Natural Projectors in Tensor Spaces*

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Abstract. The aim of this paper is to introduce a method of invariant decompositions of the tensor space $T^r_s R^n = R^n \otimes R^n \otimes \cdots \otimes R^n \otimes R^{n*} \otimes R^{n*} \otimes \cdots \otimes R^{n*}$ ($r$ factors $R^n$, $s$ factors the dual vector space $R^{n*}$), endowed with the tensor representation of the general linear group $GL_n(\mathbb{R})$. The method is elementary, and is based on the concept of a natural ($GL_n(\mathbb{R})$-equivariant) projector in $T^r_s R^n$. The case $r = 0$ corresponds with the Young-Kronecker decompositions of $T^0_s R^n$ into its primitive components. If $r, s \neq 0$, a new, unified invariant decomposition theory is obtained, including as a special case the decomposition theory of tensor spaces by the trace operation.

As an example we find the complete list of natural projectors in $T^1_2 R^n$. We show that there exist families of natural projectors, depending on real parameters, defining new representations of the group $GL_n(\mathbb{R})$ in certain vector subspaces of $T^1_2 R^n$.

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1. Introduction

In this paper we give basic definitions and prove basic results of natural projector theory in tensor spaces over the field or real numbers $\mathbb{R}$. The tensor space of type $(r, s)$ over the

vector space $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ ($n$ factors $\mathbb{R}$) is denoted by $T_x^s \mathbb{R}^n = \mathbb{R}^n \otimes \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n \otimes T_x^s \mathbb{R}^{n*} \otimes T_x^s \mathbb{R}^{n*} \otimes \cdots \otimes T_x^s \mathbb{R}^{n*}$ ($r$ factors $\mathbb{R}^n$, $s$ factors the dual vector space $\mathbb{R}^{n*}$). We always suppose $n \geq 2$. $\mathbb{R}^n$ is considered with the canonical left action of the general linear group $GL_n(\mathbb{R})$, and the tensor space $T_x^s \mathbb{R}^n$ is endowed with the induced (tensor) action. Since our discussions are $GL_n(\mathbb{R})$-invariant, the results apply in the well-known sense to any real, $n$-dimensional vector space $E$, and to the tensor space $T_x^s E$ of type $(r, s)$ over $E$.

We wish to describe a method allowing us to find all $GL_n(\mathbb{R})$-invariant vector subspaces of the vector space $T_x^s \mathbb{R}^n$; indeed, this is equivalent to finding all $GL_n(\mathbb{R})$-equivariant projectors $P : T_x^s \mathbb{R}^n \to T_x^s \mathbb{R}^n$. In accordance with the terminology of the differential invariant theory, $GL_n(\mathbb{R})$-equivariant projectors are also called natural.

This method complements our previous results on decompositions of tensor spaces, which are not based on the group representation theory (see [4, 5]). It can be applied effectively for any concrete $r$ and $s$. However, a general formula for the decomposition has not been found.

It seems that the idea to apply the theory of projectors to the problem of decomposing a tensor space of type $(r, 0)$, or $(0, s)$ into its primitive components belongs to H. Weyl [7]. However, this idea has never been developed to a complete theory, or used to an analysis of concrete cases. Later, the same author gives preference to the group representation theory over the ideas of the pure projector theory [6]; a standard restrictive assumption in this approach is usually applied from the very beginning, namely the assumption that the representation space is a vector space over an algebraically closed field.

For basic ideas and generalities on natural projectors in tensor spaces we refer to Krupka (see [3], Sections 4.4 and 7.3).

Let us now recall briefly main concepts. A tensor $t \in T_x^s \mathbb{R}^n$ is said to be invariant, if $g \cdot t = t$ for all $g \in GL_n(\mathbb{R})$. A theorem of Gurevich says that an invariant tensor of type $(r, s)$, where $r \neq s$, is always the zero tensor, and, if $r = s$, an invariant tensor is always a linear combination $\sum c_\sigma \delta_{i_1}^{1(1)} \delta_{i_2}^{2(2)} \cdots \delta_{i_n}^{N(N)}$ of products of $r$ factors of the Kronecker $\delta$-tensor, where $c_\sigma \in \mathbb{R}$, and $\sigma$ runs through all permutations of the set $\{1, 2, \ldots, r\}$ (see [1]). Consider a real, $N$-dimensional vector space $E$ endowed with a left action of $GL_n(\mathbb{R})$. A linear mapping $F : E \to E$ is called $GL_n(\mathbb{R})$-equivariant, or natural, if $F(g \cdot x) = g \cdot F(x)$ for all $x \in E$ and all $g \in GL_n(\mathbb{R})$. It is a simple observation that $F$ is natural if and only if its components form an invariant tensor [3]. A natural linear mapping $P : E \to E$ which is a projector, i.e., satisfies the projector equation $P^2 = P$, is called a natural projector.

In Section 2 we collect standard definitions and facts of the theory of projectors in a vector space (see e.g. [2]). Section 3 is devoted to natural linear operators in a vector space endowed with a left action of $GL_n(\mathbb{R})$. In Section 4 we introduce natural projectors in tensor spaces and related concepts such as natural projector equations, decomposability, reducibility, and primitivity. Section 5 is concerned with the trace decomposition theory; it is shown that the trace decomposition of a tensor is related to a natural projector determined uniquely by certain conditions. Finally, in Section 6 we describe all natural projectors in the tensor space $T_x^s \mathbb{R}^n$.

It should be pointed out that the method of natural projectors allows us to treat in a unique way the case of tensors of type $(r, s)$, where not necessarily $r = 0$, or $s = 0$. In this sense the natural projector theory represents a generalization of the classical Young–Kronecker decomposition theory (see e.g. [6]), as well as of the trace decomposition theory [4, 5].
2. Projectors

This introductory section contains a brief formulation of standard results of the projector theory in a finite-dimensional, real vector space $E$ (see e.g. [2]).

Let $E^*$ be the dual of $E$, and let $E \times E^* \ni (x, y) \mapsto y(x) = \langle x, y \rangle \in \mathbb{R}$ be the natural pairing. The dual $A^* : E^* \to E^*$ of a linear mapping $A : E \to E$ is defined by the condition $\langle Ax, y \rangle = \langle x, A^*y \rangle$ for all $x \in E$, $y \in E^*$. If $A, B : E \to E$ are two linear mappings, then $(AB)^* = B^*A^*$.

A linear operator $P : E \to E$ is said to be a projector, if $P^2 = P$. Clearly, the zero mapping $0$, and the identity mapping $id_E$, are projectors.

**Lemma 1.** Let $E$ be a finite-dimensional, real vector space.

(a) A projector $P : E \to E$ defines the direct sum decomposition $E = \ker P \oplus \text{im } P$.

(b) A linear mapping $P : E \to E$ is a projector if and only if $\text{id}_E - P$ is a projector.

(c) If $P : E \to E$ is a projector, then $Q = \alpha P$, where $\alpha \in \mathbb{R}$, is a projector if and only if $\alpha = 0, 1$.

(d) Let $P, Q : E \to E$ be two projectors such that $\text{im } P = \text{im } Q = F$. Then there exists a unique linear isomorphism $U : F \to F$ such that $P = U \circ Q$.

Let $u^* : E^* \to E^*$ denote the dual of a linear mapping $u : E \to E$. We say that two projectors $P, Q : E \to E$ are orthogonal, if $\langle Px, Q^*y \rangle = 0$ and $\langle Qx, P^*y \rangle = 0$ for all $x \in E$, $y \in E^*$. Obviously, $P$ and $Q$ are orthogonal if and only if $QP = 0$ and $PQ = 0$. For every projector $P$, the projectors $P$ and $\text{id}_E - P$ are orthogonal.

