A Classification of Contact Metric 3-Manifolds with Constant $\xi$-sectional and $\phi$-sectional Curvatures

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Abstract. We study the 3-dimensional contact metric manifolds equipped with constant $\xi$-sectional curvature and $\phi$-sectional curvature or constant norm of the Ricci operator.

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1. Introduction

D. E. Blair in [2], [3] constructed a family of examples of $(3 - \tau)$-manifolds which do not satisfy the condition $Q\phi = \phi Q$. The existence of these examples depends on the constancy of the $\xi$-sectional curvature. After this remark the following question raises:

Question 1: Does every $(3 - \tau)$-manifold with constant $\xi$-sectional curvature satisfy the condition $Q\phi = \phi Q$?

S. Tanno in [16] stated the problem about the existence of $(2n+1)$-dimensional contact metric manifolds of constant $\phi$-sectional curvature, which are not Sasakian. Positive answers have been given by D. E. Blair, Th. Koufogiorgos and R. Sharma in [5], for 3-dimensional contact metric manifolds satisfying $Q\phi = \phi Q$, Th. Koufogiorgos in [14], for $(\kappa, \mu)$-contact metric manifolds.
manifolds of dimension greater than 3 and D. E. Blair, Th. Koufogiorgos and B. Papantoniou in [4] for \((\kappa, \mu)\)-contact metric manifolds of dimension 3. In [4] the authors, extending the Tanno’s problem showed that there exist \((\kappa, \mu)\)-contact metric manifolds of dimension 3 which do not belong to the class of the manifolds satisfying \(Q\phi = \phi Q\).

Extending Tanno’s problem and the result of [4] we can state the following:

**Question 2:** Do there exist 3-dimensional contact metric manifolds of constant \(\phi\)-sectional curvature, which do not belong to the class of \((\kappa, \mu)\)-contact metric manifolds?

Combination of the above mentioned questions leads us to the study of 3-dimensional contact metric manifolds of constant \(\xi\)-sectional and \(\phi\)-sectional curvature.

The main goal of the present paper (Theorem 15) is the proof of the existence of two new classes of 3-dimensional contact metric manifolds with constant \(\xi\)-sectional and constant \(\phi\)-sectional curvatures, which do not belong to the up to date well known classes ([4], [5]).

D. E. Blair, Th. Koufogiorgos and R. Sharma in [5] proved that a 3-dimensional contact metric manifold satisfying \(Q\phi = \phi Q\) is flat or Sasakian or a manifold with constant \(\phi\)-sectional curvature \(k\) and constant \(\xi\)-sectional curvature \(-k\). In the present paper we prove the converse and so we can state the argument: A non-flat, non-Sasakian 3-dimensional contact metric manifold satisfies \(Q\phi = \phi Q\) if and only if it has constant \(\phi\)-sectional curvature \(k\) and constant \(\xi\)-sectional curvature \(-k\).

Complete, conformally flat Riemannian manifolds with constant scalar curvature and the norm of the Ricci tensor bounded (respectively constant) were classified by Goldberg ([8]) in general dimension (respectively, by Cheng, Ishikawa and Shiohama [7] in dimension 3). On the other hand the first author and R. Sharma in [10] proved that a conformally flat, contact metric 3-manifold with Ricci curvature vanishing along the characteristic vector field \(\xi\) and the norm of its Ricci tensor being constant, is flat. Therefore, it is interesting to study 3-dimensional contact metric manifolds equipped with more general conditions: constant \(\xi\)-sectional curvature and constant norm of the Ricci operator along \(\xi\).

2. Preliminaries

A contact metric manifold \(M^{2n+1} = M^{2n+1}(\phi, \xi, \eta, g)\) is a \((2n + 1)\)-dimensional Riemannian manifold on which has been defined globally a \((1, 1)\) tensor field \(\phi\), a vector field \(\xi\) (characteristic vector field), a 1-form \(\eta\) (contact form) and a Riemannian metric \(g\) (associated metric) which satisfy:

\[
\begin{align*}
\phi^2 &= -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta(X) = g(X, \xi), \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad d\eta(X, Y) = g(X, \phi Y)
\end{align*}
\]

for all vector fields \(X\) and \(Y\) on \(M^{2n+1}\). The structure \((\phi, \xi, \eta, g)\) is called contact metric structure.

