Geometric Probabilities for Convex Bodies of Large Revolution in the Euclidean Space $E_3$ (II)

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Abstract. In this paper we solve problems of Buffon type for an arbitrary convex body of revolution and four different types of lattices.

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Buffon’s problem for an arbitrary convex body $K$ and a lattice of parallelograms in the Euclidean space $E_2$ has been investigated in [1]. In [5] this problem is considered for two different types of lattices in the space $E_2$ namely, for those lattices whose fundamental cell is a triangle or a regular hexagon. Buffon’s Needle Problem for a lattice of right-angled parallelepipeds in the $n$-dimensional Euclidean space was solved in [9]. In her dissertation, E. Bosetto has answered the corresponding questions for other types of lattices in the 3-dimensional space and for test bodies like the needle or the sphere. In [7] Buffon’s problem is solved for a lattice of right-angled parallelepipeds in the 3-dimensional space (which will be denoted here by $R_1$) and an arbitrary convex body of revolution. In the present paper we prove results of this type for arbitrary convex bodies of revolution and four types of lattices in $E_3$, considered also by E. Bosetto.

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Let \( K \) be an arbitrary convex body of revolution with centroid \( S \) and oriented axis of rotation \( d \). Clearly, the axis \( d \) is determined by the angle \( \theta \) between \( d \) and the \( z \)-axis and by the angle \( \varphi \) between the projection of \( d \) on the \( xy \)-plane and the \( x \)-axis and we express this by writing \( d = d(\theta, \varphi) \). If for a given \( d = d(\theta, \varphi) \), the body \( K \) is tangent to the \( xy \)-plane such that the centroid \( S \) lies in the upper half-space, we denote by \( p(\theta, \varphi) \) the distance from \( S \) to the \( xy \)-plane. Then the length of the projection of \( K \) on the \( z \)-axis is given by \( L(\theta, \varphi) = p(\theta, \varphi) + p(\pi - \theta, \varphi) \). Note that \( p(\theta, \varphi) \) does actually depend only on the angle \( \theta \) and moreover, since \( K \) is a body of revolution about the axis \( d \) the value \( p(\theta, \varphi) \) is invariant to any rotation about this axis, say by an \( \psi \). Now let \( F \) be a fundamental cell of the lattice \( \mathcal{R} \) and assume that the two 3-dimensional random variables defined by the coordinates of \( S \) and by the triple \((\theta, \varphi, \psi)\) are uniformly distributed in the cell \( F \) and in \([0, \pi] \times [0, 2\pi] \times [0, 2\pi]\) respectively. We are interested in the probability \( p_{K, \mathcal{R}} \) that the body \( K \) intersects the lattice \( \mathcal{R} \). Furthermore, we will assume, as it is done in all papers cited here, that the body \( K \) is small with respect to the lattice \( \mathcal{R} \). In order to recall briefly this concept, consider for fixed \((\theta, \varphi) \in [0, \pi] \times [0, 2\pi]\) the set of all points \( P \in F \) for which the body \( K \) with centroid \( P \) and rotation axis \( d = d(\theta, \varphi) \) does not intersect the boundary \( \partial F \) and let \( F(\theta, \varphi) \) be the closure of this open subset of \( F \). We say that the body \( K \) is small with respect to \( \mathcal{R} \), if the polyhedrons sides of \( F(\theta, \varphi) \) and \( F \) are then clearly pairwise parallel.

Denote by \( M_F \) the set of all test bodies \( K \) whose centroid \( S \) lies in \( F \) and by \( N_F \) the set of bodies \( K \) that are completely contained in \( F \). Of course, we can identify these sets with subsets of \( \mathbb{R}^6 \) and if \( \mu \) denotes the Lebesgue measure then the probability is given by

\[
(1) \quad p_{K, \mathcal{R}} = 1 - \frac{\mu(N_F)}{\mu(M_F)}.
\]

Using the cinematic measure (see [6])

\[
(2) \quad dK = dx \wedge dy \wedge dz \wedge d\Omega \wedge d\psi,
\]

where \( x, y, z \) are the coordinates of \( S \), \( d\Omega = \sin \theta d\theta \wedge d\varphi \) and \( \psi \) is an angle of rotation about \( d \) we can compute

\[
(3) \quad \mu(M_F) = \int_{\{x,y,z\in F\}} \sin \theta \, dx \wedge dy \wedge dz = 8\pi^2 \, \text{Vol}(F),
\]

\[
(4) \quad \mu(N_F) = \int_{\{x,y,z\in F(\theta,\varphi)\}} \sin \theta \, dx \wedge dy \wedge dz = 2\pi \int_{\{x,y\leq 0\}} \text{Vol}(\mathcal{F}(\theta,\varphi)) \cdot \sin \theta d\theta \, d\varphi,
\]