**Lemma 2.** Let $P, Q : E \to E$ be projectors.

(a) $P + Q$ is a projector if and only if $P$ and $Q$ are orthogonal.

(b) $P - Q$ is a projector if and only if $PQ = QP = 0$.

(c) If $P$ and $Q$ commute, $PQ - QP = 0$, then $R = PQ = QP$ is a projector, and $\text{im } R = \text{im } P \cap \text{im } Q$.

(d) $\ker P = \ker (\text{id} - P)$.

**Remark 1.** If $P + Q$ is a projector, then condition (a) implies $PQ = QP = 0$ hence by (c), $\text{im } P \cap \text{im } Q = \{0\}$. Thus $\text{im } (P + Q) = \text{im } P + \text{im } Q$ is the direct sum of its subspaces $\text{im } P$ and $\text{im } Q$.

**Remark 2.** If $P - Q$ is a projector, condition (b) together with (c) imply that $\text{im } Q \subset \text{im } P$.

3. Natural linear operators in tensor spaces

Let $E$ be a finite-dimensional, real vector space, endowed with a left action of the general linear group $GL_n(\mathbb{R})$, denoted multiplicatively. A linear operator $F : E \to E$ is said to be $GL_n(\mathbb{R})$-equivariant, or natural, if $F(A \cdot x) = A \cdot F(x)$ for every $x \in E$ and every $A \in GL_n(\mathbb{R})$. The vector space of natural linear operators on $E$ is denoted $\mathcal{N}E$.

The kernel and the image of a natural linear operator $F : E \to E$ are $GL_n(\mathbb{R})$-invariant vector subspaces of $E$. 
Our aim in this section is to study natural linear operators in the tensor space $T^s_q \mathbb{R}^n$. If the canonical basis of $\mathbb{R}^n$ is denoted by $e_i$, and $e^i$ is the dual basis of $\mathbb{R}^{n*}$, then any tensor $t \in T^s_q \mathbb{R}^n$ is uniquely expressible in the form

$$t = \delta_{j_1 j_2 \ldots j_r}^{i_1 i_2 \ldots i_r} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes e^{j_2} \otimes \cdots \otimes e^{j_r}, \quad (3.1)$$

where the real numbers $t = \delta_{j_1 j_2 \ldots j_r}^{i_1 i_2 \ldots i_r}$ are the components of $t$. We usually write $t = \delta_{j}^{i_1 i_2 \ldots i_r}$.

Let $(A, x) \to \bar{x} = A \cdot x$ be the canonical left action of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$; in the canonical basis of $\mathbb{R}^n$, $\bar{x}^i = A^i_j x^j$, where $A = A^i_j$. If $B = A^{-1}$, $B = B^j_i$, the tensor action of $GL_n(\mathbb{R})$ on $T^s_q \mathbb{R}^n$ is given by

$$\bar{t} = A \cdot t = \delta_{j_1 j_2 \ldots j_r}^{i_1 i_2 \ldots i_r} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes e^{j_2} \otimes \cdots \otimes e^{j_r}, \quad (3.2)$$

where

$$\delta_{j_1 j_2 \ldots j_r}^{i_1 i_2 \ldots i_r} = A^{i_1}_{k_1} A^{i_2}_{k_2} \cdots A^{i_r}_{k_r} B^{j_1}_{i_1} B^{j_2}_{i_2} \cdots B^{j_r}_{i_r} k_1 k_2 \ldots k_r. \quad (3.3)$$

A tensor $t \in T^s_q \mathbb{R}^n$ is said to be invariant, if $A \cdot t = t$ for all $A \in GL_n(\mathbb{R})$. The following theorem describes all invariant tensors (see [1], and [3]).

Let $S_r$ denote the group of permutations $\sigma$ of the set $\{1, 2, \ldots, r\}$.

**Lemma 3.** Let $t \in T^s_q \mathbb{R}^n$.

(a) Assume that $r \neq s$. Then $t$ is invariant if and only if $t = 0$.

(b) Assume that $r = s$. Then $t$ is invariant if and only if

$$t_{j_1 j_2 \ldots j_r}^{i_1 i_2 \ldots i_r} = \sum_{\sigma \in S_r} a^\sigma \delta_{j_1}^{i_{\sigma(1)}} \delta_{j_2}^{i_{\sigma(2)}} \cdots \delta_{j_r}^{i_{\sigma(r)}} \quad (3.4)$$

for some $a^\sigma \in \mathbb{R}$.

Invariant tensors in $T^s_q \mathbb{R}^n$ form a real vector space. This vector space is spanned by the invariant tensors

$$E_{\sigma} = \delta_{j_1}^{i_{\sigma(1)}} \delta_{j_2}^{i_{\sigma(2)}} \cdots \delta_{j_r}^{i_{\sigma(r)}} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes e^{j_2} \otimes \cdots \otimes e^{j_r}$$

$$= e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r} \otimes e^{j_1} \otimes e^{j_2} \otimes \cdots \otimes e^{j_r}. \quad (3.5)$$

Note that any invariant tensor can be expressed, instead of (3.4), by

$$t = \sum_{\sigma \in S_r} a^\sigma E_{\sigma}. \quad (3.6)$$

Now we apply Lemma 3 to natural linear mappings $F : T^s_q \mathbb{R}^n \to T^q_s \mathbb{R}^n$. We have the following simple observation ([3], Section 4.4).

**Lemma 4.** Let $F : T^s_q \mathbb{R}^n \to T^q_s \mathbb{R}^n$ be a linear mapping,

$$\bar{t}_{j_1 j_2 \ldots j_r}^{i_1 i_2 \ldots i_r} = F_{j_1 j_2 \ldots j_r}^{i_1 i_2 \ldots i_r} k_1 k_2 \ldots k_q \ell_1 \ell_2 \ldots \ell_p k_1 k_2 \ldots k_q \quad (3.7)$$

its expression relative to the canonical basis of $\mathbb{R}^n$. $F$ is natural if and only if its components $F_{j_1 j_2 \ldots j_r}^{i_1 i_2 \ldots i_r} k_1 k_2 \ldots k_q \ell_1 \ell_2 \ldots \ell_p$ are components of an invariant tensor.
If $F$ is identified with a tensor, $F$ becomes an element of the tensor space $T^{r+s}_{r+s} \mathbb{R}^n$. Thus by Lemma 3, a nontrivial natural linear mapping $F : T^r_s \mathbb{R}^n \to T^q_q \mathbb{R}^n$ exists if and only if $r + q = s + p$.

Let us discuss the case $p = r$, $q = s$. Then by Lemma 3 (b), $F$ has an expression

$$F_{1i_1j_12\ldots j_s, \bar{s}+1k_1k_2\ldots k_q} = \sum_{\sigma \in S_{r+s}} a_{\sigma} \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \ldots \delta^{i_{r+s}}_{j_{\sigma(r+s)}}$$  \hspace{1cm} (3.8)

where $a_{\sigma} \in \mathbb{R}$. Clearly, the same is expressed by the equation

$$\tilde{t}^{i_1i_2\ldots i_r}_{(j_1j_2\ldots j_s, \emptyset)} = \sum_{\sigma \in S_{r+s}} a_{\sigma} t^{\mu(1)\nu(2)\ldots \nu(r)}_{k_{\sigma(1)}k_{\sigma(2)}\ldots k_{\sigma(s)}} + \tau^{i_1i_2\ldots i_r}_{j_1j_2\ldots j_s},$$  \hspace{1cm} (3.9)

where the summation takes place through $\sigma \in S_{r+s}$ of the form of the product of two permutations $\sigma = \mu \nu$, $\nu \in S_r$, $\mu \in S_s$ and $\tau^{i_1i_2\ldots i_r}_{j_1j_2\ldots j_s}$ contains all the remaining terms. Note that each term in $\tilde{t}^{i_1i_2\ldots i_r}_{j_1j_2\ldots j_s}$ contains at least as one factor the Kronecker $\delta$-tensor multiplied by an expression obtained from $t^{i_1i_2\ldots i_r}_{j_1j_2\ldots j_s}$ by the trace operation in one superscript and one subscript.