Denoting by \(L\) and \(R\) the Lie derivation and the curvature tensor respectively, we define the operators \(l\) and \(h\) by

\[
l := R(\cdot, \xi)\xi, \quad \eta := \frac{1}{2}L_{\xi}\phi.
\]
The tensors \( l \) and \( h \) are self-adjoint and satisfy

\[
h \xi = l \xi = 0, \quad \eta \circ h = 0, \quad Tr h = Tr h \phi = 0, \quad h \phi + \phi h = 0.
\]

On every contact metric manifold \( M^{2n+1} \) the following formulas hold

\[
\begin{align*}
\eta \circ \phi &= 0, \quad \phi \xi = 0, \quad d \eta (\xi, X) = 0, \quad \nabla_\xi \phi = 0, \\
\nabla_\xi h &= \phi - \phi l - \phi h^2, \quad Tr l = g(Q \xi, \xi) = 2n - tr h^2,
\end{align*}
\]

where \( \nabla \) is the Riemannian connection. On \( M^{2n+1} \times \mathbb{R} \) we can define an almost complex structure \( J \) by

\[
J(X, f \frac{d}{dt}) = (\phi X - f \xi, \eta(X) \frac{d}{dt}),
\]

where \( f \) is a real-valued function. If \( J \) is integrable, then the contact metric structure is said to be normal and \( M^{2n+1} \) is called Sasakian. A 3-dimensional contact metric manifold is Sasakian if and only if \( h = 0 \), \([1]\).

The sectional curvature \( K(X, \xi) \) of a plain section spanned by \( \xi \) and a vector field \( X \) orthogonal to \( \xi \) is called \( \xi \)-sectional curvature. The sectional curvature \( K(X, \phi X) \) of a plain section spanned by the vector field \( X \) (orthogonal to \( \xi \)) and \( \phi X \) is called \( \phi \)-sectional curvature.

It is well known that on every 3-dimensional Riemannian manifold the curvature tensor \( R(X, Y)Z \) is given by

\[
R(X, Y)Z = g(Y, Z)Q X - g(X, Z)Q Y + g(Q Y, Z)X - g(Q X, Z)Y - \frac{S}{2} [g(Y, Z)X - g(X, Z)Y],
\]

where \( Q \) is the Ricci operator, \( S(= Tr Q) \) is the scalar curvature and \( X, Y \) and \( Z \) are arbitrary vector fields.

A 3-dimensional contact metric manifold satisfying \( \nabla_\xi \tau = 0 \), \( \tau = L_\xi g \) is called \((3 - \tau)-manifold\), \([11]\).

A contact metric manifold \( M^{2n+1}(\phi, \xi, \eta, g) \) is called \((\kappa, \mu)\)-contact metric manifold \([4]\) if it satisfies the condition

\[
R(X, Y) \xi = \kappa [\eta(Y)X - \eta(X)Y] + \mu [\eta(Y)h X - \eta(X)h Y],
\]

where \( \kappa \) and \( \mu \) are real constants and \( X, Y \) are vector fields on \( M^{2n+1} \).

3. Auxiliary results

Let \( M^3 \) be a 3-dimensional contact metric manifold. If \( e \in \ker(\eta) \) is a unit eigenvector of \( h \) with eigenvalue \( \lambda \), then \( \phi e \) is also an eigenvector of \( h \) with eigenvalue \(-\lambda \). Hence, \( (e, \phi e, \xi) \) is an orthonormal frame on \( M^3 \).

Since \( e \) and \( \phi e \) are unit vector fields orthogonal to \( \xi \), we see that

\[
\nabla_\xi e = a \phi e, \quad \nabla_\xi \phi e = -ae,
\]

for some function \( a \) on \( M^3 \). The orthogonality of \( e, \phi e \) and \( \xi \) implies

\[
\nabla_\xi e = b \phi e, \quad \nabla_{\phi e} \phi e = ce, \quad \nabla_\xi \phi e = -be + (\lambda + 1) \xi, \quad \nabla_\phi e = -c \phi e + (\lambda - 1) \xi,
\]
where \( b \) and \( c \) are functions on \( M^3 \). Finally, from (1) we have
\[
\nabla_{\xi} e = -(1 + \lambda)\phi e, \quad \nabla_{\phi e} \xi = (1 - \lambda)e.
\]
Therefore, we can state the following

**Lemma 1.** Let \( M^3 \) be 3-dimensional contact metric manifold. Then, the following formulas hold:
\[
\begin{align*}
\nabla_{\xi} e &= a\phi e, \quad \nabla_{\phi e} \xi = -(1 + \lambda)\phi e, \\
\nabla_{\phi e} e &= \phi e, \quad \nabla_{\phi e} \phi e = ce, \\
\nabla_{\xi} \phi e &= -b\phi e + (\lambda + 1)\xi, \\
\nabla_{\phi e} e &= -c\phi e + (\lambda - 1)\xi,
\end{align*}
\]
where \( a, b \) and \( c \) are functions on \( M^3 \).

