The Densest Packing of
12 Congruent Circles in a Circle

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Abstract. The densest packings of $n$ congruent circles in a circle are known for $n \leq 11$ and $n = 19$. In this paper we exhibit the densest packing of 12 congruent circles in a circle. In fact, we show that the optimal configuration is the same as the one Kravitz [10] conjectured. We use a technique developed from a method of Bateman and Erdős [1].

1. Preliminaries and results

We shall denote the points of the Euclidean plane $\mathbb{E}^2$ by capitals, sets of points by script capitals, and the distance of two points by $d(P, Q)$. We use $PQ$ for the line through $P, Q$, and $PQ$ for the segment with endpoints $P, Q$. $\angle POQ$ denotes the convex angle determined by the three points $P, O, Q$ in this order. $C(r)$ means the closed disc of radius $r$ with center $O$. By an annulus $r < \rho \leq s$ we mean all points $P$ such that $r < d(P, O) \leq s$. A sector $POQ$ is a closed convex subset of $\mathbb{E}^2$ which is bounded by the rays $OP$ and $OQ$. We also make use of the linear structure of $\mathbb{E}^2$ by identifying each point $P$ with the vector $\overrightarrow{OP}$, where $O$ is the origin. For a point $P$ and a vector $\vec{a}$ by $P + \vec{a}$ we always mean the vector $\overrightarrow{OP} + \vec{a}$.

The problem of finding the densest packing of congruent circles in a circle arose in the 1960s. The question was phrased the following way. What is the smallest circle in which we can pack $n$ congruent unit circles, or equivalently, what is the smallest circle in which we can place $n$ points with mutual distances at least 1? In 1967 dense circle packings were given by Kravitz [10] for $n = 2, \ldots, 16$. In 1969 Pirl [13] proved that in fact the arrangements given by Kravitz are optimal for $n \leq 9$ and he also found the optimal configuration for $n = 10$. Pirl also conjectured dense configurations for $11 \leq n \leq 19$. For
$n \leq 6$ proofs were given independently by Graham in response to a problem posed by Coxeter in the American Mathematical Monthly [3]. A proof for $n = 6$ and 7 was also given by Crilly and Suen [4]. Subsequent improvements were presented by Goldberg [7] for $n = 14, 16$ and 17. He also found a new packing with 20 circles. In 1975 Reis [14] used an ingenious mechanical contraption to generate remarkably good packings up to 25 circles. Recently, Graham et al. [8], [9] using modern computers established packings with more than 100 circles and improved the packing of 25 circles. In 1994 Melissen [11] proved Pirl's conjecture for $n = 11$ and the author [6] proved it for $n = 19$. We shall find the optimal configuration for $n = 12$. Our goal is to prove the following theorem.

**Theorem 1.** The smallest circle $C$ which can accommodate 12 points with mutual distances at least 1 has radius $R = 1.5148\ldots$. Moreover, the 12 points form the unique configuration shown in Figure 1.

![Figure 1. The optimal configuration for $n = 12$](image)

Note that two points are connected with a straight line segment on Figure 1 if and only if their distance is exactly 1. In Theorem 1, $R = \frac{1}{\sqrt{3x_0}}$, where $x_0$ is the smallest positive root of $9x^5 - 15x^4 + 7x^3 - 3x + 1 = 0$.

The proof of Theorem 1 consists of three parts. We shall divide $C(R)$ into a smaller circle $C(S)$ and an annulus $S < \rho \leq R$, choosing $S$ such that there can be at most 5 points in $C(S)$ and at most 9 points $S < \rho \leq R$. This leaves us three different cases. Namely when there are 3, 4, 5 points in $C(S)$. We deal with the latter two cases first showing that they cannot occur. Then we prove that the three points in $C(S)$ form a regular triangle of unit side length centered at $O$.

In the course of our proof we shall utilize the following two statements. Lemma 1, slightly modified here, originates from a paper of Bateman and Erdős [1] where they used it to prove an improved version of a theorem of Besicovitch. Lemma 2 is used by the author [6] to show Pirl's conjecture for $n = 19$.