Since $F_{1i_1j_12\ldots j_s, \bar{s}+1k_1k_2\ldots k_q}$ are components of an invariant tensor, $F$ can also be expressed by means of (3.6) as

$$F = \sum_{\sigma \in S_{r+s}} a_{\sigma} E_{\sigma}.$$  \hspace{1cm} (3.10)

If $F, G : T^r_s \mathbb{R}^n \to T^q_q \mathbb{R}^n$ are two natural linear operators, given in components by

$$\tilde{t}^{i_1i_2\ldots i_r}_{j_1j_2\ldots j_s} = F_{1i_1j_12\ldots j_s, \bar{s}+1k_1k_2\ldots k_q} t^{k_1k_2\ldots k_q}_{1i_1j_12\ldots j_s}, \quad \tilde{t}^{i_1i_2\ldots i_r}_{j_1j_2\ldots j_s} = G^{i_1i_2\ldots i_r}_{j_1j_2\ldots j_s, 1i_1j_12\ldots j_s} t^{k_1k_2\ldots k_q}_{1i_1j_12\ldots j_s}$$

then the composed natural linear operator is given by

$$t^{i_1i_2\ldots i_r}_{j_1j_2\ldots j_s} = C^{i_1i_2\ldots i_r}_{j_1j_2\ldots j_s, 1i_1j_12\ldots j_s} F^{1i_1j_12\ldots j_s, \bar{s}+1k_1k_2\ldots k_q} F^{k_1k_2\ldots k_q}_{1i_1j_12\ldots j_s} t^{k_1k_2\ldots k_q}_{1i_1j_12\ldots j_s}.$$  \hspace{1cm} (3.11)

To obtain an explicit formula, one should substitute from (3.8) into (3.12); indeed, this cannot be done effectively in general, but in every concrete case.

4. Natural projectors in tensor spaces

Let $E$ be a finite-dimensional, real vector space, endowed with a left action of the general linear group $GL_n(\mathbb{R})$. By a natural projector on $E$ we mean a natural linear operator $F : E \to E$ which is a projector. A natural linear operator $F$ is a natural projector if and only if it satisfies the projector equation $F^2 = F$. The projector equation represents a system of quadratic equations for the components of $F$.

If $P : E \to E$ is a natural projector, then both vector subspaces $\text{im} P$, $\ker P$ of $E$ are $GL_n(\mathbb{R})$-invariant ([2], § 43).

A natural projector $P : E \to E$ is said to be decomposable, if there exist a natural projector $Q \neq 0$, $P$ and a natural projector $R$, such that $P = Q + R$. In this case $Q$ and $R$ are orthogonal (Lemma 2 (a)). A natural projector which is not decomposable is called indecomposable.

$P$ is said to be reducible, if there exists a natural projector $Q \neq 0$ such that $\text{im} Q \subset \text{im} P$ and $\text{im} Q \not= \text{im} P$. If $P$ is not reducible, it is called irreducible, or primitive.
Remark 3. Examples show that there exist reducible natural projectors which are not decomposable. Consider the family \( P_\lambda \) of natural linear operators in \( T^*_qR^n \) defined by the equations

\[
\tilde{t}_{jk}^p = \delta^p_{kj} - t_{pq}^p + \lambda \delta^p_{kj}(-nt_{pq}^p + t_{jp}^p).
\]

One can easily verify that (4.1) consists of natural projectors. Indeed, contracting (4.1) we obtain \( t_{pq}^p = t_{pq}^p + \lambda(-nt_{pq}^p + t_{jp}^p), \) \( t_{jp}^p = nt_{pq}^p + \lambda(-nt_{pq}^p + t_{jp}^p), \) and then

\[
\begin{align*}
t_{jk}^p & = \delta^p_{kj} + \lambda \delta^p_{kj}(-nt_{pq}^p + t_{jp}^p) \\
& = \delta^p_{kj} + \lambda(-nt_{pq}^p + t_{jp}^p)) - \lambda n \delta^p_{kj}(-nt_{pq}^p + t_{jp}^p)) \\
& + \lambda \delta^p_{kj}(-nt_{pq}^p + \lambda(-nt_{pq}^p + t_{jp}^p)) + \lambda \delta^p_{kj}(-nt_{pq}^p + t_{jp}^p)) \\
& = \delta^p_{kj} + \lambda \delta^p_{kj}(-nt_{pq}^p + t_{jp}^p)) - \lambda n \delta^p_{kj}(-nt_{pq}^p + t_{jp}^p)) + \lambda \delta^p_{kj}(-nt_{pq}^p + t_{jp}^p)) = \tilde{t}_{jk}^p
\end{align*}
\]

verifying the projector equations \( P_\lambda^2 = P_\lambda. \) Note that the family (4.1) includes the natural projector \( \tilde{t}_{jk}^p = \delta^p_{kj}t_{pq}^p \), and the natural projector \( \tilde{t}_{jk}^p = (1/n)\delta^p_{kj}t_{pq}^p \) defined by taking \( \lambda = 1/n. \) The family \( \lambda \delta^p_{kj}(-nt_{pq}^p + t_{jp}^p) \) in (4.1) does not consist of projectors, because \( \lambda \) serves as a multiplicative parameter, and two non-zero projectors cannot differ by a factor different from 1. Indeed, writing \( t_{pq}^p = \lambda \delta^p_{kj}(-nt_{pq}^p + t_{jp}^p), \) we get \( t_{pq}^p = \lambda(-nt_{pq}^p + t_{jp}^p), \) \( t_{jp}^p = \lambda n(-nt_{pq}^p + t_{jp}^p) \) hence \( t_{jk}^p = \lambda \delta^p_{kj}(-nt_{pq}^p + t_{jp}^p) = -\lambda n \delta^p_{kj}(-nt_{pq}^p + t_{jp}^p) + \lambda^2 \delta^p_{kj}(-nt_{pq}^p + t_{jp}^p) = 0 \neq \tilde{t}_{jk}^p. \)

From now on we consider natural projectors on a tensor space \( T^*_qR^n. \)

**Theorem 1.** Let \( P : T^*_qR^n \to T^*_qR^n \) be a natural projector.

(a) \( P \) is decomposable if and only if there exists a natural projector \( Q \neq 0, P \) such that

\[
PQ = Q, \quad QP = Q.
\]

(b) \( P \) is reducible if and only if there exists a natural projector \( Q \neq 0, P \) such that

\[
PQ = Q, \quad \text{im} Q \neq \text{im} P.
\]

**Proof.** (a) If \( P \) is decomposable, we have two natural projectors \( Q \) and \( R \) such that \( R = P - Q \) and \( QR = 0, RQ = 0 \) (Lemma 2 (a)). Thus, \( Q(P - Q) = (P - Q)Q = 0, \) i.e., \( QP = PQ = Q. \) Conversely, assume that we have a natural projector \( Q \) satisfying (4.2). Define \( R = P - Q; \) \( R \) is a natural linear operator (Lemma 3, Lemma 4), and \( R^2 = P - PQ - QP + Q = P - Q - Q = P - Q = R \) as required.

