h^2} \int_0^h \int_0^h [2f(x+u+v) - f(x+2(u+v)) - f(x)] \, dudv.$$  

It is obvious that

$$|f_h(x) - f(x)| \leq \omega_2(f; h)$$

and

$$\|f_h - f\|_{C_B} \leq \omega_2(f; h). \quad (5.2)$$

If \(f\) is continuous, then \(f'_h \in C_B\) and

$$f'_h(x) = \frac{4}{h^2} \left[ 2 \int_0^h \left( f\left(x + \frac{v + \frac{h}{2}}{2}\right) - f\left(x + \frac{h}{2}\right) \right) dv 
\right. 
- \frac{1}{2} \int_0^h \left( f(x + h + 2v) - f(x + v) \right) dv \right].$$

Thus we have

$$\|f'_h\|_{C_B} \leq \frac{5}{h} \omega_1(f; h). \quad (5.3)$$

Similarly \(f''_h \in C_B\) and

$$\|f''_h\|_{C_B} \leq \frac{9}{h^2} \omega_2(f; h). \quad (5.4)$$

**Theorem 5.1.** Let \(q \in (0, 1)\) and \(p \in (q, 1]\). The operator \(D_{n}^{p,q}\) maps space \(C_B\) into \(C_B\) and

$$\|D_{n}^{p,q}(f)\|_{C_B} \leq \|f\|_{C_B}. $$
Proof. Let \( q \in (0, 1) \) and \( p \in (q, 1) \). From Lemma 3.3, we have

\[
|D_{n}^{p,q}(f; x)| \leq [n - 1]_{p,q} \sum_{k=0}^{\infty} B_{n,k}^{p,q}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \\
\int_{0}^{\infty} \left[ n + k - 1 \right] \left( 1 \oplus pt \right)_{p,q}^{k+n} \frac{t^k}{d_{p,q}t} f(p^k t) dt \\
\leq \sup_{x \in [0, \infty)} |f(x)| [n - 1]_{p,q} \sum_{k=0}^{\infty} B_{n,k}^{p,q}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \\
\int_{0}^{\infty} \left[ n + k - 1 \right] \frac{t^k}{d_{p,q}t} d_{p,q}t \\
= \sup_{x \in [0, \infty)} |f(x)| D_{n}^{p,q}(1; x) = \|f\|_{C_B}.
\]

We are going to study the degree of approximation in terms of \( \omega_1(f; \delta) \) and \( \omega_2(f; \delta) \), first and second order modulus of continuity.

**Theorem 5.2.** Let \( q \in (0, 1) \) and \( p \in (q, 1) \). If \( f \in C_B \), then

\[
|D_{n}^{p,q}(f; x) - f(x)| \\
\leq 5 \omega_1 \left( f; \frac{1}{\sqrt{|n - 2|}_{p,q}} \right) \\
\times \left( \frac{1}{qp^2 \sqrt{|n - 2|}_{p,q}} + \frac{[2]_{p,q}}{p^2 q^2 \sqrt{|n - 2|}_{p,q}} x + \left( \frac{1}{p^n} - 1 \right) \sqrt{|n - 2|}_{p,q} x \right) \\
+ \frac{9}{2} \omega_2 \left( f; \frac{1}{\sqrt{|n - 2|}_{p,q}} \right) \left( \frac{p^{7+n} - 2p^{7+n} - 1}{p^{7+2n}} \right) [n - 2]_{p,q} x^2 \\
+ \frac{q^2 + pq + p^2 - 2p^{8+n} - 2qp^{7+n}}{p^{9+n}q^2} x^2 \\
+ \frac{[3]_{p,q} [n - 2]_{p,q}}{p^{10+nq^3} [n - 3]_{p,q}} x^2 + \frac{(p^{n+2}[3]_{p,q} + q[2]_{p,q}[3]_{p,q})}{p^{12+nq^6}[n - 3]_{p,q}} x^2 \\
+ \left( \frac{(p^5 q(q + 2p) + 1) [3]_{p,q}}{p^6 q^4 [n - 3]_{p,q}} + \frac{p^5 q(q + 2p) + 1}{p^3 q^2} \right) x + \frac{[2]_{p,q}}{q^3 [n - 3]_{p,q}}
\]

Proof. We use the Stieltjes function \( f_h \) defined by (5.1). For \( x \geq 0 \) and \( n \in \mathbb{N} \), we have

\[
|D_{n}^{p,q}(f; x) - f(x)| \leq D_{n}^{p,q}(|f - f_h; x|) + |D_{n}^{p,q}(f_h - f_h(x); x)| \\
+ |f_h(x) - f(x)|.
\]

By (5.2) we can write

\[
D_{n}^{p,q}(|f - f_h; x|) \leq \|D_{n}^{p,q}(f - f_h)\| \leq \|f - f_h\|_{C_B} \leq \omega_2(f; h).
\]
Since $D_{n}^{p,q}$ is a linear positive operator we get

$$|D_{n}^{p,q}(f_{h} - f_{h}(x); x)| \leq \left| f_{h}'(x) \right| D_{n}^{p,q}(t - x; x) + \frac{1}{2} \left\| f'' \right\|_{C_{B}} D_{n}^{p,q}( (t - x)^{2}; x).$$

By Lemma 3.3, (5.3) and (5.4) we have

$$|D_{n}^{p,q}(f_{h} - f_{h}(x); x)|$$

\begin{align*}
\leq & \frac{5}{h} \omega_{1}(f; h) \left( \frac{1}{qp^{2}[n - 2]_{p,q}} + \frac{[2]_{p,q}}{p^{2}q^{2}[n - 2]_{p,q}} x + \left( \frac{1}{p^{n} - 1} \right) x \right) \\
& + \frac{9}{2h^{2}} \omega_{2}(f; h) D_{n}^{p,q}( (t - x)^{2}; x),
\end{align*}

where

$$D_{n}^{p,q}( (t - x)^{2}; x)$$

\begin{align*}
= & \left( \frac{p^{7+2n} - 2p^{7+n} - 1}{p^{7+2n}q^{2}[n - 2]_{p,q}} \right) x^{2} + \frac{q^{2} + pq + p^{2} - 2p^{8+n} - 2qp^{7+n}}{p^{9+n}q^{2}[n - 2]_{p,q}} x^{2} \\
& + \frac{[3]_{p,q}}{p^{1+q+n}[n - 3]_{p,q}} x^{2} + \frac{(p^{n+2}[3]_{p,q} + q[2]_{p,q}[3]_{p,q})}{p^{1+q+n}[n - 2]_{p,q}[n - 3]_{p,q}} x^{2} \\
& + \frac{\left( p^{q}q(q + 2p) + 1 \right) [3]_{p,q}}{p^{q}[n - 2]_{p,q}[n - 3]_{p,q}} x^{2} + \frac{p^{q}q(q + 2p) + 1}{p^{3+q+n}[n - 2]_{p,q}} - \frac{2}{qp^{2}[n - 2]_{p,q}} x
\end{align*}

for $x \geq 0$, $h > 0$. Setting $h = \sqrt{\frac{1}{n-2}_{p,q}}$, we have desired result. \hfill \Box

**Remark 5.3.** From Theorem 5.2 we can say that that the order of approximation of $D_{n}^{p,q}(f; x)$ to $f(x)$ is at least as good as the order of approximation to $f(x)$ by classical Baskakov–Durrmeyer operators for any $x \in [0, \infty)$ as a depending on selection of $q_{n}$ and $p_{n}$. If we choose $p$ and $q$ as in Remark 4.3, we have an approximation process with the aid of operator (3.2).

**References**


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