Koszul Cohomology and $k$-Normality of a Projective Variety

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Abstract. Let $X$ be a smooth projective variety and let $L$ be a very ample divisor of $X$ embedding it in $\mathbb{P}^N$. In this paper we use the Koszul groups of $X$ to get information about the $k$-normality of $X$ (i.e. the surjectivity of the map $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \to H^0(X, kL)$ via an upper bound for the degree of the generators of $\oplus_{i \geq 0} H^0(X, tL)$. The above idea is applied to some scrolls over curves and surfaces and to some other varieties, by using also results due to Green and Butler.

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1. Introduction

It is well-known that “there are fewer ways to compute Koszul cohomology groups than reasons to compute them” (see [4]). In this paper we want to give another reason to compute them: almost every work on Koszul cohomology of a smooth projective subvariety $X$ of $\mathbb{P}^N$ considers only the case in which $X$ is projectively normal, (p.n.) (see [6], [7], [8], [13]) in which case Koszul groups give immediately a free resolution of the ideal sheaf $\mathcal{I}_X$ of $X$ in $\mathbb{P}^N$ which is the main interest of the above papers. We know only Blumenhake’s works [10] and [11] treating the case in which $X$ is not linearly normal. When $X$ is not p.n. Koszul groups give only an upper bound for the degree of the generators of the ring $R(X) = \oplus_{t \geq 0} H^0(X, tL)$ where $L$ is the very ample line bundle of $X$ giving the embedding of $X$ in $\mathbb{P}^N$. Note that $R(X)$
is the coordinate ring of $X$ if $X$ is projectively normal. In some cases, e.g. for some scrolls, the information given by Koszul groups on $R(X)$ is sufficient to establish the $k$-normality of $X$.

More precisely in this paper we prove that for a scroll $X$ over a smooth curve, whose dimension is at least three, the ring $R(X)$ is generated in degree 2 if a condition weaker than Butler’s one (see [9]) is satisfied. The same fact is true for varieties, 4-dimensional at least, which are fibered in hypersurfaces of degree 2 and 3 over a smooth curve. Hence these varieties are projectively normal if and only if they are 2-normal, moreover, this fact is true for scrolls over a genus 2 curve without any other assumptions.

As a consequence of a suitable use of corollary 1.d.4 of [4], we get that for a regular surface $(X, L)$ such that there exists a smooth curve in $|L - K_X|$, $R(X)$ is generated in degree 2 and 3. We also obtain some conditions assuring the projective normality of scrolls on surfaces.

The paper is organized as follows: In Section 2 we fix notation and recall some facts about Koszul cohomology; in Section 3 we use Butler’s work to compute some Koszul vanishings for scrolls and varieties which are fibered in hypersurfaces; in Section 4 we show some other vanishings for Koszul cohomology of scrolls; in Section 5 we consider another method to compute vanishings and we apply it to regular surfaces and to scrolls over surfaces.

2. Notation and background material

\begin{align*}
P^N & \quad \text{$N$-dimensional projective space over } \mathbb{C} \\
S & \quad \mathbb{C}[x_0, x_1, \ldots, x_N] \text{ the coordinate ring of } P^N \\
S(a) & \quad \text{the graded ring } S \text{ twisted by the integer } a \\
X & \quad \text{smooth } n \text{-dimensional projective subvariety of } P^N \\
K_X & \quad \text{canonical divisor of } X \\
L & \quad \text{very ample line bundle embedding } X \text{ in } P^N \text{ via } H^0(X, L) \\
I_X & \quad \text{the homogeneous ideal of } X \text{ in the ring } S \\
\mathcal{O}_X & \quad \text{the ideal sheaf of } X \text{ in } P^N \\
\mathcal{O}_X & \quad \text{the structural sheaf of } X \\
\Omega_X & \quad \text{cotangent bundle of } X \\
R(X) & \quad \text{the graded ring } \oplus_{t \geq 0} H^0(X, tL) \text{ which is an } S \text{-module} \\
C & \quad \text{smooth algebraic curve of genus } g \\
E & \quad \text{rank } r \text{ vector bundle over a smooth variety } X \\
E^* & \quad \text{its dual} \\
\mu(E) & \quad \text{slope of } E \\
\mu^{-}(E) & \quad \text{minimal slope of a quotient vector bundle of } E \text{ over } C \\
\mu^{+}(E) & \quad \text{maximal slope of a subbundle of } E \text{ over } C \\
P(E) & \quad \text{projectivized of } E \\
p & \quad \text{natural projection from } P(E) \text{ to } X \\
T & \quad \text{tautological line bundle of } P(E) \\
F & \quad \text{numerical class of a fibre in } P(E) \text{ or generic fibre of } p \\
\sim & \quad \text{linear equivalence among divisors} \\
\equiv & \quad \text{numerical equivalence among divisors}
\end{align*}
Let $(X,L)$ be as above, i.e. a smooth, linearly normal subvariety of $\mathbb{P}^N$, embedded by $H^0(X,L)$, where $N = h^0(X,L) - 1$. $R_t = H^0(X,tL)$, $R_0 = H^0(X,\mathcal{O}_X)$, then $R(X) = \oplus_{t \geq 0} R_t$ is a graded $S$-module having a minimal free resolution $\cdots \to E_{p+1} \to E_p \to \cdots \to E_1 \to E_0 \to R \to 0$ in which $E_0 = \oplus_{q \geq 0} (B_{0,q} \otimes S(-q))$, $E_1 = \oplus_{q \geq 0} (B_{1,q} \otimes S(-q))$ and so on, where $B_{p,q}$ are $\mathbb{C}$ vector spaces whose dimensions $b_{p,q}$ keep track of how many $S(-q)$ appear in $E_p$; the $b_{p,q}$ do not depend on the choice of the minimal free resolution, (see [6]). We will write $b_{p,q}$ instead of $b_{p,q}(X)$, $R$ instead of $R(X)$, when any confusion is impossible.

Note that $S = \oplus_{t \geq 0} H^0(X,\mathcal{O}_{\mathbb{P}^N}(t))$, so we have a natural graded map $\rho : S \to R$ and the $S$-module structure on $R$ is given by $sr = \rho(s)r$. $X$ is p.n. if every graded piece of $\rho$ is surjective.