(b) Let \( P \) be reducible. Then there exists a natural projector \( Q \neq 0 \) such that \( \text{im} Q \subset \text{im} P \) and \( \text{im} Q \neq \text{im} P. \) Thus, to any \( t \in T^*_qR^n, \) there exists \( t' \in T^*_qR^n \) such that \( Qt = Pt' = PQt \) hence \( PQ = Q. \) Conversely, assume that we have a natural projector \( Q \neq 0 \) satisfying (4.3). Then \( \text{im} Q = Q(T^*_qR^n) = P(Q(T^*_qR^n)) \subset P(T^*_qR^n) = \text{im} P. \) as required.

Equations from Theorem 1 (a) for a projector \( Q \)

\[
PQ = Q, \quad QP = Q, \quad Q^2 = Q
\]

are equivalent with the equations

\[
PQP = Q, \quad Q^2 = Q.
\]
Indeed, (4.4) implies (4.5), and vice versa: \[ QP = PQPP = PQP = Q, PQ = PPQP = PQP = Q. \] Each of the systems (4.4) and (4.5) is called the decomposability equation of \( P \). Equation \( PQ = Q \) from Theorem 1 (b) is called the reducibility equation.

Now we study indecomposability, and primitivity.

**Theorem 2.** Let \( P : T^*_s \mathbb{R}^n \to T^*_s \mathbb{R}^n \) be a natural projector.

(a) \( P \) is indecomposable if and only if the decomposability equation of \( P \) has exactly one nontrivial solution, \( Q = P \).

(b) \( P \) is primitive if and only if the reducibility equation of \( P \) has no nontrivial solution.

**Proof.** Both assertions are immediate consequences of Theorem 1.

(a) If \( P \) is indecomposable, there is no \( Q \neq 0, P \) such that \( PQ = Q, QP = Q \), which means that the decomposability equations have only one nontrivial solution, \( Q = P \). The converse is obvious.

(b) If \( P \) is primitive, then by definition, (4.3) has only the trivial solution, and vice versa.

Now we consider properties of primitive natural projectors.

**Theorem 3.**

(a) Any two different primitive natural projectors in \( T^*_s \mathbb{R}^n \) are orthogonal.

(b) The number of different nontrivial natural projectors in \( T^*_s \mathbb{R}^n \) is finite.

(c) The sum of any two primitive natural projectors is a natural projector.

(d) Let \( M \) be the number of different nontrivial primitive natural projectors in \( T^*_s \mathbb{R}^n \). If a natural projector in \( T^*_s \mathbb{R}^n \) admits a decomposition \( P = p_1 + p_2 + \cdots + p_K \), where \( p_1, p_2, \ldots, p_K \) are primitive natural projectors, then \( K \leq M \), the primitive natural projectors \( p_1, p_2, \ldots, p_K \) are mutually different, and this decomposition is unique.

(e) The identity natural projector \( \text{id} : T^*_s \mathbb{R}^n \to T^*_s \mathbb{R}^n \) admits the decomposition

\[
\text{id} = p_1 + p_2 + \cdots + p_M
\]

where \( \{p_1, p_2, \ldots, p_M\} \) is the set of nonzero primitive natural projectors.

**Proof.** (a) If \( P_1, P_2 \) are two different primitive natural projectors, then \( \text{im } P_1 P_2 = \text{im } P_2 P_1 = 0 \) hence \( P_1 P_2 = P_2 P_1 = 0 \).

(b) Since \( \dim T^*_s \mathbb{R}^n \) is finite, this assertion follows from (a).

(c) By (a), any two different primitive natural projectors \( p_1, p_2 \) are orthogonal. Thus, by Lemma 2 (a), \( p_1 + p_2 \) is always a projector; \( p_1 + p_2 \) is obviously a natural projector (Lemma 4).

(d) Assume that \( P = p_1 + p_2 + \cdots + p_K = q_1 + q_2 + \cdots + q_L \). Then by orthogonality, \( p_i^2 = p_i = p_i(q_1 + q_2 + \cdots + q_L) \), where at most one term on the right is nonzero. But \( p_i \neq 0 \) hence exactly one term on the right, say \( p_iq_k \), is nonzero, and is equal to \( p_i \), i.e., \( p_i = p_iq_k = q_kp_i \). Since different primitive projectors are orthogonal (see (a)), we have \( q_k = p_i \). In particular, the two sums \( p_1 + p_2 + \cdots + p_K, q_1 + q_2 + \cdots + q_L \) may differ only by the order of the summation.

(e) If \( P = p_1 + p_2 + \cdots + p_M \neq \text{id} \), we have a nonzero natural projector \( Q = \text{id} - P \), which is a contradiction with maximality of the set \( \{p_1, p_2, \ldots, p_M\} \).
5. The trace decomposition

For basic notions of the trace decomposition theory as used in this section, we refer to [4], [5]. The following assertion can be used when calculating the trace decomposition of concrete tensor spaces.

**Theorem 4.** Let \( r, s \geq 1 \). There exists a unique natural linear operator \( Q : T^*_s \mathbb{R}^n \rightarrow T^*_s \mathbb{R}^n \) satisfying the following two conditions:

1. \( Qt \) is traceless for every \( t \in T^*_s \mathbb{R}^n \).
2. \( (\text{id} - Q)t = t - Qt \) is \( \delta \)-generated for every \( t \in T^*_s \mathbb{R}^n \).

\( Q \) is a natural projector.

**Proof.** Existence and uniqueness of \( Q \) follows from the decomposition \( t = Qt + (\text{id} - Q)t \), and from the trace decomposition theorem. We prove that \( Q \) is a projector. By hypothesis, \( Qt \) is traceless for every \( t \in T^*_s \mathbb{R}^n \), hence \( Q^2t = Q(Qt) \) is also traceless for every \( t \). Similarly, since \( t - Qt \) is \( \delta \)-generated for every \( t \in T^*_s \mathbb{R}^n \), the formula

\[
(\text{id} - Q^2)t = (\text{id} - Q + Q - Q^2)t = (\text{id} - Q)t + (\text{id} - Q)Qt
\]

shows that \( (\text{id} - Q^2)t \) must also be \( \delta \)-generated. Since \( t = Q^2t + (\text{id} - Q^2)t \), then by uniqueness, \( Q^2 = Q \).

In a concrete case, the natural projector \( Q \) can be determined from the conditions (1) and (2) of Theorem 4. Clearly, given \( Q \), the trace decomposition of a tensor \( t \in T^*_s \mathbb{R}^n \) is obtained by the formula

\[
t = Qt + (\text{id} - Q)t.
\]

6. Natural projectors in \( \mathbb{R}^n \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \)

As an application of the natural projector theory, we find the complete list of natural projectors in the space of tensors of type (1,2) \( T^1_2 \mathbb{R}^n \). Since our discussions are \( GL_n(\mathbb{R}) \)-invariant, the results apply in the well-known sense to any real, finite-dimensional vector space \( E \), and to the tensor space of type (1,2) over \( E \).