which leads to

\[
(1') \quad p_{K, \mathcal{R}} = 1 - \frac{1}{4\pi \, \text{Vol}(F)} \int_{\{x,y\leq 0\}} \left( \int_{\{x,y\leq 0\}} \text{Vol}(\mathcal{F}(\theta,\varphi)) \cdot \sin \theta d\theta \right) d\varphi.
\]
The above reasoning is valid for all lattices $\mathcal{R}$ provided $K$ is small with respect to the lattice. Our purpose here is “only” to show that for four different types of lattices that we denote as $\mathcal{R}_{2}, \mathcal{R}_{3}, \mathcal{R}_{4}, \mathcal{R}_{5},$ the volume of $\mathcal{F}(\theta, \varphi)$ can be expressed in terms of the well known support- and width-function ($p$ and $L$) associated to the body $K$ and to compute some of the integrals involved.

1. The lattice $\mathcal{R}_{2}$

The fundamental cell $\mathcal{F}_{2}$ of the lattice $\mathcal{R}_{2}$ is the parallelepiped spanned by the vectors $a$, $b$, $c$, where $c = (0,0,c)$ is perpendicular on $a = (a \sin \alpha, a \cos \alpha, 0)$ and $b = (0,b,0)$. We can assume without loss that the angle $\alpha$ between $a$ and $b$ belongs to $\left[0, \frac{\pi}{2}\right]$. One checks that $K$ is small with respect to $\mathcal{R}_{2}$ if and only if its diameter is less than $\min(a \sin \alpha, b \sin \alpha, c)$.

Recall that given $d = d(\theta, \varphi)$, $L(\theta, \varphi)$ denotes the length of the orthogonal projection of $K$ onto the $z$-axis. In order to simplify the expression for $\text{Vol } F_{2}(\theta, \varphi)$ we use the functions $\theta_{1}$, $\varphi_{1}$ and $\theta_{2}$, $\varphi_{2}$ defined as follows:

$$
\theta_{1}(\theta, \varphi) := \arccos(\sin \theta \cos \varphi), \quad \varphi_{1}(\theta, \varphi) := \arctan\left(\frac{\cot \theta}{\sin \varphi}\right),
$$

$$
\theta_{2}(\theta, \varphi) := \arccos\left(\sin \theta \sin \left(\varphi + \alpha - \frac{\pi}{2}\right)\right), \quad \varphi_{2}(\theta, \varphi) := \arctan\left(\tan(\sin(\varphi + \alpha))\right).
$$

Thus, for $d = d(\theta, \varphi)$, the length of the orthogonal projection of $K$ onto the $x$-axis is given by $L(\theta_{1}(\theta, \varphi), \varphi_{1}(\theta, \varphi))$ and also, the distance between the two planes that are parallel to the plane spanned by the vectors $a$ and $c$ and tangent to $K$ equals $L(\theta_{2}(\theta, \varphi), \varphi_{2}(\theta, \varphi))$. This implies

$$
\text{Vol } F_{2}(\theta, \varphi) = \left(a \sin \alpha - L\left(\theta_{1}(\theta, \varphi), \varphi_{1}(\theta, \varphi)\right)\right)\left(b - \frac{1}{\sin \alpha} L\left(\theta_{2}(\theta, \varphi), \varphi_{2}(\theta, \varphi)\right)\right)
$$

$$
\cdot \left(c - L(\theta, \varphi)\right)
$$

$$
= abc \sin \alpha - ab \sin \alpha \cdot \left(L(\theta, \varphi) - bc \cdot L\left(\theta_{1}(\theta, \varphi), \varphi_{1}(\theta, \varphi)\right)\right)
$$

$$
- ca \cdot L\left(\theta_{2}(\theta, \varphi), \varphi_{2}(\theta, \varphi)\right) + a \cdot L\left(\theta_{2}(\theta, \varphi), \varphi_{2}(\theta, \varphi)\right) \cdot L(\theta, \varphi)
$$

$$
+ b \cdot L(\theta, \varphi) \cdot L\left(\theta_{1}(\theta, \varphi), \varphi_{1}(\theta, \varphi)\right) + \frac{c}{\sin \alpha} \cdot L\left(\theta_{1}(\theta, \varphi), \varphi_{1}(\theta, \varphi)\right) \cdot L\left(\theta_{2}(\theta, \varphi), \varphi_{2}(\theta, \varphi)\right)
$$

$$
- \frac{1}{\sin \alpha} \cdot L(\theta, \varphi) \cdot L\left(\theta_{1}(\theta, \varphi), \varphi_{1}(\theta, \varphi)\right) \cdot L\left(\theta_{2}(\theta, \varphi), \varphi_{2}(\theta, \varphi)\right).
$$

