On Rhombic Dodecahedra

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Abstract. In this note we prove that the intrinsic $i$-volume of any $d$-dimensional zonotope generated by $d+1$ (resp. $d$) line segments and containing a $d$-dimensional unit ball in $\mathbb{E}^d$ is at least as large as the intrinsic $i$-volume of the $d$-dimensional regular zonotope generated by $d+1$ line segments having inradius 1, where $i = 1, \ldots, d-1, d$.

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0. Introduction

According to a well-known theorem of Gauss [1] the density of any lattice packing of unit spheres in the 3-dimensional Euclidean space $\mathbb{E}^3$ is at most $\frac{\pi}{\sqrt{12}} = 0.7404 \ldots$ and equality holds for the lattice packing in which the unit spheres are centered at the points $(a\sqrt{2}, b\sqrt{2}, c\sqrt{2})$, where $a$, $b$ and $c$ are integers and their sum is even. One can easily see that in this case the Voronoi cells are regular rhombic dodecahedra that generate a face-to-face lattice tiling of $\mathbb{E}^3$. In general, a convex $d$-dimensional polytope of the $d$-dimensional Euclidean space $\mathbb{E}^d$ that tiles $\mathbb{E}^d$ by translation is called a parallelotope. Venkov [6] and later independently, McMullen [3] proved that any $d$-dimensional parallelotope admits (uniquely) a face-to-face lattice tiling of $\mathbb{E}^d$. Putting these results together one can claim that the volume of any 3-dimensional parallelotope of inradius at least 1 is at least as

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large as the volume of a regular rhombic dodecahedron of inradius 1. Recall (see for example [5]) that there are 5 combinatorial types of parallelohedra in $\textbf{E}^3$ namely, affine cubes, hexagonal prisms, rhombic dodecahedra, elongated dodecahedra and truncated octahedra (Figure 1). (Also, recall that a rhombic dodecahedron whose faces are congruent rhombi

![Figure 1](image)

and whose vertex figures are regular polygons is called a regular rhombic dodecahedron.) Finally, we mention the well-known fact (see [5]) that all five parallelohedra in $\textbf{E}^3$ are zonotopes (that is they are vector sums of line segments). We raise the following problem, an affirmative answer to which would obviously imply the above classical result of Gauss.

**The Rhombic Dodecahedral Conjecture.** The surface area of any 3-dimensional parallelohedron of inradius at least 1 in $\textbf{E}^3$ is at least as large as $12\sqrt{2} = 16.9705\ldots$ the surface area of a regular rhombic dodecahedron of inradius 1.

In order to phrase the main result of this note properly we have to recall the following. Let $K \subset \textbf{E}^d$ be a convex body (i.e. a compact convex set with nonempty interior in $\textbf{E}^d$). Let $\omega_i$ denote the $i$-dimensional volume of the unit $i$-ball, $0 \leq i \leq d$. Then the *intrinsic $i$-volume* $V_i(K)$ of $K$ can be defined via Steiner’s formula

$$
\text{Vol}_d(K + \rho B^d) = \sum_{i=0}^{d} \omega_i V_{d-i}(K) \rho^i,
$$

where $\rho > 0$ is an arbitrary positive real number and $\rho B^d$ denotes the closed ball of radius $\rho$ centered at the origin $\mathbf{o}$ of $\textbf{E}^d$ and $K + \rho B^d$ denotes the vector sum of the convex bodies $K$ and $\rho B^d$ with $d$-dimensional volume $\text{Vol}_d(K + \rho B^d)$. It is well-known (see for example [4]) that $\text{Vol}_d(K)$ is the $d$-dimensional volume of $K$, $2\text{Vol}_{d-1}(K)$ is the surface area of $K$ and $\frac{2\omega_{d-1}}{d\omega_d} V_1(K)$ is equal to the mean width of $K$. (Moreover, $V_0(K) = 1$.) Finally, the $d$-dimensional zonotope $Z$ generated by $d + 1$ line segments in $\textbf{E}^d$ is called *regular* if $Z$
can be generated by the segments connecting the center of a regular \(d\)-dimensional simplex with its vertices.