First let us describe natural linear operators in \( T^1_2 \mathbb{R}^n \). Using the canonical basis \( e_i \) of \( \mathbb{R}^n \) and the dual basis \( e^i \) of \( \mathbb{R}^{n*} \), we usually express a tensor \( t \in T^1_2 \mathbb{R}^n \) in terms of its components as \( t = t^i_{jk} e_i \otimes e^j \otimes e^k \), and we write \( t = t^i_{jk} \). If \( P : T^1_2 \mathbb{R}^n \rightarrow T^1_2 \mathbb{R}^n \) is a linear operator, we write \( P = P^i_{jk \ pq} \), where \( P^i_{jk \ pq} \) are the components of \( P \), and the indices \( i, j, k, p, q, r \) run through the set \( \{1, 2, \ldots, n\} \). The equations of \( P \) are usually written in the form \( t^i_{jk} = P^i_{jk \ pq} t^p_{qr} \). \( P \) is natural if and only if

\[
P^i_{jk \ pq} = a \delta^i_j \delta^q_p + b \delta^i_j \delta^q_p \delta^r_k + c \delta^i_k \delta^q_p \delta^r_j + d \delta^i_k \delta^q_p \delta^r_j + e \delta^i_k \delta^q_p \delta^r_j + f \delta^i_p \delta^q_k \delta^r_j,
\]

where \( a, b, c, d, e, f \) are some real numbers. In view of (6.1), we also write

\[
P = (a, b, c, d, e, f).
\]

We denote by \( N(T^1_2 \mathbb{R}^n) \) the real vector space of natural linear operators \( P : T^1_2 \mathbb{R}^n \rightarrow T^1_2 \mathbb{R}^n \); by (6.1), \( \dim N(T^1_2 \mathbb{R}^n) = 6 \).
Proof. Since for any \( \mathbf{Q} \) where
\[
Q_{bc}^{p} = a\delta_{b}^{p}q_{c}^{r} + b\delta_{c}^{p}q_{b}^{r} + c\delta_{c}^{r}q_{b}^{p} + d\delta_{b}^{r}q_{c}^{p} + e\delta_{c}^{r}q_{b}^{p} + f\delta_{b}^{p}q_{c}^{r}.
\]

(6.3)

**Lemma 6.** The composed natural linear operator \( R = PQ = R_{jk}^{i} q_{r} \) is expressed by
\[
R_{jk}^{i} q_{r} = a''\delta_{j}^{i}q_{k}^{r} + b''\delta_{k}^{i}q_{j}^{r} + c''\delta_{k}^{r}q_{j}^{i} + d''\delta_{k}^{r}q_{j}^{i} + e''\delta_{j}^{r}q_{k}^{i} + f''\delta_{j}^{i}q_{k}^{r},
\]
where
\[
a'' = a'a + nd'a + e'a + na'b + f'b + d'b + a'e + d'f, \\
b'' = b'a + nc'a + f'a + nb'b + c'b + e'b + c'f, \\
c'' = nb'c + c'c + e'c + b'd + nc'd + f'd + c'e + b'f, \\
d'' = na'c + d'c + f'c + a'd + nd'd + e'd + d'e + a'f, \\
e'' = e'e + f'f, \\
f'' = f'e + e'f.
\]

Proof. Since for any \( t \in T_{1}^{*} \mathbf{R}^{n} \), \( t = qr \), \( R_{jk}^{i} = P_{jk}^{i} \mathbf{P} = P_{jk}^{i} Q_{bc}^{a} q_{r} = R_{jk}^{i} q_{r} \), the coefficients \( R_{jk}^{i} q_{r} \) are obtained from the formula
\[
R_{jk}^{i} q_{r} = P_{jk}^{i} Q_{bc}^{a} q_{r}.
\]

(6.6)

Now we derive the equations for natural projectors in \( T_{2}^{1} \mathbf{R}^{n} \).

**Lemma 7.** A natural linear operator \( P : T_{2}^{1} \mathbf{R}^{n} \rightarrow T_{2}^{1} \mathbf{R}^{n} \) expressed by (6.1), is a natural projector if and only if
\[
a^{2} + (nb + nd + 2e - 1) a + bd + (b + d)f = 0, \\
nb^{2} + (a + c + 2e - 1) b + nca + (a + c)f = 0, \\
c^{2} + (nb + nd + 2e - 1) c + bd + (b + d)f = 0, \\
nb^{2} + (a + c + 2e - 1) d + nca + (a + c)f = 0, \\
e = e^{2} + f^{2},
\]
\[
f = 2ef.
\]

(6.7)

Proof. The components of \( P \) satisfy the projector equation \( P_{jk}^{i} q_{r} = P_{jk}^{i} q_{r} \), which can be obtained by substituting \( Q = P \) and \( R = P \) in (6.5).

Equations (6.7) are referred to as the natural projector equations. These equations represent a system of six quadratic equations for six unknowns \( (a, b, c, d, e, f) \).

**Remark 4.** If \( P \) is a natural projector, then the complementary projector \( \text{id} - P \) is also natural. Thus, if \( P (6.1) \) satisfies (6.7), then \( \text{id} - P \) also satisfies (6.7). Indeed,
\[
\text{id} - P = a'\delta_{j}^{i}q_{k}^{r} + b'\delta_{k}^{i}q_{j}^{r} + c'\delta_{k}^{r}q_{j}^{i} + d'\delta_{k}^{r}q_{j}^{i} + e'\delta_{j}^{r}q_{k}^{i} + f'\delta_{j}^{i}q_{k}^{r},
\]
where
\[
a' = -a, b' = -b, c' = -c, d' = -d, e' = 1 - e, f' = -f.
\]

(6.9)

The transformation (6.9) leaves invariant the system (6.7).
It is easily seen that the formulas
\begin{align*}
a' &= c, \quad b' = b, \quad c' = a, \quad d' = d, \quad e' = e, \quad f' = f, \\
a' &= a, \quad b' = d, \quad c' = c, \quad d' = b, \quad e' = e, \quad f' = f
\end{align*}
(also define invariant transformations of (6.7). Consequently, if \((a, b, c, d, e, f)\) is a natural projector, then also \((-a, -b, -c, -d, 1 - e, -f)\), \((c, b, a, d, e, f)\), \((a, d, c, b, e, f)\), \((a, c, b, d, f, e)\), and \((b, c, a, d, e, f)\), are natural projectors.

We are now in a position to find all solutions of the natural projector equations (6.7). We write these solutions in the form of their equations \(\bar{t}_j^i = P_{jk}^i \frac{q^r}{p^r t_{qr}}\), i.e., as
\begin{align*}
\bar{t}_j^i &= a \delta_j^i t_{ks} + b \delta_j^i t_{sk} + c \delta_j^i t_{sj} + d \delta_j^i t_{js} + e t_j^i + f t_k^i.
\end{align*}
Here \((a, b, c, d, e, f)\) are the components (6.2) of a natural projector, expressed by (6.1). Note that the list (A1), (A2),..., (D4) below includes one-, and two-parameter families of natural projectors.