From this we obtain

$$
\int_{0}^{2\pi} \int_{0}^{\pi} \text{Vol } F_{2}(\theta, \varphi) \sin \theta d\theta d\varphi = 4\pi abc \sin \alpha - ab \sin \alpha \int_{0}^{2\pi} \int_{0}^{\pi} L(\theta, \varphi) \sin \theta d\theta d\varphi
$$

$$
- bc \int_{0}^{2\pi} \int_{0}^{\pi} L\left(\theta_{1}(\theta, \varphi), \varphi_{1}(\theta, \varphi)\right) \sin \theta d\theta d\varphi - ca \int_{0}^{2\pi} \int_{0}^{\pi} L\left(\theta_{2}(\theta, \varphi), \varphi_{2}(\theta, \varphi)\right) \sin \theta d\theta d\varphi
$$
\[\begin{align*}
&+ a \int_0^{2\pi} \int_0^{2\pi} L(\theta_2(\theta, \varphi), \varphi_2(\theta, \varphi)) L(\theta, \varphi) \sin \theta d\theta d\varphi \\
&+ \frac{c}{\sin \alpha} \int_0^{2\pi} \int_0^{2\pi} L(\theta_1(\theta, \varphi), \varphi_1(\theta, \varphi)) L(\theta_2(\theta, \varphi), \varphi_2(\theta, \varphi)) \sin \theta d\theta d\varphi \\
&+ b \int_0^{2\pi} \int_0^{2\pi} L(\theta, \varphi) L(\theta_1(\theta, \varphi), \varphi_1(\theta, \varphi)) L(\theta_2(\theta, \varphi), \varphi_2(\theta, \varphi)) \sin \theta d\theta d\varphi \\
&- \frac{1}{\sin \alpha} \int_0^{2\pi} \int_0^{2\pi} L(\theta, \varphi) L(\theta_1(\theta, \varphi), \varphi_1(\theta, \varphi)) L(\theta_2(\theta, \varphi), \varphi_2(\theta, \varphi)) \sin \theta d\theta d\varphi,
\end{align*}\]

and by (1')

\[
(52) \quad p_{K, R_2} = \frac{1}{4\pi a \sin \alpha} \int_0^{2\pi} \int_0^{2\pi} L(\theta_1(\theta, \varphi), \varphi_1(\theta, \varphi)) \sin \theta d\theta d\varphi \\
+ \frac{1}{4\pi b \sin \alpha} \int_0^{2\pi} \int_0^{2\pi} L(\theta_2(\theta, \varphi), \varphi_2(\theta, \varphi)) \sin \theta d\theta d\varphi + \frac{1}{4\pi c} \int_0^{2\pi} \int_0^{2\pi} L(\theta, \varphi) \sin \theta d\theta d\varphi \\
- \frac{1}{4\pi b \sin \alpha} \int_0^{2\pi} \int_0^{2\pi} L(\theta_2(\theta, \varphi), \varphi_2(\theta, \varphi)) L(\theta, \varphi) \sin \theta d\theta d\varphi \\
- \frac{1}{4\pi ab \sin^2 \alpha} \int_0^{2\pi} \int_0^{2\pi} L(\theta_1(\theta, \varphi), \varphi_1(\theta, \varphi)) L(\theta_2(\theta, \varphi), \varphi_2(\theta, \varphi)) \sin \theta d\theta d\varphi \\
- \frac{1}{4\pi c \sin \alpha} \int_0^{2\pi} \int_0^{2\pi} L(\theta, \varphi) L(\theta_1(\theta, \varphi), \varphi_1(\theta, \varphi)) \sin \theta d\theta d\varphi \\
+ \frac{1}{4\pi abc \sin^2 \alpha} \int_0^{2\pi} \int_0^{2\pi} L(\theta, \varphi) L(\theta_1(\theta, \varphi), \varphi_1(\theta, \varphi)) L(\theta_2(\theta, \varphi), \varphi_2(\theta, \varphi)) \sin \theta d\theta d\varphi.
\]

Thus, we have proved:

**Theorem 1.** The probability \(p_{K, R_2}\) is given by the equality (52).

**Remarks.**
1) For \(\alpha = \frac{1}{2}\) one obtains (for the lattice \(R_1\)) the equality (1) in [7], since in this case the expression involved is symmetric in \(a, b\) and \(c\).
2) If \(K\) has constant width then the above result becomes

\[
\left( \frac{1}{a \sin \alpha} + \frac{1}{b \sin \alpha} + \frac{1}{c} \right) k - \left( \frac{1}{ab \sin^2 \alpha} + \frac{1}{bc \sin \alpha} + \frac{1}{ca \sin \alpha} \right) k^2 + \frac{1}{abc \sin^2 \alpha} k^3.
\]
In the case of sphere this expression is exactly the right-hand side of the formula (1.21) in
[3].