$E_0$ is the free $S$-module corresponding to the generators of $R$, $b_{0,q}$ is the number of generators of $R$ whose degree is $q$. Let us be careful: every $R_i$ is also a $\mathbb{C}$ vector space of finite dimension, but we are considering $R$ as an $S$-module: there is only one generator of degree 0, the multiplicative identity $1$ of the ring $R$, which is also the generator of the $\mathbb{C}$ vector space $R_0$. There are no generators in degree 1 because, as $X$ is linearly normal, every element of $R_1$ comes from $S$ by $\rho$, so it is the product of an element of $S$ and the generator $1$, hence $b_{0,1} = 0$.

If $X$ is p.n., for the same reason we have no other generators for $R$ as an $S$-module, so that $E_0$ is isomorphic to $S$ and the kernel of the map $E_0 \to R$ is precisely $I_X$, in this case $\cdots \to E_p \to \cdots \to E_1 \to I_X \to 0$ is a free resolution for $I_X$; this is the point of view of [6], [7], [8], [13], but what can we say when $X$ is not p.n.? Let us examine $E_0$ firstly. We have the following

**Proposition 2.1.** Let $X$ be as above, then:

- $b_{0,0} = 1$,
- $b_{0,1} = 0$,
- $X$ is 2-normal if and only if $b_{0,2} = 0$,
- for $q \geq 3$, if $X$ is $q$-normal then $b_{0,q} = 0$ (but not vice versa),
- $X$ is p.n. if and only if $b_{0,q} = 0$ for any $q \geq 2$.

**Proof.** The values of $b_{0,0}$ and $b_{0,1}$ were discussed above. As $X$ is linearly normal, the 2-normality of $X$ is equivalent to the vanishing of $b_{0,2}$: in fact if $X$ is 2-normal then $b_{0,2} = 0$ because any element of $R_2$ is a multiple of $1$ by an element of $S$. If there are no degree 2 generators in $R$, as $S$-module, every element of $R_2$ must be an $S$-linear combination of the generators of $R$ of degree 0 or 1, i.e. it must be a multiple of 1 and then it comes from $S$ by $\rho$. In any case if we consider the $\mathbb{C}$-linear map between $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(2))$ and $R_2$, $b_{0,2}$ is the $\mathbb{C}$-dimension of the cokernel of this map.

For $q \geq 3$ we have that if $X$ is $q$-normal then $b_{0,q} = 0$ for the above reason, but not vice versa because $b_{0,q}$ is always the number of the degree $q$ generators of $R$, but when we consider the $\mathbb{C}$-linear map between $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(q))$ and $R_q$, $b_{0,q}$ is only less than or equal the $\mathbb{C}$-dimension of the cokernel of this map because in the cokernel there can be also elements which are $S$-linear combinations of generators of $R$ whose degree is less than $q$.

If $X$ is p.n., $R$ is generated over $S$ by 1, hence $b_{0,q} = 0$ for all $q \geq 2$, the vice versa is obvious. □
Let us consider $E_1$, i.e. the free $S$-module of the primitive syzygies among the generators of $R$, where primitive means, according to Green [4], that every considered degree $q$ syzygy is not an $S$-linear combination of syzygies whose degree is less than $q$. Then $b_{1,q}$ is the number of the degree $q$ generators of this $S$-module. We have the following

**Proposition 2.2.** Let $X$ be as above, then:
- $b_{1,0} = b_{1,1} = 0$,
- $b_{1,2} = h^0(\mathbb{P}^N, J_X(2))$,
- if $b_{1,q} = 0$ for $q \geq k + 1$ then $b_{0,q} = 0$ for $q \geq k$.

**Proof.** The first vanishings are obvious. $b_{1,2}$ is the number of the generators of the $S$-module of degree 2 syzygies (in this degree every syzygy is primitive). As a degree $q$ syzygy can involve only generators of $R$ whose degree is less than or equal to $q - 1$, a degree 2 syzygy is always of the following type: $s1 = 0$ with $s \in S$, because there are no degree 1 generators in $R$. Moreover, the number of the generators of the $S$-module of the degree 2 syzygies coincides with the dimension of this $S$-module viewed as a $\mathbb{C}$ vector space. Note that if $q \geq 3$ this is not longer true: the submodule of the primitive degree $q$ syzygies of type $s1 = 0$ always corresponds to degree $q$ $\mathbb{C}$-independent hypersurfaces of $\mathbb{P}^N$ containing $X$, but we have only that $b_{1,q}$ is greater than or equal to the $\mathbb{C}$-dimension of the $\mathbb{C}$-vector space of irreducible hypersurfaces of degree $q$ containing $X$ (which is less than or equal to $h^0(\mathbb{P}^N, J_X(q))$: there can be primitive degree $q$ syzygies which have no links with the hypersurfaces containing $X$.

Now let us assume that $k$ is the maximal degree for the primitive syzygies among the generators of $R$: $1, x_1, x_2, \ldots, x_h$, and, by contradiction, let us assume that one of these generators, say $x_h$, belongs to $R_q$ with $q \geq k$, and thus there are no primitive syzygies involving $x_h$. However, it is easy to see that any generator is involved by a syzygy, so we have $\alpha_0 1 + \alpha_1 x_1 + \cdots + \alpha_h x_h = 0$ where $\alpha_i \in S$, $\deg(\alpha_i) \geq 1$, and this is not a primitive syzygy. Hence it must be an $S$-linear combination of primitive syzygies, but this is not possible because no primitive syzygy involves $x_h$.