**Lemma 1.** [1] Let $r, s, (s \geq \frac{1}{2})$ be two positive real numbers and suppose that we have two points $P$ and $Q$ which lie in the annulus $r \leq \rho \leq s$ and which have mutual distance at
least 1. Then the minimum $\phi(r, s)$ of the angle $\angle POQ$ has the following values:

$$
\phi(r, s) = \arccos \frac{s^2 + r^2 - 1}{2rs}, \quad \text{if} \quad 0 < s - 1 \leq r \leq s - 1/s;
$$

$$
\phi(r, s) = 2 \arcsin \frac{1}{2s}, \quad \text{if} \quad 0 < s - 1/s \leq r \leq s \text{ or } s \leq 1.
$$

**Lemma 2.** [6] Let $S$ be a set of $n \ (n \geq 2)$ points in the plane and $C$ the smallest circle containing $S$. Let $\vec{a}$ be a vector. There exist two points $P_1, P_2$ on the boundary of $S$, such that $d(P_1 + \vec{a}, O) + d(P_2 + \vec{a}, O) \geq 2r$, where $r$ is the radius and $O$ is the center of $C$.

The problem of finding the densest packing of equal circles in a circle is also stated in the book of Croft, Falconer and Guy [5] on pp. 110–111. Packings of congruent circles in hyperbolic plane were treated by K. Bezdek [2]. Analogous results of packing $n$ equal circles in an equilateral triangle and square can be traced down in the doctoral dissertation of Melissen [12].

2. Proofs

We prove Theorem 1 with a sequence of lemmas. Let $S = R - 1/R = 0.854 \ldots$

**Lemma 3.** There can be at most 9 points in the annulus $S < \rho \leq R$.

**Proof.** This follows from the fact that $\phi(S, R) = \phi(R, R) = 38^\circ, 5468^\circ \ldots$

Thus, $10\phi(R, R) > 360^\circ$. □

We know from [10] that the radius of the circumcircle of 6 points with mutual distances at least 1 is at least 1, which is larger than $S$. Therefore there cannot be fewer than 7 points in the annulus $S < \rho \leq R$, or there would be at least 6 points in $C(S)$. Thus, we have three cases to consider, namely, when there are 5, 4, 3 points in $C(S)$. We shall deal with the three cases one-by-one.

**Lemma 4.** There cannot be exactly 7 points in the annulus $S < \rho \leq R$.

**Proof.** If we have 7 points in the annulus $S < \rho \leq R$ there must 5 points in $C(S)$. Consider the following assertion. There cannot be any of these 5 points in $C(0.835)$. Because $2\phi(0.835, S) = 145^\circ, 343 \ldots$ and $\phi(S, S) = 71^\circ, 61 \ldots,$ we have that $3\phi(S, S) + 2\phi(0.835, S) = 360^\circ, 175 \ldots > 360^\circ$. Therefore we may assume that all 5 points are in the annulus $0.835 < \rho \leq S$.

Connecting the 5 points in $C(S)$ to $O$ we obtain 5 angular sectors. The minimal central angle of a sector is $\phi(S, S) = 71^\circ, 61 \ldots$. In one of these sectors, say $P_1OP_{t+1}$, there is a point $P$ of the 7 points from the annulus $S < \rho \leq R$. Then $\angle P_1OP, \angle P_{t+1}OP \geq \phi(0.835, R) = 76^\circ, 1173 \ldots$. Therefore $4\phi(S, S) + 2\phi(0.835, R) = 362^\circ, 561 \ldots > 360^\circ$. So the 5 points cannot all be in $C(S)$. □
Lemma 5. There cannot be exactly 8 points in the annulus $S < \rho \leq R$.

Proof. If we have 8 points in the annulus $S < \rho \leq R$, then there must be 4 points in $C(S)$. To accommodate 4 points in the plane with mutual distances at least 1 we need a circle of radius at least $r_4 = 1/\sqrt{2}$. Suppose that these 4 points are in the minimum radius circle $C'$ with center $O'$. The radius of $C'$ is at least $r_4$. We know from Lemma 2 that there are two of the 4 points, $P$ and $Q$ on the boundary of $C'$, such that $d(P, O) + d(Q, O) \geq 2r_4 = \sqrt{2}$. The 8 points in the annulus $S < \rho \leq R$ connected to $O$ divide $C$ into 8 angular sectors. We can assume that $P$ and $Q$ are in the interior of two different sectors, say $P$ is in $P_iO_{i+1}$ and $Q$ in $P_3OP_{j+1}$, and the angle $\angle P_iO_{i+1} \geq 2\phi(d(P, O), R)$, and $\angle P_jO_{j+1} \geq 2\phi(d(Q, O), R)$.