3) If $\mathbf{K}$ is a needle of length $l < \min(a \sin \alpha, b \sin \alpha, c)$, we have $L(\theta, \varphi) = l|\cos \theta|$, which
implies $L(\theta_2(\theta, \varphi), \varphi_2(\theta, \varphi)) = l|\sin \theta \cos(\varphi + \alpha)|$ and $L(\theta_1(\theta, \varphi), \varphi_1(\theta, \varphi)) = l|\sin \theta \cos \varphi|$ and
the computations give the same result as in formula (1.13) in [3], i.e.,

$$p_{\mathbf{K, R}_2} = \frac{ab \sin \alpha + ac + bc}{2abc \sin \alpha} \cdot \frac{l - 2}{l^2} \cdot \frac{a + b + \left[1 + (\frac{\pi}{2} - \alpha)\cot \alpha\right]c}{3\pi abc \sin \alpha} \cdot l^2 + \frac{1 + (\frac{\pi}{2} - \alpha)\cot \alpha}{4\pi abc \sin \alpha} \cdot l^3.$$

2. The lattice $\mathcal{R}_3$

The fundamental cell $\mathcal{F}_3$ of the lattice $\mathcal{R}_3$ is the parallelepiped spanned by the vectors $\mathbf{a} = (a \sin \alpha, a \cos \alpha, 0)$, $\mathbf{b} = (0, b, 0)$ and $\mathbf{c}$ (with $\|\mathbf{c}\| = c$). Let $\alpha, \beta$ and $\gamma$ the angles between $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{b}$ and $\mathbf{c}$ and $\mathbf{c}$ and $\mathbf{a}$ respectively. We can assume without loss that all three angles
belong to the interval $\left[0, \frac{\pi}{2}\right]$. We denote also by $E_1, E_2$ and $E_3$ the planes spanned by $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{c}$ and $\mathbf{a}$ and $\mathbf{a}$ and $\mathbf{b}$ respectively. Of course, $E_3$ is the $xy$-plane. Further, if $\xi_{ij}$ with
$0 < \xi_{ij} \leq \frac{\pi}{2}$ is the angle between $E_i$ and $E_j$ then $d_1 = a \sin \xi_{13} \sin \alpha = a \sin \xi_{12} \sin \gamma$, $d_2 = b \sin \xi_{12} \sin \beta = b \sin \xi_{23} \sin \gamma$ and $d_3 = c \sin \xi_{23} \sin \gamma = c \sin \xi_{13} \sin \beta$ are the heights of the
parallelepiped. Note that ($\alpha, \beta, \gamma$) is uniquely determined by $\xi_{12}, \xi_{23}, \xi_{13}$ and viceversa. Thus, we can write $\mathcal{R}_3$ as a union of lattices of parallel equidistant planes denoted by $\mathcal{E}_1^1$, $\mathcal{E}_2^2$ and $\mathcal{E}_3^3$ such that the
distance between the planes of $\mathcal{E}_i^j$ equals $d_i$. The normal vector to $E_3$ is $\mathbf{n}_3 = (0, 0, 1)$. As we did before, we denote by $\theta$ and $\varphi$ the angles between $\mathbf{d}$ and $\mathbf{n}_3$ and
between $(1, 0, 0)$ and the projection of $\mathbf{d}$ on $E_3$.

Let $\mathbf{c}'$ be the orthogonal projection of $\mathbf{c}$ on the $xz$-plane and $\mathbf{c}_1 = \frac{1}{\|\mathbf{c}'\|}\mathbf{c}' = (\cos \xi_{13}, 0, \sin \xi_{13})$.