**Proposition 2.** Let \( M^3 \) be 3-dimensional contact metric manifold of constant \( \xi \)-sectional curvature \( k \). Then, \( M^3 \) is \((3 - \tau)\)-manifold with constant \( Trl \).

**Proof.** By straightforward computation using (3) and \( \nabla_{\xi} \xi = 0 \) we obtain
\[
le = (1 - \lambda^2 - 2\alpha\lambda)e + (\xi \cdot \lambda)\phi e, \quad l\phi e = (1 - \lambda^2 + 2\alpha\lambda)\phi e + (\xi \cdot \lambda)e,
\]
and hence
\[
1 - \lambda^2 - 2\alpha\lambda = k, \quad 1 - \lambda^2 + 2\alpha\lambda = k.
\]
Adding the above two relations we obtain \( 2(1 - \lambda^2) = 2k \). Because of \( Trl = 2(1 - \lambda^2) \) ([5]) we have \( Trl = \text{constant} \). Subtracting the same relations we obtain \( \alpha\lambda = 0 \), that is \( \alpha = 0 \) or \( \lambda = 0 \).

If \( \lambda = 0 \), then \( M^3 \) is Sasakian, which is trivially \((3 - \tau)\)-manifold ([5]).

Suppose that \( a = 0 \). Taking into account that \( Trl = \text{constant} \) we obtain that \( \nabla_{\xi} h = 0 \). This relation and ([11]) complete the proof. \( \square \)

Proposition 2 and Theorem 3.2 of [12] imply the following

**Corollary 3.** Let \( M^3 \) be a 3-dimensional, conformally flat, contact metric manifold of constant \( \xi \)-sectional curvature. Then, \( M^3 \) is either flat or a Sasakian space form.

Proposition 2 and Theorem 3.1 of [14] imply the following

**Corollary 4.** Let \( M^3 \) be a 3-dimensional contact metric manifold of constant \( \xi \)-sectional curvature satisfying \( R(e,\xi) \cdot R = 0 \). Then, \( M^3 \) is either flat or a Sasakian manifold.

Proposition 2 and Theorem 3.1 of [13] imply the following

**Corollary 5.** Let \( M^3 \) be a 3-dimensional contact metric manifold of constant \( \xi \)-sectional curvature satisfying \( R(e,\xi) \cdot C = 0 \). Then, \( M^3 \) is either flat or a Sasakian manifold.

Proposition 2 and Theorem 5.1 of [11] imply the following
Corollary 6. Let $M^3$ be a 3-dimensional contact metric manifold with constant $\xi$-sectional curvature and $\eta$-parallel Ricci tensor. Then, $M^3$ is either flat or a Sasakian space form.

Proposition 2 and Theorem 6.2 of [11] imply the following

Corollary 7. Let $M^3$ be a 3-dimensional contact metric manifold with constant $\xi$-sectional curvature and cyclic $\eta$-parallel Ricci tensor. Then, $M^3$ is either flat or a Sasakian manifold with constant scalar curvature or of constant $\xi$-sectional curvature $k < 1$ and constant $\phi$-sectional curvature $-k$.

Lemma 1, Proposition 2 and [11] imply:

Lemma 8. Let $M^3$ be a 3-dimensional contact metric manifold with constant $\xi$-sectional curvature. Then, the following formulas hold:

\[
\begin{align*}
\nabla_\xi e &= \nabla_\xi \phi e = 0, \quad \nabla_e e = b\phi e, \quad \nabla_\phi \phi e = ce, \\
\nabla_e \phi e &= -be + (\lambda + 1)\xi, \quad \nabla_\phi \phi e = -c\phi e + (\lambda - 1)\xi, \\
\nabla_e \xi &= -(1 + \lambda)\phi e, \quad \nabla_\phi \xi = (1 - \lambda)e.
\end{align*}
\]

where $a$, $b$, and $c$ are functions on $M^3$ and $\lambda$ is a constant.