Suppose that $d(Q, O) \geq d(P, O)$, We may assert then that one of the points in $C(S)$ other than $Q$ is at least 0.66 from $O$. To see this we show that the three points in $C(S)$ other than $Q$ do not fit in $C(0.66)$. On the contrary let us suppose that they do. Connecting the three points to $O$ we obtain three angular sectors with angles at least $\phi(0.66, 0.66) = 98.5019 \ldots$ $\angle P_iO_{i+1}$ lies in one of these sectors, say in $P_iO_{i+1}$, Then $\angle P_iO_{i+1} \geq 2\phi(0.66, S) = 163^\circ.074 \ldots$, so $2\phi(0.66, 0.66) + 2\phi(0.66, S) = 360^\circ.079 \ldots > 360^\circ$. Therefore the three points cannot all be in $C(0.66)$.

Now the sum of the angles of the 8 sectors, determined by the points in the annulus $S < \rho \leq R$, is at least $6\phi(R, R) + 2\phi(d(P, O), R) + 2\phi(d(Q, O), R)$, where we assume that $d(P, O) + d(Q, O) = \sqrt{2}$ and, by the above argument, that $d(P, O) \geq 0.66$. It is easy to check that the minimum of the sum occurs when $d(P, O) = 0.66$ and it is $362^\circ.16 \ldots > 360^\circ$.

Therefore there cannot be exactly 8 points in the annulus $S < \rho \leq R$. \hfill $\Box$

In the subsequent paragraphs we shall deal with the remaining case when we have only 3 points, $P_1, P_2, P_3$, in $C(S)$. Let $d_i, i = 1, 2, 3$ denote the distance $d(P_i, O), i = 1, 2, 3$. In the following lemma we summarize several properties of these three points.

Lemma 6.

a) $d_i \geq \sqrt{3}/3$ for some $i$ and $0.43 \leq d_i \leq d = 0.6122$ for all $i = 1, 2, 3$.

b) $C(0.56)$ cannot contain two of $P_i, i = 1, 2, 3$.

c) A sector, determined by two points of the annulus $S < \rho \leq R$, contains at most one of $P_i, i = 1, 2, 3$.

d) A sector, determined by $P_i$ and $P_j$, contains exactly three points from the annulus $S < \rho \leq R$.

e) If $d_i \leq 0.5635$, then $P_i$ does not contribute to the minimal angle of the sector, determined by two points of the annulus $S < \rho \leq R$, in which it lies.

Proof. a) The radius of the circumcircle of three points with mutual distances 1 is $\sqrt{3}/3$, therefore at least one of $d_i$ is not less than $\sqrt{3}/3$. Suppose that $d_i \geq 0.6122$ for some $i$. Then adding up the angles of the nine sectors determined by the points in the annulus $S < \rho \leq R$ we obtain $8\phi(R, R) + 2\phi(d, R) = 308^\circ.37 \ldots + 51^\circ.66 = 360^\circ.83$.

As $d_i \leq d$ for all $i$, no side of the triangle $\triangle P_1P_2P_3$ can be longer than $2d = 1.2244 \ldots$, so all side lengths are in between 1 and 2d. It is easy to see that such a triangle is always strictly acute and all three heights are at least sin arccos(0.6122) = 0.79 \ldots. This shows that $O$ must be inside $\triangle P_1P_2P_3$ otherwise it would be farther than $d$ from one of the
vertices. Now, suppose that \( d_i \leq 0.43 \) for some \( i \). Then one of the other two \( d_j \) is at least \( \sqrt{(0.79 - 0.43)^2 + 1/4} = 0.616 \ldots > d \).

b) If \( d_i, d_j \leq 0.56 \), then adding up the angles of the three sectors determined by \( P_i \), \( I = 1, 2, 3 \) we obtain \( 2\phi(0.56, d) + \phi(0.56, 0.56) = 234^\circ.06 + 126^\circ.468 = 360^\circ.52 \).

c) Let us assume, on the contrary, that in one sector there are two of \( P_i \), \( i = 1, 2, 3 \). Then the central angle of this sector is at least \( \phi(d, d) = 109^\circ.516 \ldots \) Thus, if we sum up the angles of the 9 sectors determined by the points in the annulus \( S < \rho \leq R \) we get \( 8\phi(R, R) + \phi(d, d) = 417^\circ.8 \ldots > 360^\circ \).