Let us consider the exact sequence $0 \to M \to V \otimes \mathcal{O}_X \to L \to 0$ of vector bundles over $X$, where $M$ is the kernel of the evaluation map $V \otimes \mathcal{O}_X \to L$ and $V = H^0(X, L)$. Let $q \geq 2$; to estimate $b_{1,q}$ we have

**Proposition 2.3.** Let $X$, $L$, $M$ be as above, then $b_{1,q} = 0$ if $H^1(X, \Lambda^2 M \otimes L^q - 2) = 0$.

**Proof.** By [7], we have that $b_{1,q}$ is the dimension of the $\mathbb{C}$-vector space which is the homology at the middle level in the following piece of the Koszul complex:

\[
\cdots \to \Lambda^2(V) \otimes R_{q-2} \to V \otimes R_{q-1} \to R_q \to \cdots
\]

Let us call $\alpha_q : \Lambda^2(V) \otimes R_{q-2} \to V \otimes R_{q-1}$ and $\beta_q : V \otimes R_{q-1} \to R_q$. From it we get:

\[
0 \to \Lambda^2 M \to \Lambda^2(V) \otimes \mathcal{O}_X \to V \otimes L \to S^2(L) = L \otimes L \to 0
\]

which splits as

\[
0 \to M \otimes L \to V \otimes L \to L \otimes L \to 0 \quad \text{and} \quad 0 \to \Lambda^2 M \to \Lambda^2(V) \otimes \mathcal{O}_X \to M \otimes L \to 0.
\]

Hence we have, for any $m \in \mathbb{Z}$: $0 \to \Lambda^2 M \otimes L^m \to \Lambda^2(V) \otimes L^m \to M \otimes L^{m+1} \to 0$ and $0 \to M \otimes L^m \to V \otimes L^m \to L^{m+1} \to 0$. By choosing $m = q - 2$ in the first case we get the following exact sequence:

\[
0 \to H^0(X, \Lambda^2 M \otimes L^{q-2}) \to \Lambda^2(V) \otimes R_{q-2} \to H^0(X, M \otimes L^{q-1}) \to H^1(X, \Lambda^2 M \otimes L^{q-2}) \to \cdots
\]

\[
S^2(L) = L \otimes L
\]
By choosing $m = q - 1$ in the second case we get this exact sequence:

$$0 \to H^0(X, M \otimes L^{q-1}) \to V \otimes R_{q-1} \to R_q \to \cdots.$$ 

If we call $\gamma_q : \Lambda^2(V) \otimes R_{q-2} \to H^0(X, M \otimes L^{q-1})$, $\delta_q : H^0(X, M \otimes L^{q-1}) \to V \otimes R_{q-1}$ and $\epsilon_q : V \otimes R_{q-1} \to R_q$, we have that $\delta_q = \epsilon_q$ and $\alpha_q = \delta_q \circ \gamma_q$.

Hence if $H^1(X, \Lambda^2 M \otimes L^{q-2}) = 0$ we get that $\gamma_q$ is surjective, $\ker(\beta_q) = \text{Im}(\alpha_q)$ and therefore $b_{1,q} = 0$. \hfill \Box

**Remark 2.4.** Note that, in the same way, it is possible to get that $b_{0,q} = 0$ if $H^1(X, M \otimes L^{q-1}) = 0$, i.e. the condition $H^1(X, M \otimes L^{q-1}) = 0$ for $q \geq 2$ implies that $X$ is p.n.; in this form this condition is used by many authors (see [9], [14], [15] for instance). When $X$ is p.n. and $H^1(X, \Lambda^2 M \otimes L^{q-2}) = 0$ for $q \geq 3$ then $I_X$ is generated in degree 2, when $X$ is not p.n. the condition yields only some information about the generators of $R$.

## 3. Syzygies of scrolls

In this section we consider the vanishing of $H^1(X, \Lambda^2 M \otimes L^{q-2})$, $q \geq 2$, for $r$-dimensional scrolls $X = \mathbb{F}(\mathcal{E})$ over smooth curves $C$, $r \geq 2$, where $\mathcal{E}$ is a very ample rank $r$ vector bundle over $C$. In this case $L$ is the tautological bundle $T, p_\mathcal{E} = E$ and we have the exact sequence $0 \to M_\mathcal{E} \to H^0(C, E) \otimes \mathcal{O}_C \to E \to 0$, where $H^0(C, E) \otimes \mathcal{O}_C \to E$ is the natural evaluation map. Our strategy will be to calculate $h^i(X, \Lambda^2 M \otimes T^{q-2})$ by using $h^i(C, p_\mathcal{E}_*(\Lambda^2 M \otimes T^{q-2}))$. It is well-known that the two numbers are equal if $R^i p_\mathcal{E}_*(\Lambda^2 M \otimes T^{q-2}) = 0$, $\forall i \geq 1$ (see [12], p. 253) and this is true if $h^i(F, (\Lambda^2 M \otimes T^{q-2})|_F) = 0$, $\forall j \geq 1$. We have the following

**Lemma 3.1.** With the above notations $h^i(F, (\Lambda^2 M \otimes T^{q-2})|_F) = 0$, $\forall j \geq 1$.

**Proof.** Recall that $F \cong \mathbb{F}^{n-1}$, $(T^{q-2})|_F = \mathcal{O}_F(q - 2)$, so that $h^i(F, (\Lambda^2 M \otimes T^{q-2})|_F) = h^i(F, (\Lambda^2 M|_F \otimes \mathcal{O}_F(q - 2))$.

Now let us consider $0 \to \mathcal{O}_X(T - F) \to \mathcal{O}_X(T) \to \mathcal{O}_F(T|_F) \to 0$ and the long exact sequence $0 \to H^0(X, T - F) \to H^0(X, T) \to H^0(F, \mathcal{O}_F(1)) \to H^1(X, T - F) \to \cdots$. As $p_\mathcal{E} = E$ is generated by global sections we have that $H^0(X, T) = H^0(X, T - F) \oplus H^0(X, \mathcal{O}_F(1))$.

By considering the restriction to $F$ of $0 \to M_\mathcal{E} \to H^0(X, T) \otimes \mathcal{O}_X \to T \to 0$ we get $0 \to M|_F \to H^0(X, T) \otimes \mathcal{O}_F \to \mathcal{O}_F(1) \to 0$.

