Osculating Plane
Preserving Diffeomorphisms

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For someone familiar with the notion of self-parallel group for an immersion into euclidean space [3], [6] it is only natural to wonder what happens if, in the case of space curves, normal planes are replaced by, say, osculating planes. We give here a necessary and sufficient condition for the non-triviality of the osculating group of a simple space curve. This is, in the new situation, the group corresponding to the self-parallel group.

Problems of a similar nature have also been considered in [1] and [2].

1.

In what follows $X$ will stand for $R$ or $S^1$ and we will be dealing with smooth space curves, that is, $C^\infty$ immersions $f : X \to R^3$. In the case $X = S^1$, we will write $f$ for both $f : S^1 \to R^3$ and $f \circ \exp$, with $\exp (t) = (\cos 2\pi t, \sin 2\pi t)$. It will be clear from the context which one we are considering.

The curvature $k_f$ of $f$ is assumed not to vanish and $(T_f, N_f, B_f)$ will denote the Frenet-Serret frame. Also we do not assume parametrization by arc-length and denote by $v_f$ the velocity of $f$.

Definition 1. The osculating group $O(f)$ of $f : X \to R^3$ is the subgroup of $\text{Diff} (X)$ formed by the diffeomorphisms $\delta : X \to X$ such that, for $x \in X$, the osculating planes of $f$ at $x$ and $\delta(x)$ coincide.

If $f$ is a plane curve then $O(f)$ is $\text{Diff} (X)$ precisely.

Proposition 1. Let $f : X \to R^3$ be a smooth curve with non-vanishing curvature and torsion. Then $O(f)$ is
a) cyclic of finite order if $X = S^1$,
b) trivial or not finite if $X = R$.

Proof. The proof is almost a duplicate of proofs given in [1], [2]. It is only included for completeness.

Denote by $A^3_2$ the open Grassmannian of affine planes in $R^3$ and define $\tilde{O} : X \rightarrow A^3_2$, where $\tilde{O}(x)$ is the osculating plane at $x$. Since we are assuming non-vanishing torsion $\tilde{O}$ is an immersion. This fact implies that the action $\phi : O(f) \times X \rightarrow X$, with $\phi(\delta, x) = \delta(x)$, is properly discontinuous.

In fact let $\delta \in O(f)$ and suppose that $x \in X$ is such that $\delta(x) = x$. Since $\tilde{O}$ is an immersion there is an open neighbourhood $U$ of $x$ such that $\tilde{O} | U$ is injective. Then, for $y \in U \cap \delta^{-1}(U)$, $\delta(y) = y$ because $\tilde{O}(\delta(y)) = \tilde{O}(y)$. Therefore the fixed point set $\Delta$ of $\delta$ is open. Since $\Delta$ is also closed it follows that either $\delta$ has no fixed points or is the identity. Consequently the action of $O(f)$ on $X$ is free.

Furthermore if $U \cap \delta(U) \neq \emptyset$ then $\delta$ is the identity and $O(f)$ acts in a properly discontinuous way. Hence the projection $p : X \rightarrow X/O(f)$ is a covering projection and

\[ \pi_1(X/O(f), p(x))/\pi_1(X, x) \approx O(f) \] [4].

If $X = S^1$ then $X/O(f)$ is diffeomorphic to $S^1$ and it follows that $O(f)$ is cyclic of finite order.

Assume now that $X = R$ and that $O(f)$ is finite. Then $X/O(f)$ is either $R$ or $S^1$. Since $O(f) \approx \pi_1(X/B(f))$ it follows that it must be trivial. \hfill \Box

We will also use the tangent group $T(f)$ formed by the diffeomorphisms $\delta : X \rightarrow X$ such that, for $x \in X$, the tangent lines of $f$ at $x$ and $\delta(x)$ coincide. As above one can show that if the curvature $k_f$ never vanishes the natural action of $T(f)$ on $X$ is properly discontinuous.

2. Non-vanishing torsion

We start by recalling that a simple point for $f$ is a point $y \in X$ such that $f^{-1}(f(y)) = \{y\}$.

**Proposition 1.** Let $f : X \rightarrow R^3$ be a smooth curve with non-vanishing curvature. If $f$ has a simple point then $T(f)$ is trivial.