d) Now, we will show that each of the 3 sectors determined \( P_i \), \( i = 1, 2, 3 \) contains exactly three of the points from the annulus \( S < \rho \leq R \). To see this suppose, on the contrary, that in the sector determined by \( P_1 \) and \( P_2 \) there are four points. We may suppose that \( d_1 \geq 0.56 \). If \( d_2 \geq 0.525 \), then the angle of the sector is at least \( 3\phi(R, R) + \phi(0.56, R) + \phi(0.525, R) = 115^\circ.64 \ldots + 18^\circ.57 \ldots + 9^\circ.16 \ldots = 143^\circ.37 \ldots \) Therefore the total angle of the three sectors is at least \( 2\phi(d, d) + 140^\circ.87 \ldots = 362^\circ.4 \ldots \) If \( d_2 < 0.525 \), then the total angle of the three sectors is \( 3\phi(R, R) + \phi(0.56, R) + \phi(d, d) + \phi(0.525, d) = 115^\circ.64 \ldots + 18^\circ.57 \ldots + 109^\circ.516 \ldots + 122^\circ.9 \ldots = 366^\circ.5 \ldots \) Therefore there must be exactly three points in each of the three sectors.

e) Evaluating \( \phi(0.5635, R) \) it turns out that it is less than \( \frac{1}{2} \phi(R, R) \).

Let us introduce the following function

\[
F(d_i, d_{i+1}, s_{i+2}) = \arccos \frac{d_i^2 + d_{i+1}^2 - s_{i+2}^2}{2d_id_{i+1}},
\]

which gives the central angle of the sector \( P_i P_{i+1} \) under the assumption that the length of the side \( P_i P_{i+1} \) is \( s_{i+2} \). If \( d_i = d_{i+1} = \sqrt{3}/3 \) and \( s_{i+2} = 1 \), then \( F(d_i, d_{i+1}, s_{i+2}) = 120^\circ \). Let us further note that the \( \phi(d_i, R) \) are monotonically increasing functions of the \( d_i \).

Suppose that two sides of \( \Delta P_1 P_2 P_3 \) are longer than 1. Then we move the vertex, say \( P_1 \), where these two sides meet toward \( O \), while keeping all others points fixed, until one of the sides becomes 1.

**Lemma 7.** This operation is admissible, that is, the mutual distances remain at least 1 and the 12 points fit in \( C(R) \).

*Proof.* To see this draw circles of radius 1/2 around the 12 points. During the movement of \( P_1 \) no circles can overlap. Clearly, as we move \( P_1 \) toward \( O \) we only have to make sure that there is no circle which blocks the one around \( P_1 \) before one of the sides shrinks to 1. Assume, on the contrary, that there is a circle around a point \( P \) which blocks the circle around \( P_1 \). Then \( d(P, O) \leq \sqrt{d^2 + 1} = 1.1725 \ldots \) Suppose that \( P \) is in the sector determined by \( P_1 \) and \( P_2 \). One of \( d_1 \) and \( d_2 \) is at least 0.56. We distinguish three cases. If \( d(P, O) \leq 1.1 \) and \( d_1 \geq 0.56 \), then total angle of the three sectors determined by the three points in \( C(S) \) is at least \( 2\phi(d, d) + \phi(0.56, 1.1) + 2\phi(R, R) = 219^\circ.03 \ldots + 64^\circ.849 \ldots + 77^\circ.08 \ldots = 360^\circ.96 \ldots \). If \( d_1 < 0.56 \), then \( d_2 \geq 0.56 \), thus the total angle of the three sectors is \( 2\phi(d, d) + \phi(0.56, R) + 2\phi(R, R) + \phi(0.43, 1.1) = 219^\circ.03 \ldots + 18^\circ.57 \ldots + 77^\circ.08 \ldots + 65^\circ.52 \ldots = 379^\circ.08 \ldots \) Finally, if \( d(P, O) > 1.1 \), then we will examine the 9 sectors determined by the points in the annulus \( S < \rho \leq R \). First we note the following.
If \( d_i \leq 0.5635 \), then \( P_i \) does not contribute to the angle of the sector in which it lies, that is, \( 2\phi(0.5635, R) < \phi(R, R) \). By Lemma 2 there are two points, \( P_i, P_{i+1} \) for which \( d_i + d_{i+1} \geq 2\sqrt{3}/3 \). The total angle of the two sectors they are in is \( 2\phi(d_1, R) + 2\phi(d_2, R) \). Supposing that \( d_i + d_{i+1} = 2\sqrt{3}/3 \) and \( 2\sqrt{3}/3 - 0.5635 \leq d_i, d_{i+1} \) the minimum of the total angle occurs when one of \( d_i \) and \( d_{i+1} \) is \( 2\sqrt{3}/3 - 0.5635 \), namely \( 46^\circ.61 \ldots \). Thus, the sum of the central angles of the 9 sectors is as follows, \( 6\phi(R, R) + 2\phi(1.1, R) + 46^\circ.61 \ldots = 231^\circ.28 \ldots + 82^\circ.54 \ldots + 46^\circ.61 \ldots = 360^\circ.043 \ldots \). Therefore there cannot be a circle which blocks \( P_1 \) as we move it toward \( O \), so we can shrink one side to 1. From now on we always assume that at least two sides of \( \triangle P_1 P_2 P_3 \) are 1.