**Theorem.** The intrinsic \(i\)-volume of any \(d\)-dimensional zonotope generated by \(d + 1\) (resp., \(d\)) line segments and containing a \(d\)-dimensional unit ball in \(\mathbb{E}^d\) is at least as large as the intrinsic \(i\)-volume of the \(d\)-dimensional regular zonotope generated by \(d + 1\) line segments having inradius 1, where \(i = 1, \ldots, d - 1, d\).

The following is an immediate corollary that on the one hand, generalizes a result of Linhart [2] on the inradii of rhombic dodecahedra on the other hand, supports an affirmative answer to the Rhombic Dodecahedral Conjecture.

**Corollary.** The mean width (resp., surface area, volume) of any rhombic dodecahedron containing a ball of radius 1 in \(\mathbb{E}^3\) is at least as large as the mean width (resp., surface area, volume) of the regular rhombic dodecahedron of inradius 1.

1. **Proof of the Theorem**

The following lemma plays a key role in our proof of the theorem. The special case of the lemma having four equal line segments in \(\mathbb{E}^3\) has been proved by Linhart [2] several years ago. Our method of the proof presented below is different from the method introduced for \(d = 3\) in [2].

**Lemma.** The inradius of any \(d\)-dimensional zonotope generated by \(d + 1\) line segments of total length \(s > 0\) in \(\mathbb{E}^d\), \(d \geq 1\) is at most as large as the inradius of the \(d\)-dimensional regular zonotope generated by \(d + 1\) line segments of total length \(s\).

**Proof.** Let \(Z\) be an arbitrary \(d\)-dimensional zonotope generated by \(d + 1\) line segments of total length \(s > 0\) in \(\mathbb{E}^d\). Without loss of generality we may assume that \(Z\) is generated by the vectors \(v_1, v_2, \ldots, v_{d+1}\) of total length \(s\) that positively span \(\mathbb{E}^d\). Thus,

\[
Z = \{z \in \mathbb{E}^d \mid z = \sum_{i=1}^{d+1} \lambda_i v_i, 0 \leq \lambda_i \leq 1, 1 \leq i \leq d + 1\},
\]

\[
\sum_{i=1}^{d+1} ||v_i|| = s,
\]

\[
o \in \text{int[conv}\{v_1, v_2, \ldots, v_{d+1}\}],
\]

where \(||\ldots||, \text{o}, \text{int[...]}\), and \(\text{conv}\{\ldots\}\) stand for the norm of a vector, the origin of \(\mathbb{E}^d\), the interior of a set in \(\mathbb{E}^d\) and for the convex hull of a set in \(\mathbb{E}^d\) (Figure 2). Now \(Z\) is centrally symmetric. Moreover, it follows from the above construction that the pairs of opposite facets of \(Z\) are

\[
\{F_{ij}^i, F_{ij}^j\}, 1 \leq i < j \leq d + 1,
\]

where
\[ F_{ij}^i = v_i + F_{ij}, F_{ij}^j = v_j + F_{ij} \]

with \( F_{ij} \) being equal to the \((d - 1)\)-dimensional parallelotope generated by the vectors \( V_{ij} = \{v_1, v_2, \ldots, v_{d+1}\} \setminus \{v_i, v_j\} \), that is,

\[ F_{ij} = \{z \in \mathbb{E}^d \mid z = \sum_{v_k \in V_{ij}} \lambda_k v_k, 0 \leq \lambda_k \leq 1 \}. \]

\[ \text{Figure 2} \]

