SOME LOWER BOUNDS FOR THE NUMERICAL RADIUS OF
HILBERT SPACE OPERATORS

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Abstract. We show that if \( T \) is a bounded linear operator on a complex
Hilbert space, then
\[
\frac{1}{2} \| T \| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2}} \sqrt{w^2(T) - c^2(T)} \leq w(T),
\]
where \( w(\cdot) \) and \( c(\cdot) \) are the numerical radius and the Crawford number, respectively. We then apply it to prove that for each \( t \in [0, \frac{1}{2}) \) and natural number \( k \),
\[
\frac{(1 + 2t)^{\frac{1}{2k}}}{2^\frac{k}{2}} m(T) \leq w(T),
\]
where \( m(T) \) denotes the minimum modulus of \( T \). Some other related results
are also presented.

1. Introduction and preliminaries

Let \( \mathbb{B}(H) \) denote the \( C^* \)-algebra of all bounded linear operators on a complex
Hilbert space \( H \) with an inner product \( \langle \cdot, \cdot \rangle \) and the corresponding norm \( \| \cdot \| \). If
\( \dim H = n \), we identify \( \mathbb{B}(H) \) with the space \( \mathcal{M}_n \) of all \( n \times n \) matrices with entries
in the complex field. For \( T \in \mathbb{B}(H) \), let \( \| T \| \) and \( m(T) \) denote the usual operator
norm and the minimum modulus of \( T \), respectively. Here \( m(T) \) is defined to be
the largest number \( \alpha \geq 0 \) such that \( \|Tx\| \geq \alpha\|x\| \) \( (x \in H) \). The numerical radius

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decomposition.
and the Crawford number of $T \in \mathcal{B}(H)$ are defined by

$$w(T) = \sup \{ |\langle Tx, x \rangle| : x \in H, \|x\| = 1 \}$$

and

$$c(T) = \inf \{ |\langle Tx, x \rangle| : x \in H, \|x\| = 1 \},$$

respectively. These concepts are useful in studying linear operators and have attracted the attention of many authors in the last few decades (e.g., see [4, 8], and their references). It is well known that $w(\cdot)$ defines a norm on $\mathcal{B}(H)$ such that for all $T \in \mathcal{B}(H)$,

$$\frac{1}{2} \|T\| \leq w(T) \leq \|T\|. \quad (1.1)$$

The inequalities in (1.1) are sharp. The first inequality becomes an equality if $T^2 = 0$. The second inequality becomes an equality if $T$ is normal. Any operator $T \in \mathcal{B}(H)$ can be represented as $T = H + iK$, the so-called Cartesian decomposition, where $H = \text{Re}(T) = \frac{T + T^*}{2}$ and $K = \text{Im}(T) = \frac{T - T^*}{2i}$ are called the real and imaginary parts of $T$. It has been shown in [7] that,

$$\sup \{ \|\alpha H + \beta K\| : \alpha, \beta \in \mathbb{R}, \alpha^2 + \beta^2 = 1 \} = w(T).$$

In particular, $\|H\| \leq w(T)$ and $\|K\| \leq w(T)$.

Concerning the inequality (1.1), Kittaneh [6] has shown the following precise estimate of $w(T)$ by using norm inequalities:

$$\frac{1}{\sqrt{2}} \sqrt{\|H^2 + K^2\|} \leq w(T) \leq \sqrt{\|H^2 + K^2\|}. \quad (1.2)$$

Obviously, (1.2) is sharper than the inequality of (1.1). Yamazaki [9] has used the Aluthge transform to improve the second inequality (1.1) so that

$$w(T) \leq \frac{1}{2} \left( \|T\| + w(\tilde{T}) \right).$$

Here $\tilde{T}$ (the Aluthge transform of $T$) is defined as $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$, where $U$ is a partial isometry of the polar decomposition of $T$ and $|T| = (T^*T)^{\frac{1}{2}}$ means the absolute value of $T$.

Further, it has been shown in [1] that,

$$\frac{1}{2} \sqrt{\|T\|^2 + \|T^*\|^2} + 2c(T^2) \leq w(T) \leq \frac{1}{2} \sqrt{\|T\|^2 + \|T^*\|^2} + 2w(T^2).$$

For more material about the numerical radius and other results on numerical radius inequality, see, e.g., [3], [5], and the references therein.