\[ \square \]

Suppose we have a triangle \( \triangle P_1 P_2 P_3 \) with fixed sides \( s_i \). We will call a position of this triangle admissible if \( F(d_i, d_{i+1}, s_{i+2}) \geq 2\phi(R, R) + \phi(d_i, R) + \phi(d_{i+1}, R) \) for all \( i \). Notice that if \( d_i = \sqrt{3}/3 \) and all three sides are of unit length, then there is equality in each of the above inequalities. If any of these conditions fails, then we will call that position of the triangle not admissible.

First we will consider the case when \( s_i = 1 \) for all \( i \).

**Lemma 8.** If the three points in \( C(S) \) form a regular triangle \( \triangle P_1 P_2 P_3 \) of side length 1, then the only admissible position is when \( O \) is at the center of this triangle.

**Proof.** Let \( d_1 \) be the largest of the \( d_i \), then \( d_1 \geq \sqrt{3}/3 \). Consider the position in which \( d_2 = d_3 \) and fix \( d_1 \). Note that in this position \( d_2 \) is a function of \( d_1 \). Notice that the \( F(d_1, d_1, 1) \) are monotonically decreasing functions of the \( d_1 \). Thus, if we show that in this symmetric position the two sides \( P_1 P_2 \) and \( P_1 P_3 \) are not admissible, then in any other position, with the same fixed \( d_1 \), one of them will not be admissible.

It is enough to consider the following function of \( d_1, F(d_1, d_2, 1) - 2\phi(R, R) - \phi(d_1, R) - \phi(d_2, R) \) in the interval \([\sqrt{3}/3, d]\). Figure 2 shows its graph in \([\sqrt{3}/3, d]\). It can be seen that it is zero at \( \sqrt{3}/3 \) and negative elsewhere.

\[
R := 1.5148
\]

\[
\text{Plot}\left[\frac{180}{\pi}\left(\text{ArcCos}\left[\frac{-\frac{1}{4} \left[ \frac{\sqrt{3}}{2} - x \right]^2 + x^2}{2x}\right] - \frac{4\text{ArcSin}\left[\frac{\sqrt{3}}{2R}\right] + \text{ArcCos}\left[\frac{-\frac{1}{4} + R^2 + \left(\frac{\sqrt{3}}{2} - x\right)^2}{2R}\right] + \text{ArcCos}\left[\frac{-1 + R^2 + x^2}{2Rx}\right]}{2R}\right)\right], \{x, \text{Sqrt}[3]/3, .6122\}\]
\]
Thus, we can conclude that a regular triangle of unit side length can only be placed in $C(S)$ exactly one way, namely, when $O$ is at the center of the triangle.

Now, we return to the general case letting one side of $\triangle P_1P_2P_3$ be greater than 1.

**Lemma 9.** If one of the sides, say $P_1P_3$, is longer than 1, then the triangle $\triangle P_1P_2P_3$ cannot be admissible in any position.

**Proof.** Suppose that $d_1 \geq d_3$. Notice that $d_2 \leq d_1, d_3$. Otherwise $d_1 \geq \left(1 + d_2^2 - 2d_2\sqrt{1 - s_2^2/4}\right)^{1/2}$ which is longer than it would be if $\triangle P_1P_2P_3$ were a regular unit triangle. Therefore the side $P_1P_2$ would not be admissible unless $d_2 = \sqrt{3}/3$ by Lemma 8.

Suppose that $d_2 = d_3$. Then $d_2\sqrt{3}/3$ or one of $P_1P_2$ and $P_2P_3$ would not be admissible. Moreover, as $d_1 \leq d_i$ it follows that $d_2 > 0.56$. If we can show that in this symmetric position both sides $P_1P_2$ and $P_1P_3$ are not admissible, then by the same monotonicity argument as in Lemma 8, $\triangle P_1P_2P_3$ is not admissible in any other position.