Proposition 2 and [6] (relations 2.16) yield

Lemma 9. Let $M^3$ be a 3-dimensional contact metric manifold with constant $\xi$-sectional curvature. Then, the following formulas hold:

\[
\begin{align*}
Qe &= (\lambda^2 + \frac{S}{2} - 1)e + 2\lambda b\xi, \quad \eta(Qe) = 2\lambda b, \\
Q\phi e &= (\lambda^2 + \frac{S}{2} - 1)\phi e + 2\lambda c\xi, \quad \eta(Q\phi e) = 2\lambda c, \\
Q\xi &= 2\lambda be + 2\lambda c\phi e + 2(1 - \lambda^2)\xi.
\end{align*}
\]

Lemma 10. Let $M^3$ be a 3-dimensional contact metric manifold with constant $\xi$-sectional curvature. Then, either $l = 0$, or the following relations are equivalent: $b = 0$, $c = 0$.

Proof. Suppose that $l$ is not identically equal to zero on $M^3$. Let $\lambda^2 \neq 1$ on an open neighborhood $U$ at a point $p \in M^3$, where $l \neq 0$. Applying the Jacobi’s identity for the vector fields $e$, $\phi e$, $\xi$ and taking into account the relation (4) we obtain

\[
\xi \cdot b = (\lambda - 1)c, \quad \xi \cdot c = (\lambda + 1)b.
\]

Let $b = 0$ (or $c = 0$) on $M^3$. Then, from the first (or the second) of (6) we conclude that $c = 0$ (or $b = 0$) on $U$. So, $c = 0$, ($b = 0$) on $M^3$.

Remark 11. On a 3-dimensional contact metric manifold $M^3$, we have $b = c = 0$ if and only if $Q\phi = \phi Q$, ([11]).
4. Main results

**Theorem 12.** Let $M^3$ be a 3-dimensional contact metric manifold with constant $\xi$-sectional curvature. Then, either $M^3$ is Sasakian or

$$\xi \cdot \xi \cdot S = 4(\lambda^2 - 1)(\xi \cdot S).$$

**Proof.** If $l = 0$ on $M^3$, then $\lambda^2 = 1$ and $\xi \cdot \xi \cdot S = 0$ ([9]).

Suppose that $M^3$ is not Sasakian and $l$ is not identically equal to zero. So, let $\lambda^2 \neq 0, 1$ on an open neighborhood $U$ of a point $p \in M^3$. Applying the second Bianchi’s identity for the vector fields $e, \phi e$ and $\xi$ we obtain

$$e \cdot b + \phi e \cdot c - \frac{1}{4\lambda} \xi \cdot S = 2bc.$$  \hspace{1cm} (8)

Differentiating the above equation along $\xi$ and taking into account (6) we obtain

$$\xi \cdot e \cdot b + \xi \cdot \phi e \cdot c - \frac{1}{4\lambda} \xi \cdot \xi \cdot S = 2(\lambda - 1)e \cdot c^2 + 2(\lambda + 1)b^2.$$  \hspace{1cm} (9)

Next, differentiating the first and the second equations of (6) with respect to $e$ and $\phi e$ respectively and adding the results we get

$$e \cdot \xi \cdot b + \phi e \cdot \xi \cdot c = (\lambda - 1)e \cdot c + (\lambda + 1)\phi e \cdot b.$$  \hspace{1cm} (10)

Hence,

$$[\xi, e]b + [\xi, \phi e]c = \frac{1}{4\lambda} \xi \cdot \xi \cdot S + 2(\lambda - 1)e\cdot c^2 + 2(\lambda + 1)b^2 + (1 - \lambda)e \cdot c - (\lambda + 1)\phi e \cdot b.$$  \hspace{1cm} (11)

The above equation using (4) yields

$$(\lambda + 1)\phi e \cdot b + (\lambda - 1)e \cdot c = \frac{1}{8\lambda} \xi \cdot \xi \cdot S + \lambda(b^2 + c^2) + b^2 - c^2.$$  \hspace{1cm} (12)

Differentiating again (9) along $\xi$ and taking into account (6) and (8) we obtain

$$(\lambda + 1)\xi \cdot \phi e \cdot b + (\lambda - 1)\xi \cdot e \cdot c = \frac{1}{8\lambda} \xi \cdot \xi \cdot S + 4(\lambda^2 - 1)bc.$$  \hspace{1cm} (13)

As $\lambda^2 \neq 1$ on $U$ we obtain from (6) and (8)

$$(\lambda + 1)\phi e \cdot \xi \cdot b + (\lambda - 1)e \cdot \xi \cdot c = (\lambda^2 - 1)[\frac{1}{4\lambda} \xi \cdot S + 2bc].$$