For $T \in \mathcal{B}(H)$, let us recall the abbreviated notations

$$|\cos|T = \inf \left\{ \frac{|\langle Tx, x \rangle|}{\|Tx\| \|x\|} : x \in H, \|Tx\| \neq 0 \right\}$$

and

$$|\sin|T = \sqrt{1 - |\cos|^2 T}.$$
In the next section, we establish some considerable improvement of the first inequality (1.1). More precisely, we prove that
\[
\frac{1}{2} \|T\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2} \sqrt{w^2(T) - c^2(T)}} \leq w(T)
\]
and
\[
\frac{1}{2} \|T\| \leq \max \left\{ \left| \sin \frac{T}{\sqrt{2}} \right|, \sqrt{w^2(T)} \right\} w(T) \leq w(T).
\]

Next, we will give some applications. Particularly, for each \( t \in [0, \frac{1}{2}) \) and natural number \( k \), we show that
\[
(1 + 2t)^{\frac{3}{2}} m(T) \leq w(T).
\]

2. Main results

In this section we present some lower bounds for the numerical radii of Hilbert space operators. We start our work with the following result.

**Theorem 2.1.** Let \( T \in \mathbb{B}(H) \). Then
\[
\frac{1}{2} \|T\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2} \sqrt{w^2(T) - c^2(T)}} \leq w(T).
\]

**Proof.** Clearly, \( \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2} \sqrt{w^2(T) - c^2(T)}} \leq w(T) \). On the other hand, let \( x \in H \) with \( \|x\| \leq 1 \). Let \( \langle Tx, x \rangle = \lambda_x |\langle Tx, x \rangle| \) for some unit \( \lambda_x \in \mathbb{C} \). Hence \( \langle \lambda_x Tx, x \rangle = |\langle Tx, x \rangle| \geq 0 \). Let \( H + iK \) be the Cartesian decomposition of \( \lambda_x T \). Then \( \langle Hx, x \rangle + i\langle Kx, x \rangle = \langle \lambda_x Tx, x \rangle \geq 0 \). Hence
\[
\langle \lambda_x Tx, x \rangle = \langle Hx, x \rangle, \quad \langle Kx, x \rangle = 0.
\]

We have
\[
\frac{1}{4} \|Tx\|^2 = \frac{1}{4} \left( \|\lambda_x Tx - \langle \lambda_x Tx, x \rangle x\|^2 + |\langle Tx, x \rangle|^2 \right)
\]
\[
= \frac{1}{4} \left( \|Hx - \langle Hx, x \rangle x + iKx\|^2 + |\langle Tx, x \rangle|^2 \right) \quad \text{(since } \langle Kx, x \rangle = 0 \text{)}
\]
\[
\leq \frac{1}{4} \left( \|Hx - \langle Hx, x \rangle x\|^2 + \|Kx\|^2 + |\langle Tx, x \rangle|^2 \right)
\]
\[
\leq \frac{1}{4} \left( \left( \sqrt{\|Hx\|^2 - |\langle Hx, x \rangle|^2} + \|Kx\| \right)^2 + |\langle Tx, x \rangle|^2 \right)
\]
\[
\leq \frac{1}{4} \left( \left( \sqrt{w^2(T) - |\langle Tx, x \rangle|^2} + w(T) \right)^2 + |\langle Tx, x \rangle|^2 \right) \quad \text{(since } \|Hx\|, \|Kx\| \leq w(T) \text{ and } |\langle Tx, x \rangle| = |\langle Hx, x \rangle| \text{)}
\]
\[
= \frac{w^2(T)}{2} + \frac{w(T)}{2} \sqrt{w^2(T) - |\langle Tx, x \rangle|^2}.
\]
Hence
\[
\frac{1}{2} \|Tx\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2} \sqrt{w^2(T) - |\langle Tx, x \rangle|^2}} \quad (\|x\| \leq 1). \tag{2.2}
\]
If we replace \(x\) by \(\frac{x}{\|x\|}\) in the above inequality, then we obtain
\[
\frac{1}{2} \|Tx\| \leq \|x\| \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2} \sqrt{w^2(T) - \langle T \left( \frac{x}{\|x\|}, x \right) \rangle}}.
\]
Thus
\[
\frac{1}{2} \|Tx\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2} \sqrt{w^2(T) - c^2(T)}}.
\]
Taking the supremum over \(x \in H\) with \(\|x\| \leq 1\) in the above inequality we deduce the desired inequality. \(\square\)

**Remark 2.2.** Let \(A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\). Then \(\|A\| = w(A) = c(A) = 1\). Thus
\[
\frac{1}{2} \|A\| = \frac{1}{2} \leq \sqrt{\frac{w^2(A)}{2} + \frac{w(A)}{2} \sqrt{w^2(A) - c^2(A)}} = \frac{\sqrt{2}}{2} < w(A) = 1.
\]
Hence the inequalities in Theorem 2.1 can be strict.