**Proof.** Let $\delta \in T(f)$. Then, for $x \in X$,

\[ < f(x) - f(\delta(x)) | N_f(x) > = < f(x) - f(\delta(x)) | B_f(x) > = 0. \]

It then follows that $k_f(x) v_f(x) < f(x) - f(\delta(x)) | T_f(x) > = 0$. Consequently $f(x) - f(\delta(x)) = 0$, for $x \in X$.

If $y \in X$ is a simple point then $y = \delta(y)$ and, since the action of $T(f)$ is properly discontinuous, $\delta = id_X$. \hfill \Box

**Proposition 2.** Let $f : X \rightarrow R^3$ be a smooth curve with non-vanishing curvature and torsion. If $f$ has a simple point then $O(f)$ is trivial.
Proof. Let $\delta \in O(f)$. From

$$f(x) - f(\delta(x)) = \alpha(x) T_f(x) + \beta(x) N_f(x)$$

one can conclude, by differentiation, that $\beta(x) = 0$, for $x \in X$. That is, $f(\delta(x))$ belongs to the tangent line of $f$ at $x$.

Since $B_f(x) = \pm B_f(\delta(x))$, for $x \in X$, we can conclude that also $T_f(x) = \pm T_f(\delta(x))$, for $x \in X$. Therefore $\delta \in T(f)$. By Proposition 1, $\delta = id_X$ and $O(f)$ is trivial. \qed

3. Plane arcs

From now on we will assume that $\tau_f$ vanishes but that the curve is not plane.

Lemma 1. Let $f : X \to \mathbb{R}^3$ be a smooth, simple curve with non-vanishing curvature. If $x_0$ is a point such that $\tau_f(x_0) \neq 0$ and $\delta \in O(f)$ then $\delta(x_0) = x_0$.

Proof. There is an open interval $I$ containing $x_0$ where $\tau_f$ does not vanish. Then, for $x \in I$, $f(\delta(x)) = f(x) + \alpha(x) T_f(x)$.

Since $B_f(x) = \pm B_f(\delta(x))$, for $x \in X$, we also have $T_f(x) = \pm T_f(\delta(x))$. By differentiation, $\alpha(x) = 0$, for $x \in I$, and, due to the injectivity of $f$, $\delta | I = id_I$. \qed

Proposition 1. Let $f : X \to \mathbb{R}^3$ be a smooth, simple curve with non-vanishing curvature and such that $\tau_f$ vanishes but not everywhere. Then $O(f)$ is non-trivial if and only if $f$ has a plane arc.

Proof. Assume that $f : [a, b] \to \mathbb{R}^3$ is a plane arc for $f$. Without loss of generality we assume $0 < a < b < 1$.

Let $\delta$ be a diffeomorphism of $[a, b]$ such that $\delta(a) = a$, $\delta(b) = b$, $\delta'(a) = \delta'(b) = 1$ and $\delta^{(k)}(a) = \delta^{(k)}(b) = 0$, for $k \geq 2$. Any such $\delta$ can be extended to a diffeomorphism of $R$ by letting the extension be the identity outside $[a, b]$ if $X = R$ or, in the case $X = S^1$, by letting the extension $\overline{\delta}$ be the identity in $[0, 1] \setminus [a, b]$ and satisfy $\overline{\delta}(x + 1) = \overline{\delta}(x) + 1$. The resulting diffeomorphism or the diffeomorphism that it induces for $S^1$, in the case $X = S^1$, is then an element of $O(f)$.

Assume now that there are no plane arcs for $f$ and that $\delta \in O(f)$. If $x_0$ is such that $\tau_f(x_0) = 0$ then $x_0$ belongs to the topological closure of $A = \{ x \in X | \tau_f(x) \neq 0 \}$. Therefore there exists a sequence $(x_n), x_n \in A$, which converges to $x_0$. The sequence $(\delta(x_n))$ converges to $\delta(x_0)$. Since by Lemma 1 $\delta(x_n) = x_n, n \in N$, it follows that $\delta(x_0) = x_0$. Using Lemma 1 again, $\delta$ must be $id_X$. \qed

Examples of curves with plane arcs can be constructed using convenient bump functions [5].
References


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