We define
\begin{align*}
A_1 &= n d + d^2 - n^2 d^2, & A_2 &= - n d + d^2 - n^2 d^2, \\
A_3 &= n^2 d^2 - d^2 - d, & A_4 &= n^2 d^2 - d^2 + d, \\
B_1 &= n^2 c^2 - c^2 + c, & B_2 &= n^2 c^2 - c^2 - c, \\
C_1 &= 4 d + 4 d^2 - 4 n^2 d^2 + 1, & C_2 &= - 4 d + 4 d^2 - 4 n^2 d^2 + 1.
\end{align*}

**Theorem 5.** The following list contains all natural projectors \(P : T_2^1 \mathbb{R}^n \to T_2^1 \mathbb{R}^n\):

\begin{align*}
\bar{t}_j^i &= 0, \\
\bar{t}_j^i &= \frac{1}{2(n-1)} (- \delta_j^i t_{ks} + \delta_j^i t_{sk} - \delta_j^i t_{sj} + \delta_j^i t_{js}), \\
\bar{t}_j^i &= \frac{1}{2(n-1)} (\delta_j^i t_{ks} + \delta_j^i t_{sk} + \delta_j^i t_{sj} + \delta_j^i t_{js}), \\
\bar{t}_j^i &= - \frac{1}{n-1} \delta_j^i t_{ks} + \frac{1}{n^2-1} \delta_j^i t_{sk} - \frac{1}{n^2-1} \delta_j^i t_{sj} + \frac{1}{n^2-1} \delta_j^i t_{js},
\end{align*}

\((A1)\)

\begin{align*}
\bar{t}_j^i &= \frac{d + \sqrt{4 c + 1}}{n} \delta_j^i t_{ks} + n + 2 d - 2 d^2 - 2 \sqrt{4 c + 1} \delta_j^i t_{sk} + \frac{d + \sqrt{4 c + 1}}{n} \delta_j^i t_{sj} + \frac{n}{n^2-1} \delta_j^i t_{js}, \\
\bar{t}_j^i &= \frac{d + \sqrt{4 c + 1}}{n} \delta_j^i t_{ks} + n + 2 d - 2 d^2 + 2 \sqrt{4 c + 1} \delta_j^i t_{sk} - \frac{d + \sqrt{4 c + 1}}{n} \delta_j^i t_{sj} + \frac{n}{n^2-1} \delta_j^i t_{js},
\end{align*}

\((A2)\)

\begin{align*}
\bar{t}_j^i &= (1 - c - 2 n (-n c + \sqrt{4 c + 1})) \delta_j^i t_{ks} + (n c + \sqrt{4 c + 1}) \delta_j^i t_{sk} + c \delta_j^i t_{sj} + (n c + \sqrt{4 c + 1}) \delta_j^i t_{js}, \\
&\quad + (n c + \sqrt{4 c + 1}) \delta_j^i t_{js}, \\
\bar{t}_j^i &= (1 - c + 2 n (n c + \sqrt{4 c + 1})) \delta_j^i t_{ks} - (n c + \sqrt{4 c + 1}) \delta_j^i t_{sk} + c \delta_j^i t_{sj} - (n c + \sqrt{4 c + 1}) \delta_j^i t_{js},
\end{align*}

\((A3)\)

\begin{align*}
\bar{t}_j^i &= (1 - n b) \delta_j^i t_{ks} + b \delta_j^i t_{sk}, \\
\bar{t}_j^i &= \frac{d - n d - c d}{d + n c} \delta_j^i t_{ks} + \frac{-c^2 - n c d}{d + n c} \delta_j^i t_{sk} + c \delta_j^i t_{sj} + d \delta_j^i t_{js}, & d + n c \neq 0,
\end{align*}

\((A4)\)
\[
\begin{align}
\vec{v}_{jk} &= t_{jk}, \\
\overline{v}_{jk} &= -\frac{1}{2(n-1)}(\delta^i_{jk}s_{ks} - \delta^i_{jk}s_{sk} + \delta^i_{jk}t_{sj} - \delta^i_{jk}t_{js} + \frac{1}{2}t_{jk}), \\
\tilde{v}_{jk} &= -\frac{1}{2(n-1)}(\delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \delta^i_{jk}t_{sj} + \delta^i_{jk}t_{js} + \frac{1}{2}t_{jk}), \\
\end{align}
\]

(B1)

\[
\begin{align}
\overline{v}_{jk} &= -\frac{d+\sqrt{n}}{n}(\delta^i_{jk}s_{ks} - \delta^i_{jk}s_{sk} + \delta^i_{jk}t_{sj} - \delta^i_{jk}t_{js} + \frac{1}{2}t_{jk}), \\
\overline{v}_{jk} &= -\frac{d+\sqrt{n}}{n}(\delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \delta^i_{jk}t_{sj} + \delta^i_{jk}t_{js} + \frac{1}{2}t_{jk}), \\
\overline{v}_{jk} &= \frac{d}{d+nc}(\delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \delta^i_{jk}t_{sj} + \delta^i_{jk}t_{js} + \frac{1}{2}t_{jk}), \\
\overline{v}_{jk} &= -\frac{d+\sqrt{n}}{n}(\delta^i_{jk}s_{ks} - \delta^i_{jk}s_{sk} + \delta^i_{jk}t_{sj} - \delta^i_{jk}t_{js} + \frac{1}{2}t_{jk}), \\
\end{align}
\]

\[
d \in [-n/(n^2 - 1), 0],
\]

(B2)

\[
\begin{align}
\overline{v}_{jk} &= \left(1 - c \right) - 2n \left(-nc + \sqrt{B_2} \right) \delta^i_{jk}s_{ks} + \left(-nc + \sqrt{B_2} \right) \delta^i_{jk}s_{sk} + c \delta^i_{jk}t_{sj} + \frac{1}{2}t_{jk}, \\
\overline{v}_{jk} &= \left(1 - c \right) - 2n \left(-nc - \sqrt{B_2} \right) \delta^i_{jk}s_{ks} - \left(nc + \sqrt{B_2} \right) \delta^i_{jk}s_{sk} + c \delta^i_{jk}t_{sj} + \frac{1}{2}t_{jk}, \\
\overline{v}_{jk} &= (1 - n) \delta^i_{jk}s_{ks} + b \delta^i_{jk}s_{sk} + \frac{1}{2}t_{jk},
\end{align}
\]

\[
c \in (-\infty, 0] \cup [1/(n^2 - 1), \infty),
\]

(B3)

\[
\begin{align}
\overline{v}_{jk} &= -\frac{1}{2(n-1)}(\delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \delta^i_{jk}t_{sj} + \delta^i_{jk}t_{js} + \frac{1}{2}t_{jk}), \\
\overline{v}_{jk} &= -\frac{1}{2(n-1)}(\delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \delta^i_{jk}t_{sj} + \delta^i_{jk}t_{js} + \frac{1}{2}t_{jk}), \\
\overline{v}_{jk} &= \frac{1}{2(n+nc)}\delta^i_{jk}s_{ks} + \frac{1}{2(n+nc)}\delta^i_{jk}s_{sk} + \frac{1}{2(n+nc)}\delta^i_{jk}t_{sj} + \frac{1}{2(n+nc)}\delta^i_{jk}t_{js} + \frac{1}{2}t_{jk}, \\
\overline{v}_{jk} &= \frac{1}{2(n+nc)}\delta^i_{jk}s_{ks} + \frac{1}{2(n+nc)}\delta^i_{jk}s_{sk} + \frac{1}{2(n+nc)}\delta^i_{jk}t_{sj} + \frac{1}{2(n+nc)}\delta^i_{jk}t_{js} + \frac{1}{2}t_{jk},
\end{align}
\]

\[
d \in [-\frac{1}{2}(n+1), \frac{1}{2}(n+1)],
\]

(B4)

\[
\begin{align}
\overline{v}_{jk} &= \left(-nd + \sqrt{A_3} \right) \delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \left(-nd + \sqrt{A_3} \right) \delta^i_{jk}t_{sj} + \delta^i_{jk}t_{js} + \frac{1}{2}t_{jk}, \\
\overline{v}_{jk} &= \left(-nd + \sqrt{A_3} \right) \delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \left(-nd + \sqrt{A_3} \right) \delta^i_{jk}t_{sj} + \delta^i_{jk}t_{js} + \frac{1}{2}t_{jk}, \\
\overline{v}_{jk} &= \left(-nd + \sqrt{A_3} \right) \delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \delta^i_{jk}t_{sj} + \delta^i_{jk}t_{js} + \frac{1}{2}t_{jk} + \frac{1}{2}t_{jk},
\end{align}
\]

\[
d \in (-\infty, 0] \cup [1/(n^2 - 1), \infty),
\]