The vector $\mathbf{n}_1 = (\sin \xi_{13}, 0, -\cos \xi_{13})$ is orthogonal to $E_1$ and $(\mathbf{b}, \mathbf{c}_1, \mathbf{n}_1)$ is a (positively
oriented) triple of orthonormal vectors. Let $\theta_1$ and $\varphi_1$ be the angles formed by $\mathbf{d}$ and $\mathbf{n}_1$ and the
projection of $\mathbf{d}$ on $E_1$ and $\mathbf{b}$. We have

$$\theta_1 = \theta_1(\theta, \varphi) = \arccos(\sin \xi_{13} \sin \theta \cos \varphi - \cos \xi_{13} \cos \theta),$$

$$\varphi_1 = \varphi_1(\theta, \varphi) = \arctan\left(\cos \xi_{13} \cot \varphi + \frac{\sin \xi_{13} \cot \theta}{\sin \varphi}\right).$$

$x \sin \xi_{23} \cos \alpha - y \sin \xi_{23} \sin \alpha + z \cos \xi_{23} = 0$ is an equation for the plane $E_2$. The corresponding
normal vector is $\mathbf{n}_2 = (\sin \xi_{23} \cos \alpha, -\sin \xi_{23} \sin \alpha, \cos \xi_{23})$. The vectors $\mathbf{c}_2 = (-\cos \xi_{23} \cos \alpha, 
\cos \xi_{23} \sin \alpha, \sin \xi_{23})$, $\mathbf{a}$ and $\mathbf{n}_2$ form a positively oriented triple of orthogonal vectors. If we
consider the angles $\theta_2$ and $\varphi_2$ between $\mathbf{d}$ and $\mathbf{n}_2$ and between the projection of $\mathbf{d}$ on $E_2$ and $\mathbf{c}_2$ we have

$$\theta_2 = \theta_2(\theta, \varphi) = \arccos(-\sin \xi_{23} \sin \theta \cos(\varphi + \alpha) - \cos \xi_{23} \cos \theta),$$

$$\varphi_2 = \varphi_2(\theta, \varphi) = \arctan\left(\frac{\sin \theta \sin(\alpha + \varphi)}{\sin \xi_{23} \cos \theta - \sin \theta \cos \xi_{23} \cos(\alpha + \varphi)}\right).$$

The parallelepiped $\mathcal{F}_3$ has the volume

$$\text{Vol } \mathcal{F}_3 = ab \sin \alpha \cdot d_3 = abc \sin \alpha \sin \gamma \sin \xi_{23}$$

$$= \frac{d_1 d_2 d_3}{\sin \xi_{13}} = \frac{abc}{\sin \xi_{13} \sin \xi_{23} \sin \alpha}.$$
Now when $K$ is small with respect to $R_3$, that is, when the diameter \( \sup L(\theta, \varphi) \) of $K$ is smaller than \( \min(d_1, d_2, d_3) \), then $F_3(\theta, \varphi)$ is at its turn a parallelepiped whose faces and sides are parallel to the corresponding faces and sides of $F_3$ for all values $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi]$. The heights of $F_3(\theta, \varphi)$ are given by

$$d_1(\theta, \varphi) = d_1 - L(\theta_1, \varphi_1), \quad d_2(\theta, \varphi) = d_2 - L(\theta_2, \varphi_2), \quad d_3(\theta, \varphi) = d_3 - L(\theta, \varphi).$$

Then $\text{Vol}_3(\theta, \varphi) = \frac{d_1(\theta, \varphi)d_2(\theta, \varphi)d_3(\theta, \varphi)}{\sin \xi_{13} \sin \xi_{23} \sin \alpha}$ and from (1') we get

$$p_{K,R_3} = 1 - \frac{1}{4\pi} \int_0^\pi \int_0^\pi \text{Vol}_3(\theta, \varphi) \sin \theta d\theta d\varphi$$

$$= 1 - \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \left[ 1 - \frac{L(\theta_1, \varphi_1)}{d_1} - \frac{L(\theta_2, \varphi_2)}{d_2} - \frac{L(\theta, \varphi)}{d_3} + \frac{L(\theta_1, \varphi_1)L(\theta_2, \varphi_2)}{d_1d_2} + \frac{L(\theta_1, \varphi_1)L(\theta_2, \varphi_2)}{d_1d_2} \right] \sin \theta d\theta d\varphi.$$

We have proved

**Theorem 2.** If $K$ is small with respect to $R_3$, the probability $p_{K,R_3}$ is given by

$$p_{K,R_3} = \frac{1}{4\pi} \left[ \frac{1}{d_1} \int_0^\pi L(\theta_1, \varphi_1) \sin \theta d\theta d\varphi + \frac{1}{d_2} \int_0^\pi L(\theta_2, \varphi_2) \sin \theta d\theta d\varphi \right.$$}

$$+ \frac{1}{d_3} \int_0^\pi L(\theta, \varphi) \sin \theta d\theta d\varphi - \frac{1}{d_1d_2} \int_0^\pi \int_0^\pi L(\theta_1, \varphi_1)L(\theta_2, \varphi_2) \sin \theta d\theta d\varphi$$

$$- \frac{1}{d_2d_3} \int_0^\pi \int_0^\pi L(\theta_2, \varphi_2)L(\theta, \varphi) \sin \theta d\theta d\varphi - \frac{1}{d_3d_1} \int_0^\pi \int_0^\pi L(\theta, \varphi) L(\theta_1, \varphi_1) \sin \theta d\theta d\varphi$$

$$+ \frac{1}{d_1d_2d_3} \int_0^\pi \int_0^\pi L(\theta, \varphi)L(\theta_1, \varphi_1)L(\theta_2, \varphi_2) \sin \theta d\theta d\varphi \right].$$