**Corollary 2.3.** Let \(T \in \mathcal{B}(H)\). Then
\[
\|Tx\|^2 + |\langle Tx, x \rangle|^2 \leq 4w^2(T) \quad (x \in H, \|x\| \leq 1).
\]
**Proof.** Let \(x \in H\) with \(\|x\| \leq 1\). By (2.1) it follows that
\[
\frac{1}{4} \|Tx\|^2 \leq \frac{1}{4} \left( \left( \sqrt{w^2(T) - |\langle Tx, x \rangle|^2} + w(T) \right)^2 + |\langle Tx, x \rangle|^2 \right)
\]
\[
\leq \frac{1}{4} \left( 2 \left( \sqrt{w^2(T) - |\langle Tx, x \rangle|^2} \right)^2 + 2w^2(T) + |\langle Tx, x \rangle|^2 \right)
\]
(by the arithmetic geometric mean inequality)
\[
= \frac{1}{4} \left( 4w^2(T) - |\langle Tx, x \rangle|^2 \right),
\]
which gives \(\|Tx\|^2 + |\langle Tx, x \rangle|^2 \leq 4w^2(T)\). \(\square\)

**Corollary 2.4.** Let \(A = [a_{ij}] \in \mathcal{M}_n\). Then
\[
\sum_{k=1}^n |a_{ki}|^2 \leq w^2(A) + w(A)\sqrt{w^2(A) - |a_{ii}|^2} \quad (1 \leq i \leq n).
\]
Proof. Let \( x = [0, \cdots, 0, 1, 0, \cdots, 0]^t \) with 1 in place of \( i \). Then \( Ax = [a_{1i}, a_{2i}, \cdots, a_{ni}]^t \) and \( \langle Ax, x \rangle = a_{ii} \). So, by (2.2) we obtain

\[
\frac{1}{2} \sum_{k=1}^n |a_{ki}|^2 = \frac{1}{2} \|Ax\| \leq \sqrt{\frac{w^2(A)}{2} + \frac{w(A)}{2} \sqrt{w^2(A) - |\langle Ax, x \rangle|^2}} = \sqrt{\frac{w^2(A)}{2} + \frac{w(A)}{2} \sqrt{w^2(A) - |a_{ii}|^2}}.
\]

This yields

\[
\sum_{k=1}^n |a_{ki}|^2 \leq w^2(A) + w(A) \sqrt{w^2(A) - |a_{ii}|^2}.
\]

□

Theorem 2.5. Let \( T \in \mathfrak{B}(H) \). Then

\[
\frac{1}{2} \|T\| \leq \max \left\{ |\sin T, \sqrt{\frac{2}{2}} | \right\} w(T) \leq w(T).
\]

Proof. Clearly, \( \max \left\{ |\sin T, \sqrt{\frac{2}{2}} | \right\} w(T) \leq w(T) \). On the other hand, let \( x \in H \) with \( \|x\| \leq 1 \). By (2.2) we have

\[
\frac{1}{2} \|Tx\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2} \sqrt{w^2(T) - \|Tx, x\|^2}}.
\]

Hence

\[
\frac{1}{2} \|Tx\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2} \sqrt{w^2(T) - \|Tx\|^2} \cos^2 T},
\]

or equivalently,

\[
\|Tx\|^2 - 2w^2(T) \leq 2w(T) \sqrt{w^2(T) - \|Tx\|^2} \cos^2 T. \quad (2.3)
\]

We consider two cases.

Case 1. \( \|Tx\|^2 - 2w^2(T) \leq 0 \). So we get \( \|Tx\| \leq \sqrt{2} w(T) \) and hence

\[
\frac{1}{2} \|T\| \leq \frac{\sqrt{2}}{2} w(T). \quad (2.4)
\]

Case 2. \( \|Tx\|^2 - 2w^2(T) > 0 \). It follows from (2.3) that

\[
\|Tx\|^4 - 4\|Tx\|^2 w^2(T) + 4w^4(T) \leq 4w^4(T) - 4w^2(T) \|Tx\|^2 |\cos^2 T|.
\]

This implies

\[
\|Tx\|^2 \leq 4 \left( 1 - |\cos^2 T| \right) w^2(T)
\]

which yields

\[
\frac{1}{2} \|Tx\| \leq |\sin Tw(T)|.
\]

Taking the supremum over \( x \in H \) with \( \|x\| \leq 1 \) in the above inequality we get

\[
\frac{1}{2} \|T\| \leq |\sin Tw(T)|. \quad (2.5)
\]
Finally, by (2.4) and (2.5) we conclude the desired inequality.