(C1)

\[
\begin{align}
\overline{v}_{jk} &= \left(-\frac{1}{2} + \frac{1}{2} \sqrt{n} \right) \delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \frac{1}{2}t_{jk} + \frac{1}{2}t_{jk}, \\
\overline{v}_{jk} &= \left(-\frac{1}{2} + \frac{1}{2} \sqrt{n} \right) \delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \frac{1}{2}t_{jk} + \frac{1}{2}t_{jk}, \\
\overline{v}_{jk} &= \left(-\frac{1}{2} + \frac{1}{2} \sqrt{n} \right) \delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \frac{1}{2}t_{jk} + \frac{1}{2}t_{jk} + \frac{1}{2}t_{jk}, \\
\overline{v}_{jk} &= \left(-\frac{1}{2} + \frac{1}{2} \sqrt{n} \right) \delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \frac{1}{2}t_{jk} + \frac{1}{2}t_{jk} + \frac{1}{2}t_{jk},
\end{align}
\]

\[
d \in [-\frac{1}{2}(n+1), \frac{1}{2}(n+1)],
\]

(C2)

\[
\begin{align}
\overline{v}_{jk} &= \left(-\frac{1}{2} + \frac{1}{2} \sqrt{n} \right) \delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \frac{1}{2}t_{jk} + \frac{1}{2}t_{jk}, \\
\overline{v}_{jk} &= \left(-\frac{1}{2} + \frac{1}{2} \sqrt{n} \right) \delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \frac{1}{2}t_{jk} + \frac{1}{2}t_{jk} + \frac{1}{2}t_{jk},
\end{align}
\]

\[
d \in (-\infty, 0] \cup [1/(n^2 - 1), \infty),
\]

(C3)

\[
\begin{align}
\overline{v}_{jk} &= \left(-\frac{1}{2} + \frac{1}{2} \sqrt{n} \right) \delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \frac{1}{2}t_{jk} + \frac{1}{2}t_{jk}, \\
\overline{v}_{jk} &= \left(-\frac{1}{2} + \frac{1}{2} \sqrt{n} \right) \delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \frac{1}{2}t_{jk} + \frac{1}{2}t_{jk} + \frac{1}{2}t_{jk},
\end{align}
\]

\[
d \in (-\infty, 0] \cup [1/(n^2 - 1), \infty),
\]

(C4)

\[
\begin{align}
\overline{v}_{jk} &= \left(-\frac{1}{2} + \frac{1}{2} \sqrt{n} \right) \delta^i_{jk}s_{ks} + \delta^i_{jk}s_{sk} + \frac{1}{2}t_{jk} + \frac{1}{2}t_{jk},
\end{align}
\]
\[
\begin{align*}
\bar{t}_{jk} &= \frac{1}{2}t_{jk} - \frac{1}{2}t_{kj}, \\
\tilde{t}_{jk} &= \frac{1}{2(n-1)}(\delta^j_{jk} - \delta^k_{jk} + \delta^i_{j}t^s_{jk} - \delta^i_{k}t^s_{jk}) + \frac{1}{2}t_{jk} - \frac{1}{2}t_{kj}, \\
\bar{t}_{jk} &= \frac{1}{2(n-1)}(\delta^j_{jk} - \delta^k_{jk} + \delta^i_{j}t^s_{jk} + \delta^i_{k}t^s_{jk}) + \frac{1}{2}t_{jk} - \frac{1}{2}t_{kj}, \\
\tilde{t}_{jk} &= \frac{n}{n-1}\delta^s_{jk} t^s_{jk} - \frac{1}{n-1}\delta^j_{jk} t^s_{jk} + \frac{2}{n-1}\delta^k_{jk} t^s_{jk} + \frac{1}{2}t_{jk} - \frac{1}{2}t_{kj}, \\

\bar{t}_{jk} &= \frac{1}{2}t_{jk} + \frac{1}{2}t_{kj}, \\
\tilde{t}_{jk} &= \frac{1}{2}t_{jk} + \frac{1}{2}t_{kj},
\end{align*}
\]

(D1)

Proof. (6.7) splits into the following 16 cases to be considered separately:

\[(e, f) = (0, 0), \quad a = c, \quad b = d, \]  

(A1)

\[(e, f) = (0, 0), \quad a = c, \quad b = -d - \frac{1}{n}(a + c + 2e - 1), \]  

(A2)

\[(e, f) = (0, 0), \quad a = -c - nb - nd - 2e + 1, \quad b = d, \]  

(A3)

\[(e, f) = (0, 0), \quad a = -c - nb - nd - 2e + 1, \quad b = -d - \frac{1}{n}(a + c + 2e - 1), \]  

(A4)

\[(e, f) = (1, 0), \quad a = c, \quad b = d, \]  

(B1)

\[(e, f) = (1, 0), \quad a = c, \quad b = -d - \frac{1}{n}(a + c + 2e - 1), \]  

(B2)

\[(e, f) = (1, 0), \quad a = -c - nb - nd - 2e + 1, \quad b = d, \]  

(B3)

\[(e, f) = (1, 0), \quad a = -c - nb - nd - 2e + 1, \quad b = -d - \frac{1}{n}(a + c + 2e - 1), \]  

(B4)

\[(e, f) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad a = c, \quad b = d, \]  

(C1)

\[(e, f) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad a = c, \quad b = -d - \frac{1}{n}(a + c + 2e - 1), \]  

(C2)

\[(e, f) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad a = -c - nb - nd - 2e + 1, \quad b = d, \]  

(C3)
(e, f) = (1/2, 1/2), a = -c - nb - nd - 2e + 1, b = -d - \frac{1}{n}(a + c + 2e - 1) \quad (C4)

(e, f) = (1/2, -1/2), a = c, b = d, \quad (D1)

(e, f) = (1/2, -1/2), a = c, b = -d - \frac{1}{n}(a + c + 2e - 1), \quad (D2)

(e, f) = (1/2, -1/2), a = -c - nb - nd - 2e + 1, b = d, \quad (D3)

(e, f) = (1/2, -1/2), a = -c - nb - nd - 2e + 1, b = -d - \frac{1}{n}(a + c + 2e - 1). \quad (D4)

Each of these cases is subject to the conditions

\begin{align*}
a^2 + c^2 + (nb + nd + 2e - 1)(a + c) + 2bd + 2(b + d)f &= 0, \\
nb^2 + nd^2 + (a + c + 2e - 1)(b + d) + 2nac + 2(a + c)f &= 0. \quad (6.13)
\end{align*}