Subtracting the above equation from (10) and using (4) the seeking formula follows at once. □

**Theorem 13.** Let $M^3$ be a 3-dimensional contact metric manifold with constant $\xi$-sectional curvature. If the norm of the Ricci operator is constant along $\xi$, then either $Q\phi = \phi Q$ or $l = 0$ with constant scalar curvature and $\eta(QX) = 0$ for all eigenvectors $X \in \ker(\eta)$ of $h$ with eigenvalue 1.
Proof. The square of the norm of the Ricci operator $Q$ is $\text{Tr}Q^2 = g(Q^2 e, e) + g(Q^2 φe, φe) + g(Q^2 ξ, ξ)$ and is computed using (5) and turns out to be

\[(λ^2 + \frac{S}{2} - 1)^2 + 4λ^2(b^2 + c^2) + 2(1 - λ^2)^2 = ψ,\]  

(11)

where $ψ$ is a smooth function on $M^3$ being constant along $ξ$.

Suppose that $l = 0$. Then, $λ^2 = 1$ and (11) yields

\[\frac{S^2}{4} + 4(b^2 + c^2) = ψ.\]  

(12)

Differentiating three times the equation (12) along $ξ$ and taking into account (6) and (7) for $λ = 1$ we obtain respectively

\[S(ξ · S) + 32bc = 0,\]  

\[S(ξ · ξ · S) + (ξ · S)^2 + 64b^2 = 0,\]  

(13)

\[(ξ · S)(ξ · ξ · S) = 0.\]

Therefore, $ξ · S = 0$. or $ξ · ξ · S = 0$.

Supposing $ξ · S = 0$ from the first of (13) we have $b = 0$ or $c = 0$.

If $b = 0$, from (5) we obtain $η(Qe) = 0$.

If $c = 0$ then (6) implies $b = 0$ that is $Qφ = φQ$. In this case the manifold is flat.

If $ξ · ξ · S = 0$ then from (13) we have $ξ · S = 0$ and $b = 0$.

If $M^3$ is Sasakian then it is known that we have $Qφ = φQ$.

Suppose that $M^3$ is not Sasakian with $l$ not identically equal to zero. So, let be $λ^2 ≠ 0, 1$ on an open neighborhood $U$ of a point $p ∈ M^3$. Hence, we can write the equation (11) in the form

\[b^2 + c^2 = \frac{ψ}{4λ^2} + \frac{(λ^2 - 1)^2}{2λ^2} - \frac{(λ^2 + \frac{S}{2} - 1)^2}{4λ^2}.\]

Differentiating the above equation along $ξ$ and taking into account (6) we obtain

\[bc = -\frac{1}{16λ^2}(λ^2 + \frac{S}{2} - 1)(ξ · S).\]  

(14)

Differentiating two times the relation (14) with respect to $ξ$ and using (6) and (14) we have

\[(ξ · S)[8(1 - λ^2)(λ^2 + \frac{S}{2} - 1) - 1 - ξ · ξ · S] = 0.\]

Hence,

\[ξ · S = 0 \text{ or } ξ · ξ · S = 8(1 - λ^2)(λ^2 + \frac{S}{2} - 1) - 1.\]  

(15)

Supposing $ξ · S = 0$, the equation (14) yields $b = 0$ or $c = 0$ on $U$ and hence $b = 0$ or $c = 0$ on $M^3$. Both cases using (6) imply $Qφ = φQ$.

If the second of (15) holds on $U$, differentiating this relation along $ξ$ and using Theorem 12 we obtain $ξ · S = 0$ and therefore $Qφ = φQ$. □
Proposition 14. Let $M^3$ be a 3-dimensional non-Sasakian contact metric manifold with constant $\xi$-sectional curvature. If $l$ is not identically equal to zero then the following formulas hold:

\[ e \cdot b = \frac{1}{8\lambda} \xi \cdot S + bc + \Phi, \]  
(16)

\[ \phi e \cdot b = \frac{1}{16\lambda} \xi \cdot \xi \cdot S + \frac{1}{2} (1 - \lambda) (\lambda^2 + \frac{S}{2} - 1) + b^2, \]  
(17)

\[ e \cdot c = -\frac{1}{16\lambda} \xi \cdot S + \frac{1}{2} (1 + \lambda) (\lambda^2 + \frac{S}{2} - 1) + c^2, \]  
(18)