For every $d_2$ in $[0.5635, \sqrt{3}/3]$ there is an $f(d_2)$ defined by $6\phi(R, R) + 4\phi(d_2, R) + 2\phi(f(d_2), R) = 360^\circ$, which means that $\sqrt{3}/3 \leq d_1 \leq f(d_2)$. For $d_2$ in $[0.56, 0.5635]$ define $f(d_2) = d$. Assume that $\triangle P_1'P_2P_3$ is a regular triangle of unit side length, then $\sqrt{3}/3 \leq d_1' < d_1$. However, from the previous lemma we already know that the side $P_1'P_2$ is not admissible unless $d_1' = d_2 = \frac{\sqrt{3}}{3}$. Therefore $P_1P_2$ is not admissible.

Now we have only to show that $P_1P_3$ is not admissible as well. Notice that for a fixed $d_2$ the $\angle P_1OP_2 + \angle P_1OP_3 = \text{const.}$ Therefore $\frac{\partial}{\partial d_1} F(d_1, d_2, s_2) = -\frac{\partial}{\partial d_1} F(d_1, d_2, 1)$. Using this fact simple calculus shows that

$$\frac{\partial}{\partial d_1} F(d_1, d_2, s_2) = \left(\frac{(2d_1R)^2 - (d_1^2 + R^2 - 1)^2}{(2d_1d_2)^2 - (d_1^2 + d_2^2 - 1)^2}\right)^{1/2} \frac{1 - d_2^2 + d_1^2}{R^2 - 1 - d_1^2}.$$

It can be easily seen that if $d_1$ is in $[\sqrt{3}/3, d]$ and $d_2$ in $[0.56, \sqrt{3}/3]$, then both the numerators and denominators are monotone functions of $d_1$ as well as $d_2$. Thus, plugging in concrete values for $d_1$ and $d_2$ we can see that the expression is larger than one. So $F(d_1, d_2, s_2)$
is increasing faster that \( \phi(d_1, R) \). We already know that \( F(d_1', d_2, 1) \leq 2\phi(R, R) + \phi(d_1', R) + \phi(d_2, R) \) with equality only if \( d_2 = \sqrt{3}/3 \). Thus, if \( F(f(d_2), d_2, s_2) = 2\phi(R, R) - \phi(f(d_2), R) - \phi(d_2, R) < 0 \), then the same relation holds everywhere in \( [d_1', f(d_2)] \). Figure 3 shows the graph of the above function for \( d_2 \in [0.5635, \sqrt{3}/3] \) drawn by Mathematica. It can be seen that it is zero at \( d_2 = \sqrt{3}/3 \) and negative elsewhere.

\[
R := 1.5148
\]

\[
F[x_] := -R\cos[2\text{ArcCos}[\frac{-1+R^2+x^2}{2Rx}]] + 6\text{ArcSin}[\frac{1}{2R}] - \sqrt{\left(1 - R^2 + R^2\cos[2\text{ArcCos}[\frac{-1+R^2+x^2}{2Rx}]] + 6\text{ArcSin}[\frac{1}{2R}]\right)^2}
\]

Plot[180/Pi(\text{ArcCos}[(x^2 + F[x]^2 - (2 - 2\cos[\text{ArcCos}[1/(2x)] + \text{ArcCos}((x^2 - F[x]^2 + 1)/(2x)])))/(2xF[x]]) - (4\text{ArcSin}[1/(2R)] + \text{ArcCos}((x^2 + R^2 - 1)/(2xR)) + \text{ArcCos}((F[x]^2 + R^2 - 1)/(2F[x]R)))),
{x, .5635, Sqrt[3]/3}]

If \( d_2 \in [0.56, 0.5635] \) it is not hard to check that \( F(d, d_2, s_2) \) takes on its maximum if \( d_2 = 0.5635 \) and \( \phi(d_2, R) \) attains its minimum if \( d_2 = 0.56 \). Thus, by plugging in the actual numbers it turns out that the same relation holds as in \( [0.5635, \sqrt{3}/3] \).

Therefore we can conclude that the triangle \( \triangle P_1P_2P_3 \) is not admissible in any position if any of the sides is longer than 1. Thus, we have finished the proof of Theorem 1. \( \square \)

References


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