Remark 2.6. Let \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 + i \end{bmatrix} \). Simple computations show that \( \|A\| = w(A) = \sqrt{2} \) and \( |\sin |A = \sqrt{2} - 1 \). Thus
\[
\frac{1}{2} \|A\| = \frac{\sqrt{2}}{2} < \max \left\{ |\sin |A, \frac{\sqrt{2}}{2} \right\} w(A) = \frac{\sqrt{2}}{2} \times \sqrt{2} = 1 < w(A) = \sqrt{2}.
\]
Hence the inequalities in Theorem 2.5 can be strict.

As a consequence of Theorem 2.5 we have the following result.

Corollary 2.7. Let \( T, S \in \mathbb{B}(H) \). Then
\[
w(TS) \leq 4 \max \left\{ |\sin |T, \frac{\sqrt{2}}{2} \right\} \max \left\{ |\sin |S, \frac{\sqrt{2}}{2} \right\} w(T)w(S) \leq 4w(T)w(S).
\]

Proof. Applying the second inequality of (1.1) and Theorem 2.5, we get
\[
w(TS) \leq \|TS\| \leq \|T\|\|S\|
\]
\[
\leq 2 \max \left\{ |\sin |T, \frac{\sqrt{2}}{2} \right\} w(T) \times 2 \max \left\{ |\sin |S, \frac{\sqrt{2}}{2} \right\} w(S)
\]
\[
= 4 \max \left\{ |\sin |T, \frac{\sqrt{2}}{2} \right\} \max \left\{ |\sin |S, \frac{\sqrt{2}}{2} \right\} w(T)w(S) \leq 4w(T)w(S).
\]

A fundamental inequality for the numerical radius is the power inequality, which says that for \( T \in \mathbb{B}(H) \),
\[
w(T^k) \leq w^k(T)
\]
for \( k = 1, 2, \cdots \) (see, e.g., [5]). We are now in a position to establish one of our main results.

Theorem 2.8. Let \( T \in \mathbb{B}(H) \). For each \( t \in [0, \frac{1}{2}] \) and natural number \( k \),
\[
\frac{(1 + 2t)^{\frac{\pi}{4}}}{2^t} m(T) \leq w(T).
\]

Proof. Let \( t \in [0, \frac{1}{2}] \) and \( k \in \mathbb{N} \). Let \( x \in H \) with \( \|x\| \leq 1 \). We consider two cases.

Case 1. \( \|Tx\|^2 - 2w^2(T) \leq 0 \). So we have
\[
w^2(T) - 2tw(T)\Re\langle Tx, x \rangle + (t^2 - \frac{1}{4})\|Tx\|^2
\]
\[
\geq w^2(T) - 2tw(T)\Re\langle Tx, x \rangle + 2(t^2 - \frac{1}{4})w^2(T)
\]
\[
= 2w^2(T) \left| t - \frac{\langle Tx, x \rangle}{2w(T)} \right|^2 + \frac{w^2(T) - |\langle Tx, x \rangle|^2}{2} \geq 0.
\]
Hence
\[ w^2(T) - 2tw(T)\text{Re}\langle Tx, x \rangle + (t^2 - \frac{1}{4})\|Tx\|^2 \geq 0. \] (2.6)

Case 2. \(\|Tx\|^2 - 2w^2(T) > 0\). It follows from (2.2) that
\[ \frac{1}{2}\|Tx\| \leq \sqrt{\frac{w^2(T)}{2} + \frac{w(T)}{2}\sqrt{w^2(T) - |\langle Tx, x \rangle|^2}}. \]
This implies
\[ \left(\frac{1}{4}\|Tx\|^2 - \frac{w^2(T)}{2}\right)^2 \geq \frac{w^2(T)}{4} \left(w^2(T) - |\langle Tx, x \rangle|^2\right) \]
which yields
\[ 4w^2(T)\|Tx\|^2 - \|Tx\|^4 - 4w^2(T)|\langle Tx, x \rangle|^2 \geq 0. \] (2.7)