Thus,

\[ \text{dist}(F_{ij}^i, F_{ij}^j) \leq \text{dist}(v_i, v_j) = ||v_i - v_j|| \text{ for all } 1 \leq i < j \leq d + 1, \]

where \( \text{dist}(\ldots, \ldots) \) stands for the distance between two sets (resp., two points) in \( \mathbb{E}^d \). Now let \( V = \text{conv}\{v_1, v_2, \ldots, v_{d+1}\} \). Then (7) implies that the diameter of the insphere of \( Z \) is at most as large as the minimum edge length of the \( d \)-dimensional simplex \( V \), that is, it is at most \( \min\{||v_i - v_j|| \mid 1 \leq i < j \leq d + 1\} \). As a result in order to finish the proof of the lemma it is sufficient to show that among the \( d \)-dimensional simplices \( V = \text{conv}\{v_1, v_2, \ldots, v_{d+1}\} \) satisfying (2) and (3) in \( \mathbb{E}^d \) the regular one with center \( o \) has the largest possible shortest edge length. This we prove as follows.

Obviously, there is an extremal \( d \)-dimensional simplex \( V^* = \text{conv}\{v_1^*, v_2^*, \ldots, v_{d+1}^*\} \) satisfying

\[ \sum_{i=1}^{d+1} ||v_i^*|| = s, \]

\[ o \in \text{int}[\text{conv}\{v_1^*, v_2^*, \ldots, v_{d+1}^*\}] \]

with the largest possible value of \( \min\{||v_i^* - v_j^*|| \mid 1 \leq i < j \leq d + 1\} \). Suppose that \( V^* \) is not a regular simplex. Let \( m > 0 \) be the length of the shortest edge of \( V^* \). Then \( V^* \) must have a vertex say, \( v_k^* \) with some edges having length equal to \( m \) and some edges having length > \( m \). As the total number of edges meeting at \( v_k^* \) is \( d \) there exists a hyperplane \( H_k \)
of $\mathbf{E}^d$ passing through $\mathbf{o}$ as well as $\mathbf{v}_k^*$ such that it separates the edges of $\mathbf{v}_k^*$ of length $m$ from the edges of $\mathbf{v}_k^*$ of length $> m$. Let $H_k^+$ be the closed half-space of $\mathbf{E}^d$ bounded by $H_k$ that contains all the edges of $\mathbf{v}_k^*$ of length $> m$. Now it is clear that if we rotate the vertex $\mathbf{v}_k^*$ about the origin $\mathbf{o}$ towards $H_k^+$ with initial tangent vector being perpendicular to $H_k$ by a small angle, then all the edges of $\mathbf{v}_k^*$ will have length $> m$. Repeating this transformation at some other vertices of $V^*$ we can increase $\min \{ ||\mathbf{v}_i^* - \mathbf{v}_j^*|| \mid 1 \leq i < j \leq d + 1 \}$ without changing the norms of the vectors $\mathbf{v}_1^*, \mathbf{v}_2^*, \ldots, \mathbf{v}_{d+1}^*$, a contradiction. Thus $V^*$ must be a regular $d$-dimensional simplex of $\mathbf{E}^d$. Finally, let $F_i^*$ be the facet of $V^*$ opposite to the vertex $\mathbf{v}_i^*$ and let $U_i^* = \text{conv}(F_i^* \cup \{\mathbf{o}\}), 1 \leq i \leq d + 1$. If $I = \{1, 2, \ldots, d + 1\}$ and $v^* = \text{Vol}_{d-1}(F_1^*) = \text{Vol}_{d-1}(F_2^*) = \ldots = \text{Vol}_{d-1}(F_{d+1}^*)$, then it is easy to see that

$$\sum_{j \in I \setminus \{i\}} \text{Vol}_d(U_j^*) \leq \frac{1}{d} \cdot ||\mathbf{v}_i^*|| \cdot v^* \text{ for all } i \in I. \quad (8)$$

Thus, (8) implies in a straightforward way that

$$d \cdot \sum_{j \in I} \text{Vol}_d(U_j^*) \leq \frac{1}{d} \cdot s \cdot v^* \quad (9)$$

that is

$$\frac{d^2}{v^*} \cdot \text{Vol}_d(V^*) \leq s \quad (10)$$

with equality if and only if $\mathbf{o}$ is the center of the regular $d$-dimensional simplex $V^*$. Hence, if $\mathbf{o}$ were not the center of the regular $d$-dimensional simplex $V^*$, then using (10) we could move $\mathbf{o}$ to the center of $V^*$ thereby shortening the total length $\sum_{i=1}^{d+1} ||\mathbf{v}_i^*||$ of the spanning vectors of $V^*$, a contradiction. This completes the proof of the lemma. \hfill $\square$

Now we turn to the proof of the theorem. We distinguish two cases.