\[ \phi e \cdot c = \frac{1}{8\lambda} \xi \cdot S + bc - \Phi. \]  
(19)

where $\Phi$ is a smooth function on $M^3$ such that

\[ \xi \cdot \Phi = 0, \]  
(20)

\[ e \cdot \Phi = \frac{1}{16\lambda} [\phi e \cdot \xi \cdot S - 2b(\xi \cdot \xi \cdot S) + 2(e \cdot \xi \cdot S) - 4c(\xi \cdot S) - \]  
\[ - 4\lambda(\lambda + 1)(\phi e \cdot S)] + (\lambda + 1)(\lambda^2 + \frac{S}{2} - 3)b + 4c\Phi, \]  
(21)

\[ \phi e \cdot \Phi = \frac{1}{16\lambda} [e \cdot \xi \cdot \xi \cdot S - 2c(\xi \cdot \xi \cdot S) - 2(\phi e \cdot \xi \cdot S) + 4b(\xi \cdot S) + \]  
\[ + 4\lambda(1 - \lambda)(e \cdot S)] + (\lambda - 1)(\lambda^2 + \frac{S}{2} - 3)c + 4b\Phi. \]  
(22)

Proof. Calculating $R(e, \phi e)\xi$ firstly by straightforward computation using Lemma 8 and secondly from the relation (2) we obtain

\[ \phi e \cdot b + e \cdot c = b^2 + c^2 + \lambda^2 - 1 + \frac{S}{2}. \]  
(23)

From (23) and (9) the relations (17) and (18) follow at once.

Differentiating (17) first with respect to $\xi$ (respectively with respect to $e$) and secondly with respect to $e$ (respectively with respect to $\xi$) and using (6) we have

\[ \xi \cdot e \cdot \phi e \cdot b = \frac{\lambda - 1}{4\lambda} e \cdot \xi \cdot S + 2(\lambda - 1)[e \cdot (bc)] \]  
(24)

respectively

\[ e \cdot \xi \cdot \phi e \cdot b = \frac{1}{16\lambda} (\xi \cdot e \cdot \xi \cdot S) + \frac{1 - \lambda}{4} (\xi \cdot e \cdot S) + \]  
\[ + 2b(\xi \cdot e \cdot b) + 2(\lambda - 1)c(e \cdot b). \]  
(25)
Differentiation of the relation (7) along $e$ implies

$$\frac{1}{16\lambda}(e \cdot \xi \cdot \xi \cdot S) = \frac{\lambda^2 - 1}{4\lambda}(e \cdot \xi \cdot S).$$

(26)

Adding (25) and (26) and using Lemma 8 we obtain

$$\xi \cdot e \cdot \phi e \cdot b = \frac{\lambda + 1}{16\lambda}(\phi e \cdot \xi \cdot \xi \cdot S) + \frac{\lambda^2 - 1}{4\lambda}(e \cdot \xi \cdot S) +$$

$$+ \frac{1 - \lambda}{4}(\xi \cdot e \cdot S) + 2b(\xi \cdot e \cdot b) + 2(\lambda - 1)c(e \cdot b).$$

(27)

Subtraction of (24) from (27) yields

$$\xi \cdot e \cdot \phi e \cdot b = \frac{\lambda + 1}{16\lambda}(\phi e \cdot \xi \cdot \xi \cdot S) + \frac{1 - \lambda^2}{4}(\phi e \cdot S) +$$

$$+ 2b(\xi \cdot e \cdot b) + 2(1 - \lambda)b(e \cdot c).$$

(28)

On the other hand differentiation of (17) with respect to $\phi e$ using $\lambda^2 \neq 1$ (since $l \neq 0$) implies

$$(\lambda + 1)(\phi e \cdot \phi e \cdot b) = \frac{\lambda + 1}{16\lambda}(\phi e \cdot \xi \cdot \xi \cdot S) + \frac{1 - \lambda^2}{4}(\phi e \cdot S) + 2(\lambda + 1)b(\phi e \cdot b).$$

Comparing the above relation with (28) we obtain

$$b = 0, \quad \xi \cdot e \cdot b = (\lambda + 1)\phi e \cdot b + (\lambda - 1)e \cdot c.$$  

(29)

If $b = 0$ Lemma 10 implies $c = 0$, therefore from Remark 11 we obtain $Q\phi = \phi Q$. In this case it has been proved ([5]) that $S = constant$, which means that (16) and (19) are trivial ($\Phi = 0$).