To complete the proof, we solve the system (6.13) of two quadratic equations for every of the possibilities (A1), (A2), \ldots, (D4). We get, using MAPLE,

\begin{align*}
(0, 0, 0, 0, 0, 0), \\
&\left(-\frac{1}{2(n-1)}, \frac{1}{2(n-1)}, -\frac{1}{2(n-1)}, \frac{1}{2(n-1)}, 0, 0\right), \\
&\left(-\frac{1}{n^2-1}, \frac{n}{n^2-1}, -\frac{1}{n^2-1}, 0, 0\right), \\
&\left(-\frac{d+\sqrt{\Delta_1}}{n}, \frac{n+2d-n^2d-2\sqrt{\Delta_1}}{n^2}, -\frac{d+\sqrt{\Delta_1}}{n}, d, 0, 0\right), \\
&\left(-\frac{d+\sqrt{\Delta_1}}{n}, \frac{n+2d-n^2d+2\sqrt{\Delta_1}}{n^2}, -\frac{d+\sqrt{\Delta_1}}{n}, d, 0, 0\right), \quad (A1)
\end{align*}

\begin{align*}
\left(1 - c - 2n\left(-nc + \sqrt{B_1}\right), -nc + \sqrt{B_1}, c, -nc + \sqrt{B_1}, 0, 0\right), \\
\left(1 - c + 2n\left(nc + \sqrt{B_1}\right), -\left(nc + \sqrt{B_1}\right), c, -\left(nc + \sqrt{B_1}\right), 0, 0\right), \\
c \in (-\infty, -1/(n^2 - 1)] \cup [0, \infty), \quad (A2)
\end{align*}

\begin{align*}
(1 - nb, b, 0, 0, 0, 0), \\
\left(d - \frac{nd^2 - cd}{d + nc}, \frac{c - c - nd - cd}{d + nc}, c, d, 0, 0\right), \quad d + nc \neq 0, \\
-\left(-nb, b, -\frac{1}{n^2-1}, \frac{n}{n^2-1}, 0, 0\right), \quad (A4)
\end{align*}

\begin{align*}
(0, 0, 0, 0, 1, 0), \\
&\left(-\frac{1}{2(n-1)}, -\frac{1}{2(n-1)}, \frac{1}{2(n-1)}, -\frac{1}{2(n-1)}, 1, 0\right), \\
&\left(-\frac{1}{2(n-1)}, -\frac{1}{2(n-1)}, -\frac{1}{2(n-1)}, -\frac{1}{2(n-1)}, 1, 0\right), \\
&\left(\frac{1}{n^2-1}, \frac{n}{n^2-1}, -\frac{n}{n^2-1}, 1, 0\right), \quad (B1)
\end{align*}
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\[
\begin{align*}
&\left( -\frac{d+\sqrt{A_3}}{n}, -\frac{n+2d-n^2d-2\sqrt{A_3}}{n^2}, -\frac{d+\sqrt{A_3}}{n}, d, 0, 0 \right), \\
&\left( \frac{d+\sqrt{A_3}}{n}, -\frac{n+2d-n^2d+2\sqrt{A_3}}{n^2}, -\frac{d+\sqrt{A_3}}{n}, d, 0, 0 \right), \\
&d \in \left[-n/(n^2-1), 0 \right],
\end{align*}
\]

\[
\begin{align*}
&\left( 1 - c - 2n \left( -nc + \sqrt{B_2} \right), -nc + \sqrt{B_2}, c, -nc + \sqrt{B_2}, 1, 0 \right), \\
&\left( 1 - c - 2n \left( -nc - \sqrt{B_2} \right), -nc - \sqrt{B_2}, c, -nc - \sqrt{B_2}, 1, 0 \right), \\
&c \in (-\infty, 0] \cup \left[ 1/(n^2-1), \infty \right),
\end{align*}
\]

\[
\begin{align*}
&(1 - nb, b, 0, 0, 1, 0), \\
&(\frac{d+nd^2+cd}{d+nc}, -\frac{c+e^2+ncd}{d+nc}, c, d, 1, 0), & d + nc \neq 0, \\
&(\frac{1}{n^2-1}, -\frac{1}{n^2-1}, 0),
\end{align*}
\]

\[
\begin{align*}
&(0, 0, 0, 0, 1, 1), \\
&(\left( 1+2d+\sqrt{C_1} \right) 2d-n^2d+1+\sqrt{C_1}, -\frac{1+2d+\sqrt{C_1}}{n^2}, d, \frac{1}{2}, \frac{1}{2} ), \\
&(\left( 1-2d+\sqrt{C_1} \right) 2d-n^2d+1-\sqrt{C_1}, -\frac{1-2d+\sqrt{C_1}}{n^2}, d, \frac{1}{2}, \frac{1}{2} ), \\
&d \in \left[-\frac{1}{2}(n+1), \frac{1}{2}(n+1) \right],
\end{align*}
\]

\[
\begin{align*}
&\left( -nd - \sqrt{A_3}, d, -nd + \sqrt{A_3}, d, \frac{1}{2}, \frac{1}{2} \right), \\
&\left( -nd + \sqrt{A_3}, d, -nd - \sqrt{A_3}, d, \frac{1}{2}, \frac{1}{2} \right), \\
&d \in (-\infty, 0] \cup \left[ 1/(n^2-1), \infty \right),
\end{align*}
\]

\[
\begin{align*}
&(\frac{1}{2} - nb, b, -\frac{1}{2(n-1)}, \frac{1}{2(n-1)}, \frac{1}{2}, \frac{1}{2} ), \\
&(\frac{1+2d^2+2cd}{2d+2nc+1}, -\frac{d+2cd+2d^2}{2d+2nc+1}, c, d, \frac{1}{2}, \frac{1}{2} ), & 2d + 2nc + 1 \neq 0, \\
&(\frac{1}{2} - nb, b, -\frac{1}{2(n-1)}, \frac{1}{2(n-1)}, \frac{1}{2}, \frac{1}{2} ),
\end{align*}
\]

\[
\begin{align*}
&(0, 0, 0, 0, 1, 1), \\
&(\left( 1 \right) \frac{1}{2(n-1)}, \frac{1}{2(n-1)}, \frac{1}{2(n-1)}, \frac{1}{2}, \frac{1}{2} ), \\
&(\left( 1 \right) \frac{1}{2(n-1)}, \frac{1}{2(n-1)}, \frac{1}{2(n-1)}, \frac{1}{2}, \frac{1}{2} ), \\
&(\left( n \right) \frac{1}{n^2-1}, \frac{1}{n^2-1}, \frac{1}{n^2-1}, \frac{1}{2}, \frac{1}{2} ), \\
&(\left( 1-2d-\sqrt{C_2} \right) 2d-n^2d+1+\sqrt{C_2}, -\frac{1-2d-\sqrt{C_2}}{n^2}, d, \frac{1}{2}, \frac{1}{2} ), \\
&(\left( 1-2d+\sqrt{C_2} \right) 2d-n^2d+1-\sqrt{C_2}, -\frac{1-2d+\sqrt{C_2}}{n^2}, d, \frac{1}{2}, \frac{1}{2} ), \\
&d \in \left[-1/(n^2-1), 1/(n^2-1) \right],
\end{align*}
\]
\[( -nd - \sqrt{A_4}, d, -nd + \sqrt{A_4}, d, \frac{1}{2}, -\frac{1}{2}), \]
\[( -nd + \sqrt{A_4}, d, -nd - \sqrt{A_4}, d, \frac{1}{2}, -\frac{1}{2}), \quad (D3) \]
\[d \in (-\infty, -1/(n^2 - 1)] \cup [0, \infty),\]
\[( -\frac{1}{2} - nb, b, \frac{1}{2(n-1)}, \frac{1}{2(n-1)}, \frac{1}{2}, -\frac{1}{2}), \]
\[( -\frac{1}{2} - nd^2 - 2nd, d, -2nd + 2nc - 2c^2, c, d, \frac{1}{2}, -\frac{1}{2}), 2d + 2nc - 1 \neq 0, \quad (D4) \]
\[( \frac{1}{2} - nb, b, \frac{1}{2(n-1)}, -\frac{1}{2(n-1)}, \frac{1}{2}, -\frac{1}{2}). \]

Now our assertion follows from (6.11).

**Remark 5.** Note that Theorem 5 gives us a complete answer to the problem of finding all natural projectors in \(T^1_2 \mathbb{R}^n\). Properties of these natural projectors can be obtained from this list (A1), (A2), . . . , (D4) by a direct analysis.

**References**


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