Case (1): The $d$-dimensional zonotope in question is generated by $d + 1$ line segments. The proof is by induction on $d$. Clearly, the theorem holds for $d = 2$. So, we may assume that $d \geq 3$ and the theorem holds in any Euclidean space of dimension less than $d$. Then let $Z$ be an arbitrary $d$-dimensional zonotope generated by $d + 1$ line segments and containing a $d$-dimensional unit ball in $\mathbf{E}^d$. Without loss of generality we may assume that $Z$ is generated by the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1}$ of $\mathbf{E}^d$ satisfying (1) and (3). Recall the following elegant formula for the intrinsic $i$-volume $V_i(Q)$ of a $d$-dimensional convex polyhedron $Q$ in $\mathbf{E}^d$ (see for example [4]):

$$V_i(Q) = \sum_{F^{(i)}} \gamma(F^{(i)}, Q) \cdot \text{Vol}_i(F^{(i)}), \quad (11)$$

where the summation is over all $i$-dimensional faces $F^{(i)}$ of $Q$ and $\gamma(F^{(i)}, Q)$ denotes the normalized exterior angle of $Q$ at the face $F^{(i)}$. We split the proof of the theorem in two subcases according to the values of $i$. 
Subcase $1 \leq i \leq d - 1$. Let $Z^*$ be the $d$-dimensional regular zonotope of inradius 1 generated by the vectors $u_1, u_2, \ldots, u_{d+1}$ of $E^d$ satisfying the corresponding versions of (1) and (3). Take a generating vector $u_j$ of $Z^*$, $1 \leq j \leq d + 1$. Let $\text{Pr}_j(Z^*)$ denote the orthogonal projection of $Z^*$ onto a hyperplane of $E^d$ perpendicular to $u_j$. Finally, let $u_{i-1} = V_{i-1}[	ext{Pr}_1(Z^*)] = V_{i-1}[	ext{Pr}_2(Z^*)] = \ldots = V_{i-1}[	ext{Pr}_{d+1}(Z^*)]$. Now, take a generating vector $v_j$ of $Z$, $1 \leq j \leq d + 1$. Then let $Z_i^{(j)}$ denote the union (“zone”) of the $i$-dimensional faces of $Z$ that are parallel to $v_j$. Then by induction one can easily get that

$$
\sum_{F^{(i)} \in Z_i^{(j)}} \gamma(F^{(i)}; Z) \cdot \text{Vol}_i(F^{(i)}) \geq ||v_j|| \cdot u_{i-1} \text{ for all } 1 \leq j \leq d + 1.
$$

(12)

Thus, (11) and (12) imply that

$$
i \cdot V_i(Z) \geq u_{i-1} \cdot \sum_{j=1}^{d+1} ||v_j||.
$$

(13)

Hence, (13) and the lemma imply the theorem in a straightforward way.

Subcase $i = d$. As the $d$-dimensional volume of any $d$-dimensional zonotope generated by $d + 1$ line segments and containing a $d$-dimensional unit ball in $E^d$ is at least as large as $\frac{1}{d}$ times its surface area the theorem follows from the subcase $i = d - 1$ in a trivial way.

Case (2): The $d$-dimensional zonotope in question is an affine cube.

As any $d$-dimensional affine cube of inradius at least one can be approximated by $d$-dimensional zonotopes generated by $d + 1$ line segments and containing a $d$-dimensional unit ball in $E^d$ the theorem follows from case (1).

This completes the proof of the theorem. \hfill $\square$

References


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