Differentiating (18) first with respect to $\xi$ (respectively to $\phi e$) and secondly with respect to $\phi e$ (respectively to $\xi$) and following the technique used to prove the relation (29) we can show that either $Q\phi = \phi Q$ or

$$\xi \cdot \phi e \cdot c = (\lambda + 1)\phi e \cdot b + (\lambda - 1)e \cdot c.$$  

(30)

We suppose that the second of (29) and (30) hold on $M^3$.

Using (6), (17) and (18) we obtain

$$\xi \cdot e \cdot b = \xi \cdot \phi e \cdot c = \frac{1}{8\lambda}(\xi \cdot \xi \cdot S) + \xi \cdot (bc).$$

From the above relation and (23) the relations (16) and (19) follow at once.

Now we compute $[e, \phi e]b$ (respectively $[e, \phi e]c$) in two ways, first using (16) and (17) (respectively (18), (19)) as $e \cdot \phi e \cdot b - \phi e \cdot e \cdot b$ (respectively $e \cdot \phi e \cdot c - \phi e \cdot e \cdot c$), and secondly through (4), (6), (16) and (17) as $(\nabla_e \phi e - \nabla_{\phi e} e)b$ (respectively (4), (6), (18) and (19) as $(\nabla_e \phi e - \nabla_{\phi e} e)c$. Comparing the two resulting expressions we obtain (22) (respectively (21)).

□
Theorem 15. Let $M^3$ be a 3-dimensional contact metric manifold with constant $\xi$-sectional curvature $k$ and constant $\phi$-sectional curvature $m$. Then, one of the following conditions holds:

(i) $M^3$ is Sasakian,
(ii) $Q\phi = \phi Q$, and $m = -k$,
(iii) $l = 0$,
(iv) $k + m = \frac{2}{3}$,
(v) $k + m = -2$.

Proof. We suppose that $M^3$ is a non-Sasakian manifold with $l$ being not identically equal to zero.

It is known ([5]) that on every 3-dimensional contact metric manifold $K(e, \phi e) = \frac{s}{2} - Trl$. Hence, this relation and Proposition 2 imply that $S = constant$. In this case the relations (16), (17), (18), (19), (21) and (22) take the form:

$$e \cdot b = bc + \Phi,$$
$$\phi e \cdot b = b^2 + \frac{1 - \lambda}{2} (\lambda^2 + \frac{S}{2} - 1),$$
$$e \cdot c = c^2 + \frac{1 + \lambda}{2} (\lambda^2 + \frac{S}{2} - 1),$$
$$\phi e \cdot c = bc - \Phi,$$
$$e \cdot \Phi = (\lambda + 1)(\lambda^2 + \frac{S}{2} - 3)b + 4c\Phi,$$
$$\phi e \cdot \Phi = (\lambda - 1)(\lambda^2 + \frac{S}{2} - 3)c + 4b\Phi.$$  

Computing $[e, \phi e]\Phi$ in two different ways (as in the last part of the proof of Proposition 14), using (4), (20), (35) and (36) we obtain

$$8\Phi^2 = (\lambda^2 + \frac{S}{2} - 3)[-4(\lambda + 1)b^2 + 4(\lambda - 1)c^2 + (1 - \lambda^2)(\lambda^2 + \frac{S}{2} - 1)].$$  

Differentiating (37) with respect to $e$ (respectively to $\phi e$) and taking into account (31), (33), (35) and (37) (respectively (32), (34), (36) and (37)) we have

$$((\lambda^2 + \frac{S}{2} - 3)[-(\lambda + 1)b^2c + (\lambda - 1)c^3 + \frac{1 - \lambda^2}{2}(\lambda^2 + \frac{S}{2} - 1)c + (\lambda + 1)b\Phi] = 0,$$

$$((\lambda^2 + \frac{S}{2} - 3)[-(\lambda + 1)b^3c + (\lambda - 1)b^2c + \frac{1 - \lambda^2}{2}(\lambda^2 + \frac{S}{2} - 1)b + (\lambda - 1)c\Phi] = 0.$$
Hence, either
\[ \lambda^2 + \frac{S}{2} - 3 = 0, \]
or
\[ (\lambda + 1)b\Phi = c[(\lambda + 1)b^2 + (1 - \lambda)c^2 + \frac{\lambda^2 - 1}{2} (\lambda^2 + \frac{S}{2} - 1)] = 0 \tag{38} \]
and
\[ (\lambda - 1)c\Phi = b[(\lambda + 1)b^2 + (1 - \lambda)c^2 + \frac{\lambda^2 - 1}{2} (\lambda^2 + \frac{S}{2} - 1)] = 0. \tag{39} \]

Suppose that \( \lambda^2 + \frac{S}{2} - 3 = 0 \), then using \( K(e, \phi e) = \frac{S}{2} - Trl, Trl = 2(1 - \lambda^2) \) and \( K(e, \xi) = \frac{Trl}{2} \), we obtain \( k + m = -2 \).