By using the Euler sequence for $F$ we get the following commutative diagram:

\[
\begin{array}{ccccccccc}
0 & & 0 & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
0 & \longrightarrow & M|_F & \longrightarrow & H^0(X, T - F) \otimes \mathcal{O}_F & \longrightarrow & \mathcal{O}_F(1) & \longrightarrow & 0 \\
& & & & & & & & \\
& & & & & & & & \\
0 & \longrightarrow & \Omega_F(1) & \longrightarrow & H^0(X, \mathcal{O}_F(1)) \otimes \mathcal{O}_F & \longrightarrow & \mathcal{O}_F(1) & \longrightarrow & 0 \\
& & & & & & & & \\
0 & & 0 & & \\
\end{array}
\]
The left column splits so that $M_{|F} \cong H^0(X, T - F) \otimes \mathcal{O}_F \oplus \Omega_F(1)$,
$\Lambda^2(M_{|F}) = \Omega^2_F(2) \otimes \mathcal{O}_F(1) \otimes H^0(X, T - F) \otimes \mathcal{O}_F \oplus \Lambda^2(H^0(X, T - F)) \otimes \mathcal{O}_F$, and
$\Lambda^2(M_{|F} \otimes \mathcal{O}_F(q-2) = \Omega^2_F(q) \otimes \mathcal{O}_F(q-1) \otimes H^0(X, T - F) \otimes \mathcal{O}_F \oplus \Lambda^2(H^0(X, T - F)) \otimes \mathcal{O}_F(q-2)$, now it is very easy to see that $h^1(F, (\Lambda^2 M \otimes T^{r-2})_{|F}) = 0, \forall j \geq 1.$

Now we can prove

**Theorem 3.2.** Let $(X, T)$ be a scroll as above over a genus $g$ curve, $r \geq 3, q \geq 4$, then $H^1(X, \Lambda^2 M \otimes L^{q-2}) = 0$ if $\mu^-(M_E) + \mu^-(E) > g - 1$.

**Proof.** By Lemma 3.1 we have to show that $h^1(C, p_* (\Lambda^2 M \otimes T^{q-2})) = 0$. As $p_*(M \otimes M \otimes T^{q-2}) = p_* [\Lambda^2 M \otimes T^{q-2}] \oplus p_* [S^2 M \otimes T^{q-2}]$ it suffices to show that $h^1(C, p_* (M \otimes M \otimes T^{q-2})) = 0$.

This is true if $\mu^-[p_*(M \otimes M \otimes T^{q-2})] > 2g - 2$, (see [9]).

We can use Prop. 4.2 of [9], in fact $p_* T = E$ is generated by global sections, $T$ is $0$-p-regular and $M \otimes T^{q-2}$ is -1 p-regular, hence we have that $\mu^-[p_*(M \otimes M \otimes T^{q-2})] \geq \mu^-(M_E) + \mu^-[p_*(M \otimes T^{q-2})]$; (note that, in this case, the inequality given by Prop. 4.2 of [9] is very simple because for $i = r - 2$ we have that $R^i p_* (T^{-1}) = 0$, so that $\min\{\mu^-(M_E) + \mu^-[p_*(M \otimes T^{q-2})], +\infty\} = \mu^-(M_E) + \mu^-[p_*(M \otimes T^{q-2})]$; see [9] for the notion of $k$ p-regular vector bundles).

Now we can use the same proposition for $\mu^-[p_*(M \otimes T^{q-2})]$, in fact $T$ is 0-p-regular and $T^{q-2}$ is -1 p-regular. By the same previous reason we get: $\mu^-[p_*(M \otimes T^{q-2})] \geq \mu^-(M_E) + \mu^-[S^2 T^{q-2}] (E)$ and $\mu^-[S^2 T^{q-2} (E)] = (q - 2) \mu^-(E)$.

Hence we have $\mu^-[p_*(M \otimes M \otimes T^{q-2})] \geq 2 \mu^-(M_E) + (q - 2) \mu^-(E)$, but as $T$ is very ample, this inequality is satisfied for $g \geq 4$ if it is true for $g = 4$, so that the condition is simply $\mu^-(M_E) + \mu^-(E) > g - 1$.

**Remark 3.3.** Assume that $E$ is semistable, $g \geq 2$ and $\mu(E) < 2g$ then we have $\mu^-(M_E) > r [\mu^-(E) - 2g] - 2 + 2h^1(E)$ (see [9], Prop. 1.5), so the condition in Theorem 3.2 becomes: $(2 + r) \mu(E) + 2h^1(E) > (2r + 1) g + 1$.

**Remark 3.4.** Although Theorem 3.2 is stated for $E$ very ample, exactly the same proof works also when $E$ is ample and generated by global sections.

For the rest of this section we are concerned with the vanishing of $H^1(X, \Lambda^2 M \otimes L^{q-2}), q \geq 2$, when $X$ is a divisor of $W = \mathbb{P}(E)$ where $E$ is an ample, globally generated vector bundle over a smooth, genus $g$, curve $C$. We assume that $L$, the restriction to $X$ of the tautological divisor $T$ of $W$, is very ample. $X$ is fibered over $C$ and the generic fibre is a smooth hypersurface of $\mathbb{P}^{n-1}$ whose degree is fixed. We can prove the following

**Proposition 3.5.** Let $(X, L)$ be as above, assume that $X \equiv aT + bF$ with $a \geq 2$, then $H^1(X, \Lambda^2 M \otimes L^{q-2}) = 0$ if $r \geq 4, q \geq 4, \mu^-(M_E) + \mu^-(E) > g - 1$ except, possibly, for $q = a$ and $q = a + 1$.

**Proof.** As $E$ is generated by global sections we can consider the usual exact sequence

$$0 \rightarrow M_T \rightarrow H^0(W, T) \otimes \mathcal{O}_W \rightarrow T \rightarrow 0.$$ 

By restricting it to $X$ we get: $0 \rightarrow (M_T)|_X \rightarrow H^0(W, T) \otimes \mathcal{O}_X \rightarrow L \rightarrow 0$ as $L = T|_X$. On the other hand we have $0 \rightarrow M \rightarrow H^0(X, L) \otimes \mathcal{O}_X \rightarrow L \rightarrow 0$, but it is easy to see that...
$H^0(W, T) \otimes \mathcal{O}_X = H^0(X, L) \otimes \mathcal{O}_X$ so that $(M_T)_X = M$. Hence we can tensorize the exact sequence $0 \to \mathcal{O}_W(-X) \to \mathcal{O}_W \to \mathcal{O}_X \to 0$ with $\Lambda^2 M_T \otimes T^{q-2}$ and in cohomology we have 
\[ \cdots \to H^1(W, \Lambda^2 M_T \otimes T^{q-2}) \to H^1(X, \Lambda^2 M \otimes L^{q-2}) \to H^2(W, \Lambda^2 M_T \otimes \mathcal{O}_W(-X + (q - 2)T) \to \cdots \]
as $H^1(X, \Lambda^2 M \otimes L^{q-2}) = H^1(X, \Lambda^2 (M_T)|_X \otimes (T|_X)^{q-2})$. By arguing as in 3.2 and by recalling that $E$ ample implies $\mu^{-}(E) > 0$, we have $H^1(W, \Lambda^2 M_T \otimes T^{q-2}) = 0$. To deal with $H^2(W, \Lambda^2 M_T \otimes \mathcal{O}_W(-X + (q - 2)T))$ recalling Remark 3.4, we proceed as in the proof of 3.2. Thus that group vanishes if $h^2(F, (\Lambda^2 M_T \otimes \mathcal{O}_W(-X + (q - 2)T))|_F) = 0$, $\forall j \geq 1$.