**Remarks.** 1) The result is a generalization of Theorem 1 which is obtained for $\xi_{13} = \xi_{23} = \frac{\pi}{2}$, $\beta = \gamma = \frac{\pi}{2}$.

2) If $K$ has constant width $k < \min (d_1, d_2, d_3)$ we obtain the special case

$$p_{K,R_3} = \left( \frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} \right) k - \left( \frac{1}{d_1d_2} + \frac{1}{d_2d_3} + \frac{1}{d_3d_1} \right) k^2 + \frac{k^3}{d_1d_2d_3}.$$

3) For a needle of length $l < \min (d_1, d_2, d_3)$ one can find more detailed computations in [2].
3. The lattice $\mathcal{R}_4$

The fundamental cell $F_4$ of the lattice $\mathcal{R}_4$ is a right-angled prism whose base $B_4$ is a right-angled triangle with catheti $a$ and $b$. If $c$ is the height of the prism, then we can assume that the vertices of $F_4$ are $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, $(a, 0, c)$ and $(0, b, c)$. We denote $\gamma := \arctan \frac{b}{a}$ and $h := \frac{ab}{\sqrt{a^2 + b^2}}$. The body $K$ is small with respect to $\mathcal{R}_4$ if

$$\text{Diam}(K) < \min \left( \frac{3ab}{2(a + b + \sqrt{a^2 + b^2})} \right)$$

(see [6]). In this case the set $\mathcal{F}_4(\theta, \varphi)$ is also a right-angled prism with height $c - L(\theta, \varphi)$, and whose base $B_4(\theta, \varphi)$ is a right-angled triangle. We denote by $p_1$, $p_2$ and $p_3$ the lengths $p(\theta_1(\theta, \varphi), \varphi_1(\theta, \varphi))$, $p(\theta_2(\theta, \varphi), \varphi_2(\theta, \varphi))$ and $p(\theta_3(\theta, \varphi), \varphi_3(\theta, \varphi))$. Let $\theta_1, \varphi_1, \theta_2, \varphi_2, \theta_3$ and $\varphi_3$ be the functions defined by

$$\begin{align*}
\theta_1(\theta, \varphi) &:= \arccos(\sin \theta \cos \varphi), \quad \varphi_1(\theta, \varphi) := \arctan \left( \frac{\cot \theta}{\sin \varphi} \right), \\
\theta_2(\theta, \varphi) &:= \arccos(\sin \theta \sin \varphi), \quad \varphi_2(\theta, \varphi) := \arctan(\tan \theta \cos \varphi), \\
\theta_3(\theta, \varphi) &:= \arccos(-\sin \theta \sin(\varphi + \gamma)), \quad \varphi_3(\theta, \varphi) := \arccot(-\tan \theta \cos(\varphi + \gamma)).
\end{align*}$$

By a simple geometric argument (see e.g. [2]) is follows that

$$\frac{\text{Area } B_4(\theta, \varphi)}{\text{Area } B_4} = \left(1 - \frac{p_1}{a} - \frac{p_2}{b} - \frac{p_3}{h}\right)^2.$$
Using also the fact that $L(\theta, \varphi) = L$ we obtain

$$
\frac{\text{Vol } \mathcal{F}_4(\theta, \varphi)}{\text{Vol } \mathcal{F}_4} = \left(1 - \frac{p_1}{a} - \frac{p_2}{b} - \frac{p_3}{h}\right)^2 \left(1 - \frac{L}{c}\right).
$$