In this case using [16] we conclude that if \( k = -3 \) and \( m = 1 \), then \( M^3 \) is Sasakian. Also, for \( k + m = -2 \) and \( m > 1 \) we obtain a new class of contact metric 3-manifolds, which does not belong to the \((\kappa, \mu)\)-contact metric manifolds, ([4]).

Suppose now that (38) and (39) hold. If \( b = 0 \) (respectively \( c = 0 \)), then (6) implies \( c = 0 \) (respectively \( b = 0 \)) and therefore \( Q\phi = \phi Q \). In this case using [5] we have \( m = -k \). If \( bc \neq 0 \), multiplying (38) with \( b \) and (39) with \( c \) we obtain
\[ \Phi[(\lambda + 1)b^2 + (1 - \lambda)c^2] = 0. \]

Case A: \( \Phi = 0 \).

The relation (37) yields
\[ (\lambda + 1)b^2 + (1 - \lambda)c^2 + \frac{\lambda^2 - 1}{4} (\lambda^2 + \frac{S}{2} - 1) = 0. \]

On the other hand the relation (38) yields
\[ (\lambda + 1)b^2 + (1 - \lambda)c^2 + \frac{\lambda^2 - 1}{2} (\lambda^2 + \frac{S}{2} - 1) = 0. \]

Comparing the last two relations we obtain either \( \lambda^2 = 1 \), a contradiction because of the assumption that \( l \) is not identically equal to zero on \( M^3 \), or
\[ \lambda^2 + \frac{S}{2} - 1 = 0. \]

From \( \Phi = 0 \), (31), (32), (33) and (34) we obtain
\[ e \cdot b = \phi e \cdot c = bc, \quad \phi e \cdot b = b^2, \quad \phi e \cdot c = c^2. \tag{40} \]

Computing \([e, \phi e]b]\) in two ways (by use of (4) and (40)) and comparing the results we obtain \( \xi \cdot b = 0. \) Hence, from the assumption \( \lambda^2 \neq 1 \) and (6) we obtain \( b = c = 0 \), a contradiction.
Case B:

\[ \Phi \neq 0 \quad \text{and} \quad (\lambda + 1)b^2 + (1 - \lambda)c^2 = 0. \]  \hfill (41)

The relations (38), (39) and (41) with the assumption \( \lambda^2 \neq 1 \) yield

\[ b\Phi = \frac{\lambda - 1}{2}(\lambda^2 + \frac{S}{2} - 1)c, \]  \hfill (42)

\[ c\Phi = \frac{\lambda + 1}{2}(\lambda^2 + \frac{S}{2} - 1)b. \]  \hfill (43)

On the other hand (37) and (41) imply

\[ 8\Phi^2 = (\lambda^2 + \frac{S}{2} - 3)(1 - \lambda^2)(\lambda^2 + \frac{S}{2} - 1). \]

Hence, \( \Phi = \text{constant} \). This conclusion and the relations (35) and (36) yield

\[ 4b\Phi = (1 - \lambda)(\lambda^2 + \frac{S}{2} - 3)c, \]  \hfill (44)

\[ 4c\Phi = -(\lambda + 1)(\lambda^2 + \frac{S}{2} - 3)b. \]  \hfill (45)

Comparing (42) with (44) or (43) with (45) we obtain

\[ \lambda^2 + \frac{S}{2} = \frac{5}{3}. \]

Taking into account the last relation, \( K(e, \phi e) = \frac{S}{2} - Trl, Trl = 2(1 - \lambda^2) \) and \( K(e, \xi) = \frac{Trl}{2} \), we obtain \( k + m = \frac{2}{3} \).

In this case using [16] we conclude that if \( k = 1 \) and \( m = -\frac{1}{3} \), then \( M^3 \) is Sasakian. Also, for \( k + m = \frac{2}{3} \) and \( m > -\frac{1}{3} \) we obtain a new class of contact metric 3-manifolds, which does not belong to the \((\kappa, \mu)\)-contact metric manifolds, ([4]).

\[ \square \]

References


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