As in the proof of 3.1 we have that $(\Lambda^2 M_T \otimes \mathcal{O}_W(-X + (q - 2)T))|_F$ is the direct sum of some copies of $\Omega^2_F(q - a), \Omega^1_F(q - a - 1)$ and $\mathcal{O}_F(q - a - 2)$ so that we get the vanishing for $q \geq a + 2$. If $q \leq a - 1$ (if necessary, recall that $a \geq 1$ in any case) we can consider $H^{r-2}(W, [\Lambda^2 M_T \otimes \Omega^1_W(-X + (q - 2)T)] \otimes K_W)$ and we can proceed analogously as $r - 2 \geq 2$.

**Remark 3.6.** If $a = 2$ or $a = 3$ (i.e. the fibres are hypersurfaces of degree 2 or 3) Proposition 3.5 shows that under the same assumptions of 3.2 for $\mathbb{P}(E)$, with $r \geq 4$, $b_{0,q}(X) = 0$ if $q \geq 3$ and therefore $X$ is p.n. if and only if it is 2-normal. If $r = 3$ the previous proof works only for $q \geq a + 2$.

**Proposition 3.7.** Let $(X, L)$ as above, with the same assumptions of 3.5, then $b_{1,a}(X) = 0$ if $b \geq 1$.

**Proof.** We have only to show that $H^2(W, \Lambda^2 M_T \otimes \mathcal{O}_W(-X + (a - 2)T)) = 0$. By using $0 \to \Lambda^2 M_T \to \Lambda^2 (H^0(W, T)) \otimes \mathcal{O}_W \to H^0(W, T) \otimes T \to S^2(T) = T \otimes T \to 0$ tensorized by $\mathcal{O}_W(-X + (a - 2)T)$ we see that 

$H^2(W, \Lambda^2 M_T \otimes \mathcal{O}_W(-X + (a - 2)T)) = H^1(W, M_T \otimes \mathcal{O}_W(-X + (a - 1)T))$

as $M_T \otimes T$ is the kernel of $H^0(W, T) \otimes T \to T \otimes T$.

Let $B$ be a degree $b$ divisor on $C$ such that $X = aT + p^*B$, then $-X + (a - 1)T = -T + p^*(-B)$ and $H^1(W, M_T \otimes \mathcal{O}_W(-T + p^*(-B)) = H^0(C, -B)$ by using Leray’s spectral sequence as usual. If $b \geq 1$ we have the required vanishing. \qed

**Remark 3.8.** By arguing as in the previous proof we can show that 

$H^2(W, \Lambda^2 M_T \otimes \mathcal{O}_W(-X + (a - 1)T)) = H^1(C, M_E \otimes B)$

which does not vanish for $b \geq 1$, so that it is not possible to get conditions under which $b_{1,a+1}(X) = b_{1,a}(X) = 0$ by using this method.

4. Koszul groups for $X = \mathbb{P}(E)$

For any smooth $n$-dimensional $X$, embedded in $\mathbb{P}^N$ by a very ample line bundle $L$ and for any vector bundle $\mathcal{E}$ over $X$ we can consider the Koszul $\mathbb{C}$-vector spaces $K_{p,q}(X, \mathcal{E}, L)$ (see [4]). Let $k_{p,q}(X, \mathcal{E}, L)$ be the dimension of $K_{p,q}(X, \mathcal{E}, L)$. It is $k_{p,q}(X, \mathcal{O}_X, L) = b_{p,p+q}$ for any $p, q$, so that the computation of these Koszul groups is related to the minimal resolutions of $R$.

For the convenience of the reader we recall the following basic results, due to M. Green, which will be used in the sequel:
Theorem 4.1. ([4], Th.3.a.1) \(K_{p,q}(X,E,L) = 0\) if \(h^0(X,E \otimes L^q) \leq p\).

Theorem 4.2. ([4], Th.2.c.6) \(K_{p,q}(X,E,L)^* \cong K_{N-n-p,n+1-q}(X,E^* \otimes K_X,L)\) if \(h^i(X,E \otimes L^{q-i}) = 0\) and \(h^i(X,E \otimes L^{q-i-1}) = 0\) for \(i = 1, 2, \ldots, n - 1\).

In this section \(E\) is a rank \(r\) vector bundle over a smooth genus \(g\) curve \(C\), \(r \geq 2\), \(X = \mathbb{P}(E)\) is embedded in \(\mathbb{P}^N\) by a very ample line bundle \(L \sim aT + p^*B\), where \(B\) is a divisor of \(C\), \(\text{deg}(B) = b\). \(X\) is linearly normal, \(N = h^0(L) - 1\), \(n = r\), \(L \equiv aT + bF\), \(\delta = c_1(E)\). We want to compute \(k_{p,q}\) by using 4.1 and 4.2 when \(E = \mathcal{O}_X\).

First of all we consider \(h^i(X,L^{q-i})\) for \(i = 1, 2, \ldots, r - 1\) and \(q \geq 2\). Recall that \(L^{q-i} \sim (q - i)aT + p^*[(q - i)B]\) and that \(a \geq 1\) and \(a\mu^-(E) + b > 0\) as \(L\) is very ample; moreover, by using Leray’s spectral sequence and Kodaira’s vanishing we have that all cohomology groups vanish but for \(i = 1\), in this case we have \(h^1(X,L^{q-1}) = \mu^1(C,S^{(q-1)a^i}(E) \otimes (q - 1)B) = 0\) if \((q - 1)(a\mu^-(E) + b) > 2g - 2\).

Now we consider \(h^i(X,L^{q-i-1})\) for \(i = 1, 2, \ldots, r - 1\) and \(q \geq 3\). Reasoning as in the previous case we get that all groups vanish if \((q - 2)(a\mu^-(E) + b) > 2g - 2\). Note that if \(q = 2\), \(i = 1\) the corresponding group does not vanish unless \(g = 0\).