We now prove

**Theorem 3.** The probability $p_{K, R_4}$ is given by

$$
(5.4) \quad p_{K, R_4} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \left(\frac{p_1}{a} + \frac{p_2}{b} + \frac{p_3}{h} + \frac{L}{2c}\right) \sin \theta d\theta d\varphi
$$

$$
- \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \left(\frac{p_1 p_2}{ab} + \frac{p_2 p_3}{bh} + \frac{p_3 p_1}{ha} + \frac{p_1 L}{ac} + \frac{p_2 L}{bc} + \frac{p_3 L}{hc}\right) \sin \theta d\theta d\varphi
$$

$$
- \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \left(\frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} + \frac{p_3^2}{h^2}\right) \sin \theta d\theta d\varphi
$$

$$
+ \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi \left(\frac{p_1 p_2 L}{abc} + \frac{p_2 p_3 L}{bhc} + \frac{p_3 p_1 L}{hac}\right) \sin \theta d\theta d\varphi
$$

$$
+ \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \left(\frac{p_1^2 L}{a^2 c} + \frac{p_2^2 L}{b^2 c} + \frac{p_3^2 L}{h^2 c}\right) \sin \theta d\theta d\varphi.
$$

**Proof.** We have

$$
\left(1 - \frac{p_1}{a} - \frac{p_2}{b} - \frac{p_3}{h}\right)^2 \left(1 - \frac{L}{c}\right) = 1 - 2\left(\frac{p_1}{a} + \frac{p_2}{b} + \frac{p_3}{h} + \frac{L}{2c}\right)
$$

$$
+ 2\left(\frac{p_1 p_2}{ab} + \frac{p_2 p_3}{bh} + \frac{p_3 p_1}{ha} + \frac{p_1 L}{ac} + \frac{p_2 L}{bc} + \frac{p_3 L}{hc}\right) + \frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} + \frac{p_3^2}{h^2}
$$

$$
- 2\left(\frac{p_1 p_2 L}{abc} + \frac{p_2 p_3 L}{bhc} + \frac{p_3 p_1 L}{hac}\right) - \left(\frac{p_1^2 L}{a^2 c} + \frac{p_2^2 L}{b^2 c} + \frac{p_3^2 L}{h^2 c}\right)
$$

and from (1') we obtain (5.4).

**Remarks.** 1) In the case when $K$ is a needle of length $l < \min (h, c)$ one can deduce from (5.4), after some tedious calculations, the result of Theorem 1.3.3 in [3].

2) In the case when $K$ is a sphere of radius $r < \min \left(\frac{c}{2}, \frac{ab}{a + b + \sqrt{a^2 + b^2}}\right)$, one obtains the probability

$$
2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{h} + \frac{1}{c}\right) r - 2\left(\frac{1}{ab} + \frac{1}{bh} + \frac{1}{ha}\right) r^2 - 4\left(\frac{1}{ac} + \frac{1}{bc} + \frac{1}{hc}\right) r^2
$$

$$
- \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{h^2}\right) r^2 + 4\left(\frac{1}{abc} + \frac{1}{bhc} + \frac{1}{hac}\right) r^3 + 2\left(\frac{1}{a^2 c} + \frac{1}{b^2 c} + \frac{1}{h^2 c}\right) r^3,
$$
which can be shown to be equivalent to the formula (1.23) in [3].

4. The lattice $\mathcal{R}_5$

The fundamental cell $\mathcal{F}_5$ of the lattice $\mathcal{R}_5$ is a right-angled prism whose base $\mathcal{T}_5$ is a right-angled trapezoid, as it is shown in the figure below.

The convex body $K$ is small with respect to $\mathcal{R}_5$ if it satisfies the inequality $\text{Diam}(K) < \min(a - b \cot \gamma, b, c)$. In this case $\mathcal{F}_5(\theta, \varphi)$ is again a right-angled prism having the height $c - L(\theta, \varphi)$ (or in short form $c - L$) and the trapezoid $\mathcal{T}_5(\theta, \varphi)$ as a base. Using the notations from the previous section, we have again that the prism is completely determined by the distances $p_1, p_2, p_3$ and $p_2' = p(\pi - \theta_2, \varphi_2)$:
If we denote \( L := L(\theta, \varphi) \) and \( L_2 := p_2 + p'_2 \) we can write

\[
\text{Area } T_5(\theta, \varphi) = (b - p_2 - p'_2) \left( a - b \cot \gamma - p_1 - \frac{p_2 - p'_2}{2} \cot \gamma - \frac{p_3}{\sin \gamma} \right) = \\
\text{Area } T_5 - b \left( p_1 + \frac{p_3}{\sin \gamma} \right) + \frac{b}{2} (p_2 - p'_2) \cot \gamma - \left( a - \frac{b}{2} \cot \gamma \right) L_2 + \frac{1}{2} (p_2 - p'_2) \cot \gamma \\
+ L_2 \left( p_1 + \frac{p_3}{\sin \gamma} \right),
\]