By (2.7), we get
\[ w^2(T) - 2tw(T)\text{Re}\langle Tx, x \rangle + (t^2 - \frac{1}{4})\|Tx\|^2 \]
whence
\[ w^2(T) - 2tw(T)\text{Re}\langle Tx, x \rangle + (t^2 - \frac{1}{4})\|Tx\|^2 \geq 0. \] (2.8)

By (2.6) and (2.8), we obtain
\[ 2tw(T)\text{Re}\langle Tx, x \rangle \leq w^2(T) + (t^2 - \frac{1}{4})\|Tx\|^2. \]

If we replace \(T\) by \(\frac{\text{Re}\langle Tx, x \rangle}{|\text{Re}\langle Tx, x \rangle|} T\) in the above inequality, then we get
\[ 2tw(T)|\text{Re}\langle Tx, x \rangle| \leq w^2(T) + (t^2 - \frac{1}{4})\|Tx\|^2 \quad (\|x\| \leq 1). \] (2.9)
Furthermore, if we replace \(T\) by \(e^{i\theta}T\) in (2.9), then we deduce
\[ 2tw(T)|\text{Re}(e^{i\theta} \langle Tx, x \rangle)| \leq w^2(T) + (t^2 - \frac{1}{4})\|Tx\|^2. \]

Since \(\sup \{|\text{Re}(e^{i\theta} \langle Tx, x \rangle)| : \theta \in \mathbb{R}\} = |\langle Tx, x \rangle|\), by taking the supremum over \(\theta \in \mathbb{R}\) in the above inequality we reach
\[ 2tw(T)|\langle Tx, x \rangle| \leq w^2(T) + (t^2 - \frac{1}{4})\|Tx\|^2. \] (2.10)

By (2.10), we get
\[ 2tw(T)|\langle Tx, x \rangle| \leq w^2(T) + (t^2 - \frac{1}{4})\|Tx\|^2 \leq w^2(T) + (t^2 - \frac{1}{4})m^2(T). \]
Thus
\[ 2tw(T)|\langle Tx, x \rangle| \leq w^2(T) + (t^2 - \frac{1}{4})m^2(T). \] (2.11)
By taking the supremum over \(x \in H\) with \(\|x\| = 1\) in (2.11), we obtain
\[
2tw^2(T) \leq w^2(T) + (t^2 - \frac{1}{4})m^2(T),
\]
or equivalently,
\[
\frac{(1 + 2t)^{1/2}}{2}m(T) \leq w(T).
\]
Replacing \(T\) by \(T^k\) in the last inequality gives
\[
\frac{(1 + 2t)^{1/2}}{2}m(T^k) \leq w(T^k).
\]
Since \(m^k(T) \leq m(T^k)\) and \(w(T^k) \leq w^k(T)\), the above inequality becomes
\[
\frac{(1 + 2t)^{1/2}}{2}m^k(T) \leq w^k(T).
\]
Thus
\[
\frac{(1+2t)^{1/2}}{2^k}m(T) \leq w(T).\]

\textbf{Remark 2.9.} Recall that an operator \(T \in \mathcal{B}(H)\) is said to be idempotent if \(T^2 = T\) and an involution if \(T^2 = I\). It is well-known that, if \(T\) is idempotent such that \(T \neq 0\), then \(w(T) = \frac{1}{2}(1 + \|T\|)\) and if \(T\) is involution then, \(w(T) = \frac{1}{2}(\|T\| + \|T\|^{-1})\) (see, e.g., [1]). So, by Theorem 2.8 for each \(t \in [0, \frac{1}{2})\) and \(k \in \mathbb{N}\), the following statements hold:

(i) If \(T\) is an idempotent operator such that \(T \neq 0\), then
\[
2^{1-k}(1 + 2t)^{1/2} m(T) \leq 1 + \|T\|.
\]

(ii) If \(T\) is an involution operator, then
\[
2^{1-k}(1 + 2t)^{1/2} m(T) \leq \|T\| + \|T\|^{-1}.
\]

\textbf{Corollary 2.10.} Let \(T \in \mathcal{B}(H)\). For each \(t \in [0, \frac{1}{2})\),
\[
\frac{\|T\|^2}{2} \leq \sqrt{\frac{w^2(T) - 2tw(T)\mu(T)}{1 - 4t^2}},
\]
where \(\mu(T) = \inf \{|\text{Re}\langle Tx, x \rangle| : x \in H, \|x\| \leq 1\}\).