We have proved the following

Lemma 4.3. With the notation as in this section let \(q \geq 3\), \(g \geq 1\), then \(K_{p,q}(X,\mathcal{O}_X,L)^* \cong K_{N-r-p,r+1-q}(X,K_X,L)\) if \((q - 2)(a\mu^-(E) + b) > 2g - 2\).

Lemma 4.3 and Theorem 4.2 tell us that, under some conditions, for our varieties \(k_{p,q}(X,\mathcal{O}_X,L) = 0\) if \(N - r - p < 0\) and \(k_{p,q}(X,\mathcal{O}_X,L) = 0\) if \(N - r - p \geq 0\) and \(h^0(X,K_X + (r + 1 - q)L) \leq N - r - p\). If \(g \geq 1\) it is well-known that \(N \geq 2r\), hence \(N - r - p \geq r - p\), so that for \(p = 0, 1, \ldots, r\) to get \(k_{p,q}(X,\mathcal{O}_X,L) = 0\) it suffices that \(h^0(X,K_X + (r + 1 - q)L) = 0\).

We have \(K_X + (r + 1 - q)L \equiv [(r + 1 - q)a - r]T + [d + 2g - 2 + (r + 1 - q)b]F\), and such a line bundle has no sections if \((r + 1 - q)a - r < 0\) or (see [9], Lemma 1.12) if \([(r + 1 - q)a - r]\mu^+(E) + \delta + 2g - 2 + (r + 1 - q)b < 0\) and \((r + 1 - q)a - r \geq 0\). Then we have proved the following

Lemma 4.4. With the notation as in this section let \(q \geq 3\), \(r \geq p \geq 0\), \(g \geq 1\), then \(K_{p,q}(X,\mathcal{O}_X,L) = 0\) if \((q - 2)(a\mu^-(E) + b) > 2g - 2\) and \((r + 1 - q)a - r < 0\) or if \((r + 1 - q)a - r \geq 0\) and \([(r + 1 - q)a - r]\mu^+(E) + \delta + 2g - 2 + (r + 1 - q)b < 0\).

Corollary 4.5. Let \(X = \mathbb{P}(E)\) as above with \(r = 2\), \(g \geq 1\), then \(b_{0,q}(X) = 0\) for \(q \geq 3\) if \(a\mu^-(E) + b > 2g - 2\).

Corollary 4.6. Let \(X = \mathbb{P}(E)\) be a scroll over a curve of genus \(g \geq 1\) (hence \(a = 1\), \(b = 0\)) then \(b_{0,q}(X) = 0\) for \(q \geq 3\) if \(a\mu^-(E) > 2g - 2\), therefore \(X\) is p.n. if and only if it is 2-normal.

Corollary 4.7. Let \(X = \mathbb{P}(E)\) be a scroll over a curve of genus \(g = 2\) then \(X\) is p.n. if and only if it is 2-normal; in fact in this case \(\mu^-(E) > 3\), (see [2]); moreover, \(b_{p,p+q}(X) = 0\) for \(q \geq 3\), \(r \geq p \geq 0\).
Corollary 4.8. Let $X = \mathbb{P}(E)$ be a scroll over a curve of genus $g \geq 1$ with $\mu^-(E) > 2g$, then $X$ is projectively normal (see [9]) and $I_X$ is generated in degree two.

Remark 4.9. The previous results hold also if $X$ is embedded by a linear subspace $W$ of $H^0(X, L)$, i.e. if $X$ is not linearly normal. To see this it is enough to use Green’s Theorems 4.1 and 4.2, being careful to use $N' = \dim W$ instead of $N$ in the previous formulas.

5. Long exact sequences for Koszul groups and applications.

Let $X$ be a smooth variety in $\mathbb{P}^N$ as usual and let $Y$ be a smooth, one codimensional subvariety of $X$. Let $L$ be a very ample divisor of $X$. Then from the natural exact sequence given by $Y$: $0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ we get $0 \to H^0(X, qL - Y) \to H^0(X, qL) \to H^0(Y, qL|_Y) \to \cdots$ for any $q \geq 0$.

Assume that the previous sequence is exact for any $q \geq 0$, then if we put $\mathcal{A} = \oplus_{q \geq 0} H^0(X, qL - Y), \quad \mathcal{B} = \oplus_{q \geq 0} H^0(X, qL)$ and $\mathcal{C} = \oplus_{q \geq 0} H^0(Y, qL|_Y)$ we get an exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ of $S$-modules from which we deduce some long exact sequences for Koszul groups (see [4], Cor. 1.d.4):

$$\cdots \to K_{p,q}(\mathcal{B}) \to K_{p,q}(\mathcal{C}) \to K_{p+1,q}(\mathcal{A}) \to K_{p+1,q}(\mathcal{B}) \to \cdots,$$

where $K_{p,q}(\mathcal{A}) = K_{p,q}(X, \mathcal{O}_X(-Y), L), K_{p,q}(\mathcal{B}) = K_{p,q}(X, \mathcal{O}_X, L), K_{p,q}(\mathcal{C}) = K_{p,q}(Y, \mathcal{O}_Y, L|_Y)$.

In [3] the authors considered the projective normality of $(X, L)$ in the case in which $L$ is interesting from the point of view of adjunction theory, i.e. when $L = aK_X + bA$ where $A$ is a suitable divisor of $X$ and $a, b$ are integers. Notice that it is the same point of view of [8] and [14], [15], [16]. Here, by using the previous ideas we can prove the following proposition:

Proposition 5.1. Let $X$ be a regular surface and let $L$ be a very ample line bundle on $X$. Assume that there exists a smooth curve in $|L - K_X|$. Then $b_{0,q}(X) = 0$ for $q \geq 4$.

Proof. Firstly notice that the proposition is true for $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Let now $(X, L) \neq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ and let $Y$ be a smooth curve in $|L - K_X|$. From the exact sequence $0 \to K_X \to L \to L|_Y \to 0$, as $X$ is regular it is easy to see that $h^1(X, L) = h^2(X, L) = 0$ and that $Y$ is linearly normal in the embedding given by $L$. Now let

$$\mathcal{A} = \oplus_{q \geq 0} H^0(X, qL - Y), \quad \mathcal{B} = \oplus_{q \geq 0} H^0(X, qL), \quad \mathcal{C} = \oplus_{q \geq 0} H^0(Y, qL|_Y).$$

The regularity of $X$ and Kodaira’s vanishing theorem give an exact sequence of $S$-modules $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$, which in turn gives the exact sequences

$$\cdots \to K_{0,q}(\mathcal{A}) \to K_{0,q}(\mathcal{B}) \to K_{0,q}(\mathcal{C}) \to 0$$

for any $q \geq 0$.