\[
\text{Vol } F_5(\theta, \varphi) = (c - L) \text{ Area } T_5(\theta, \varphi) = \text{Vol } F_5 - bc \left( p_1 + \frac{p_3}{\sin \gamma} \right) \\
+ \frac{bc}{2} (p'_2 - p_2) \cot \gamma - \left( a - \frac{b}{2} \cot \gamma \right) c L_2 + \frac{c}{2} (p'_2 - p_2) \cot \gamma \\
- \left( a - \frac{b}{2} \cot \gamma \right) b L + b \left( p_1 + \frac{p_3}{\sin \gamma} \right) L - \frac{b}{2} (p'_2 - p_2) L \cot \gamma + c L_2 \left( p_1 + \frac{p_3}{\sin \gamma} \right) \\
+ \left( a - \frac{b}{2} \cot \gamma \right) L L_2 - \frac{1}{2} (p'_2 - p_2) L \cot \gamma - L L_2 \left( p_1 + \frac{p_3}{\sin \gamma} \right).
\]

Using now (1') and the equalities

\[
\int_{0}^{2\pi} \int_{0}^{\pi} p'_2 \sin \theta d\theta d\varphi = \int_{0}^{2\pi} \int_{0}^{\pi} p''_2 \sin \theta d\theta d\varphi, \quad i = 1, 2, \\
\int_{0}^{2\pi} \int_{0}^{\pi} p'_2 L \sin \theta d\theta d\varphi = \int_{0}^{2\pi} \int_{0}^{\pi} p''_2 L \sin \theta d\theta d\varphi, \quad i = 1, 2
\]

we obtain a proof of the following result.

**Theorem 4.** The probability \( p_{K, R_s} \) that a uniformly distributed convex body of revolution \( K \), which is small with respect to \( R_5 \), hits \( R_5 \) is

\[
(5) \quad p_{K, R_s} = \frac{1}{4\pi} \left[ - \frac{1}{a - \frac{b}{2} \cot \gamma} \int_{0}^{2\pi} \int_{0}^{\pi} \left( p_1 + \frac{p_3}{\sin \gamma} \right) \sin \theta d\theta d\varphi + \frac{1}{b} \int_{0}^{2\pi} \int_{0}^{\pi} L_2 \sin \theta d\theta d\varphi \\
+ \frac{1}{c} \int_{0}^{2\pi} \int_{0}^{\pi} L \sin \theta d\theta d\varphi - \frac{1}{a - \frac{b}{2} \cot \gamma} c \int_{0}^{2\pi} \int_{0}^{\pi} \left( p_1 + \frac{p_3}{\sin \gamma} \right) L \sin \theta d\theta d\varphi \\
- \frac{1}{a - \frac{b}{2} \cot \gamma} b \int_{0}^{2\pi} \int_{0}^{\pi} L_2 \left( p_1 + \frac{p_3}{\sin \gamma} \right) \sin \theta d\theta d\varphi - \frac{1}{bc} \int_{0}^{2\pi} \int_{0}^{\pi} LL_2 \sin \theta d\theta d\varphi \\
+ \frac{1}{a - \frac{b}{2} \cot \gamma} bc \int_{0}^{2\pi} \int_{0}^{\pi} LL_2 \left( p_1 + \frac{p_3}{\sin \gamma} \right) \sin \theta d\theta d\varphi \right].
\]

**Remarks.** 1) In the case \( K \) is a sphere of radius \( r \), the conditions for \( K \) to be small with respect to \( R_5 \) can be weakened; the upper bound \( a - b \cot \gamma \) can be replaced by the larger number \( \frac{2a - b \cot \gamma}{1 + \tan \frac{\gamma}{2}} \), and the condition in the theorem becomes

\[
2r < \min \left( \frac{2}{1 + \tan \frac{\gamma}{2}}, b, c \right).
\]
From (5.5) we obtain
\[
P_{K,R_5} = \frac{1 + \frac{1}{a-b \cot \gamma}}{a - \frac{1}{2} b \cot \gamma} r + \frac{2 \pi}{b} + \frac{2 \pi}{c} - 2 \frac{1 + \frac{1}{a-b \cot \gamma}}{a - \frac{1}{2} b \cot \gamma} b r^2
- 2 \frac{1 + \frac{1}{a-b \cot \gamma}}{a - \frac{1}{2} b \cot \gamma} c r^2 - 4 \frac{r^2}{bc} + 4 \frac{1 + \frac{1}{a-b \cot \gamma}}{a - \frac{1}{2} b \cot \gamma} bc r^3.
\]

The same result follows from the formula (1.24) from [3] after some manipulations.

2) If $K$ is a needle of length $l < \min (a - b \cot \gamma, b, c)$ then one can use (5.5) to deduce the formula (1.18) in [3], however some integrals are to be computed for this purpose.

References


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