\textbf{Proof.} Let \(t \in [0, \frac{1}{2})\) and let \(x \in H\) with \(\|x\| \leq 1\). By (2.9), we have
\[
2tw(T)|\text{Re}\langle Tx, x \rangle| \leq w^2(T) + (t^2 - \frac{1}{4})\|Tx\|^2.
\]
Since \(\mu(T) = \inf \{|\text{Re}\langle Tx, x \rangle| : x \in H, \|x\| \leq 1\}\), so by the above inequality we obtain
\[
w^2(T) - 2tw(T)\mu(T) \geq w^2(T) - 2tw(T)|\text{Re}\langle Tx, x \rangle| \geq (\frac{1}{4} - t^2)\|Tx\|^2.
\]
Hence
\[
(\frac{1}{4} - t^2)\|Tx\|^2 \leq w^2(T) - 2tw(T)\mu(T).
\]
By taking the supremum over $x \in H$ with $\|x\| = 1$ in the above inequality, we wet
\[
\frac{1}{4} - t^2 \|T\|^2 \leq w^2(T) - 2tw(T)\mu(T).
\]
Now, by the last inequality, we deduce the desired inequality. $\square$

Let us recall that by [2, Lemma 2.1] we have
\[
w(x \otimes y) = \frac{1}{2} \left( |\langle x, y \rangle| + \|x\||y\| \right),
\]
for all $x, y \in H$. Here, $x \otimes y$ denotes the rank one operator in $B(H)$ defined by $(x \otimes y)(z) := \langle z, y \rangle x$ for all $z \in H$. The following result is a reverse the Cauchy-Schwarz inequality in the setting of Hilbert spaces.

**Corollary 2.11.** Let $x, y \in H$. For each $t \in [0, \frac{1}{2})$ and $k \in \mathbb{N}$, the following statements hold.

(i) \[
\left( \frac{1}{\max \left\{ \sqrt{1 - \inf \left\{ \frac{\|\langle x, z \rangle\|^2}{\|x\|^2\|z\|^2} : z \in H, \langle z, y \rangle \neq 0 \right\}}, \frac{\sqrt{2}}{4} \right\}} - 1 \right) \|x\||y\| \leq |\langle x, y \rangle|.
\]

(ii) \[
\left( 2^{1 - \frac{k}{2}} (1 + 2t)^{\frac{k}{2}} \inf \left\{ |\langle z, y \rangle| : z \in H, \|z\| = 1 \right\} - \|y\| \right) \|x\| \leq |\langle x, y \rangle|.
\]

**Proof.** Simple computations show that
\[
|\sin |(x \otimes y)| = \sqrt{1 - \inf \left\{ \frac{|\langle x, z \rangle|^2}{\|x\|^2\|z\|^2} : z \in H, \langle z, y \rangle \neq 0 \right\}}, \quad \left( 2.12 \right)
\]
and
\[
m(x \otimes y) = \|x\| \inf \left\{ |\langle z, y \rangle| : z \in H, \|z\| = 1 \right\}. \quad \left( 2.13 \right)
\]
So, by Theorem 2.5 and (2.12), we obtain
\[
\frac{1}{2} \|x\||y\| \leq \max \left\{ \left| \sin |(x \otimes y)|, \frac{\sqrt{2}}{2} \right|, \frac{1}{2} \left( |\langle x, y \rangle| + \|x\||y\| \right) \right\},
\]
or equivalently,
\[
\left( \frac{1}{\max \left\{ \sqrt{1 - \inf \left\{ \frac{\|\langle x, z \rangle\|^2}{\|x\|^2\|z\|^2} : z \in H, \langle z, y \rangle \neq 0 \right\}}, \frac{\sqrt{2}}{4} \right\}} - 1 \right) \|x\||y\| \leq |\langle x, y \rangle|.
\]
Furthermore, for each $t \in [0, \frac{1}{2})$ and $k \in \mathbb{N}$, by Theorem 2.8 and (2.13) we get
\[
\frac{(1 + 2t)^{\frac{k}{2}}}{2^{\frac{k}{2}}} \|x\| \inf \left\{ |\langle z, y \rangle| : z \in H, \|z\| = 1 \right\} \leq \frac{1}{2} \left( |\langle x, y \rangle| + \|x\||y\| \right),
\]
or equivalently,
\[
\left( 2^{1 - \frac{k}{2}} (1 + 2t)^{\frac{k}{2}} \inf \left\{ |\langle z, y \rangle| : z \in H, \|z\| = 1 \right\} - \|y\| \right) \|x\| \leq |\langle x, y \rangle|.
\]

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