$K_{0,q}(\mathcal{C}) = 0$ for $q \geq 2$ because $Y$ is p.n. in $\mathbb{P}^{N-1}$ as it is canonically embedded by $L|_Y = K_Y$, hence it suffices to show that $b_{0,q}(\mathcal{A}) = K_{0,q}(X, \mathcal{O}_X(-Y), L) = 0$ for $q \geq 4$. Note that here $\mathcal{C}$ is considered as an $S$-module, not a $\mathbb{C}[x_0, x_1, \ldots, x_{N-1}]$-module, however, $Y$ is p.n. in $\mathbb{P}^N$ too, so that $b_{0,q}(\mathcal{C}) = k_{0,q}(\mathcal{C}) = 0, \forall q \geq 2$. 

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We can use Theorem 4.2 as \( h^1(X, -Y + (q - 1)L) = h^1(X, -Y + (q - 2)L) = 0 \) for \( q \geq 4 \), so we have to consider \( K_{N-2,3-q}(X, \mathcal{O}_X(Y + K_X), L) \). Now we can use Theorem 4.1 because \( h^0(X, (4 - q)L) \leq h^0(X, L) - 3 \) for \( q \geq 4 \). 

The previous ideas can be applied in other cases, for instance when \( X = \mathbb{P}(E) \) is the projectivized of a rank \( r \) vector bundle \( E \) over a surface \( \Sigma \). In this case let \( T \) be the tautological bundle and \( p : X \to \Sigma \) the natural projection as usual. Let \( C \) be a smooth curve on \( \Sigma \), \( C' = \pi^{-1}(C) \) and let us consider, for any \( j \geq 0 \), the exact sequences:

\[
0 \to \mathcal{O}_X(jT - p^*C) \to \mathcal{O}_X(jT) \to \mathcal{O}_{C'}(jT|_C) \to 0.
\]

If we assume that \( H^1(X, jT - p^*C) = 0 \), \( \forall j \geq 0 \), \( K_{0,q}(X, \mathcal{O}_X(-p^*C), T) = 0 \), \( \forall q \geq 2 \), and \( T|_{C'} \) embeds \( C' \) p.n. in \( \mathbb{P}^{N-1} \), then we have that \( X \) is p.n. In fact by these assumptions there is an exact sequence \( 0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0 \) of \( S \)-modules, and from the sequences \( \cdots \to K_{0,q}(\mathcal{A}) \to K_{0,q}(\mathcal{B}) \to K_{0,q}(\mathcal{C}) \to \cdots \) we have that \( b_{0,q}(\mathcal{B}) = 0 \), \( \forall q \geq 2 \), (i.e. \( X \) is p.n.) as \( b_{0,q}(\mathcal{A}) = b_{0,q}(\mathcal{C}) = 0 \), \( \forall q \geq 2 \), by assumptions. Note that here \( \mathcal{C} \) is considered as an \( S \)-module, not a \( \mathbb{C}[x_0, x_1, \ldots, x_{N-1}] \)-module, however \( C' \) is p.n. in \( \mathbb{P}^N \) too, so that \( b_{0,q}(\mathcal{C}) = 0 \), \( \forall q \geq 2 \).

Now we translate our assumptions into conditions on \( \Sigma \). The first one is simply \( H^1(\Sigma, S^i(E) \otimes \mathcal{O}_\Sigma(-C)) = 0 \), \( \forall j \geq 1 \), the third one is satisfied if we assume that \( \mu^-(E[C]) > 2g(C) \) by Butler’s results [9]; for the second one we use Green’s Theorems 4.1 and 4.2. Let us consider \( h^i(X, (q - i)T + p^*(C)) \) for \( i = 1, \ldots, n - 1 = r \), by standard calculations they vanish if \( h^i(\Sigma, S^{n-i}(E) \otimes \mathcal{O}_\Sigma(-C)) = 0 \) for \( i = 1, \ldots, r \) and \( q \geq 1 \), so we have only to assume further that \( h^2(\Sigma, S^{n-2}(E) \otimes \mathcal{O}_\Sigma(-C)) = 0 \) for \( q \geq 2 \). In order to have \( h^i(X, (q - i - 1)T + p^*(C)) = 0 \) for \( i = 1, \ldots, r \), it suffices to ask that \( h^1(\Sigma, \mathcal{O}_\Sigma(-C)) = 0 \) by similar arguments. Hence we can apply Theorem 4.2 and we consider, for \( q \geq 2 \), \( K_{N-n_n+1-q}(X, \mathcal{O}_X(p^*C + K_X), T) \). This group vanishes if \( h^0(X, p^*C + K_X + (n + 1 - q)T) \leq N - n \), i.e. \( h^0(\Sigma, C + \det(E) + K_X) \leq h^0(E) - r - 2 \), by Theorem 4.1.

Thus we have proved the following

**Theorem 5.2.** Let \( E \) be a very ample, rank \( r \), vector bundle over a smooth surface \( \Sigma \), let \( X = \mathbb{P}(E) \) and let \( T \) be the tautological bundle. Moreover, let \( C \) be a smooth genus \( g \) curve on \( S \). Then \( (X, T) \) is p.n. if

1. \( h^1(\Sigma, S^j(E) \otimes \mathcal{O}_\Sigma(-C)) = 0 \) for \( j \geq 0 \),
2. \( h^2(\Sigma, S^j(E) \otimes \mathcal{O}_\Sigma(-C)) = 0 \) for \( j \geq 0 \),
3. \( h^0(\Sigma, C + \det(E) + K_X) \leq h^0(E) - r - 2 \),
4. \( \mu^-(E[C]) > 2g \).

**Remark 5.3.** If \( C \) is a rational curve 4) is satisfied; if \( C \) is an ample divisor 1) and 2) are satisfied for \( j = 0 \); if \( -K_X \) is effective 3) is more easily satisfied.

Now we want to give some examples in which Theorem 5.2 can be applied.

**Example 5.4.** Let \( \pi : \Sigma \to \mathbb{P}^2 \) be the blowing up of \( \mathbb{P}^2 \) of \( k \) points in general position with \( 1 \leq k \leq 5 \), let \( L \) be the generator of \( \text{Pic}(\mathbb{P}^2) \), let \( E_1, \ldots, E_k \) be the exceptional divisors. \( \Sigma \) is
a well-known Del Pezzo surface and it is known that, in this range, \( -K_\Sigma \) is very ample. Let 
\( E \) be \( -K_\Sigma \oplus -K_\Sigma \) and let \( C \) be \( E_i \). Then Theorem 5.2 proves that \( (X, T) \) is p.n.

In fact by looking at the exact sequence \( 0 \to \mathcal{O}_X(-E_i) \to \mathcal{O}_X \to \mathcal{O}_{E_i} \to 0 \) we get that 1) and 2) are true for \( j = 0 \). 4) is true as \( E_i \) is a rational curve. By recalling that \( h^0(\Sigma, -K_\Sigma) = 10 - k \) we have that \( h^0(E) = 12 - 2k \). Moreover, \( h^0(\Sigma, C + \det(E) + K_\Sigma) = h^0(\Sigma, -K_\Sigma + E_i) = h^0(\Sigma, 3\pi^*L - E_2, \ldots, -E_k) = 11 - k \) as the \( k \) points are in general position, hence 3) is satisfied. Now let us consider 1) and 2) for \( j \geq 1 \). It suffices to show that \( h^j(\Sigma, -K_\Sigma - E_i) = 0 \) for \( t \geq 1, i = 1, 2 \). For \( i = 2 \) we can use Serre duality. For \( i = 1 \) we can use Kodaira vanishing because \( -K_\Sigma - E_i = K_\Sigma - (t + 1)K_\Sigma - E_i \) and \( -(t + 1)K_\Sigma - E_i \) is ample by Nakai-Moishezon criterion: \( -(t + 1)K_\Sigma - E_i)^2 > 0 \) and for any curve \( \Gamma \) on \( \Sigma \) we have:

\[
-(t + 1)K_\Sigma - E_i \Gamma = -(t + 1)K_\Sigma \Gamma - E_i \Gamma = -tK_\Sigma \Gamma - K_\Sigma \Gamma - E_i \Gamma \geq -tK_\Sigma \Gamma > 0
\]

because \( \Phi_{|K_\Sigma|} \) embeds \( E_i \) as a line so that

\[
-K_\Sigma \Gamma = \deg \left[ \Phi_{|K_\Sigma|}(\Gamma) \right] \geq \Phi_{|K_\Sigma|}(E_i) \Phi_{|K_\Sigma|}(\Gamma) = E_i \Gamma.
\]

**Remark 5.5.** Obviously, in the previous example, when \( k = 1 \) Butler’s criterion can be used (see [9], Theorem 5.1A), to get the projective normality of \( (X, T) \).

**Example 5.6.** Let \( \Sigma \) be \( \mathbb{P}^2 \), let \( C \) be a line, let \( E \) be a rank 2 very ample vector bundle on \( \mathbb{P}^2 \), let \( \delta L \) and \( c \) be, respectively, the first and second Chern classes of \( E \) (\( L \) is the generator of \( \text{Pic}(\mathbb{P}^2) \) as above), let \( p : \mathbb{P}(E) \to \mathbb{P}^2 \) be the natural projection. Under which assumptions can we apply Theorem 5.2?

First of all 4) is true as \( C \) is a rational curve. 1) and 2) are true for \( j = 0 \) as \( C \) is ample. \( h^0(\Sigma, C + \det(E) + K_\Sigma) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(\delta - 2)) = \delta(\delta - 1)/2 \), so condition 3) becomes: \( h^0(E) \geq 4 + \delta(\delta - 1)/2 \). Now let us consider 1) and 2) for \( j \geq 1 \). Let \( Y \) be a smooth element of \( [T] \), so that \( Y \) is isomorphic to the blowing up of \( \mathbb{P}^2 \) at \( c \) points. Let \( \pi \) be the blowing up and let \( E_1, \ldots, E_C \) be the exceptional divisors as before. We consider \( 0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0 \) tensorized with \( jT - p^*L \) and we have:

\[
0 \to \mathcal{O}_X((j-1)T - p^*L) \to \mathcal{O}_X(jT - p^*L) \to \mathcal{O}_Y(jT - p^*L)|_Y \to 0.
\]

We can proceed by induction on \( j \geq 1 \) as \( h^i(X, -p^*L) = 0 \) for \( i = 1, 2 \), so we have only to consider \( h^i(Y, (jT - p^*L)|_Y) = 0 \) for \( i = 1, 2 \). Recall that \( T|_Y = \delta \pi^*L - E_1 \cdots - E_C \). Now if \( i = 2 \) we can use Serre duality, if \( i = 1 \) we can use Kodaira vanishing as in Example 5.4 when \( \delta \geq 4 \) and \( 0 \leq c \leq 6 \) (or \( \delta \geq 2 \) and \( 0 \leq c \leq 2 \)).

Hence, by using Theorem 5.2, with the previously introduced notation, we get the projective normality of \( (X, T) \) if

\[
h^0(E) \geq 4 + \delta(\delta - 1)/2, \delta \geq 4 \text{ and } 0 \leq c \leq 6.
\]

**Example 5.7.** Let \( \Sigma \) be any surface, let \( C \) be any rational curve on \( \Sigma \), choose \( E = L \), a very ample line bundle \( L \). When \( r = 1 \) Theorem 5.2 is true too, moreover, condition 2) is unnecessary. So we get that \( (X, L) \) is p.n. if:

\[
h^0(E) \geq 4 + \delta(\delta - 1)/2, \delta \geq 4 \text{ and } 0 \leq c \leq 6.
\]
\[ h^1(\Sigma, jL - C) = 0 \text{ for } j \geq 0, \]
\[ h^0(\Sigma, C + (3 - q)L + K_\Sigma) \leq h^0(L) - 3. \]

Such conditions are satisfied, for example, in many cases when \( \Sigma \) is the blowing up of \( \mathbb{P}^2 \) in \( k \) points in general position and \( C = E_i \).

References


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