The Gelfand-Kirillov Dimension of Rings with Hopf Algebra Action

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Abstract. Let $k$ be a perfect field, $H$ a irreducible cocommutative Hopf $k$-algebra and $P(H)$ the space of primitive elements of $H$, $R$ a $k$-algebra on which acts locally finitely $H$ and $R\#H$ the associated smash product. Assume that $H$ is almost solvable with $P(H)$ finite-dimensional $n$ and the sequences of divided powers are all infinite. Then the Gelfand-Kirillov dimension of $R\#H$ is $GK(R) + n$.

1. Introduction

It is well known [7], that if $\delta$ is a derivation of an algebra $R$ over a field $k$, then the Gelfand-Kirillov dimension of the polynomial algebra $R[\theta, \delta]$ is equal to $GK(R) + 1$, provided $R$ is $\delta$-locally-finite. More generally, if $g$ is a finite-dimensional $k$-Lie algebra acting locally finitely on $R$, then the Gelfand-Kirillov dimension of the differential operator ring $R\#U(g)$ is $GK(R) + dim_k(g)$ where $U(g)$ is the enveloping algebra of $g$ (see [5, Corollary 1.5]). The main objective of this note is to present a generalization of the above mentioned result to the case of a irreducible cocommutative Hopf algebra action. However, we assume that $H$ is almost solvable. Note that $U(g)$ is a irreducible cocommutative Hopf algebra.

The Gelfand-Kirillov dimension of $R$ (see [6] for the basic material), denoted $GK(R)$, is defined as follows (here $V^l$ is the linear span of all products $v_1v_2\cdots v_l$ with $v_1, v_2, \ldots, v_l \in V$):

$$GK(R) = \sup\{\limsup_{n \to \infty} (\log_n dim_k V^n : V \text{ is a finite-dimensional subspace of } R)\}.$$  

Throughout the paper, $k$ is a field, $H$ is a Hopf $k$-algebra with comultiplication $\Delta$, counit $\epsilon$ and antipode $s$, and $R$ is an $H$-module algebra (the action of $h \in H$ shall be denoted by $h.r$), i.e.
This paper accomplishes the following: Let \( k \) be a field, \( H \) be an \( H \)-module such that the multiplication in \( R \) is an \( H \)-module map, i.e., \( h(ab) = \sum (h_i) (h_1, a)(h_2, b) \) for all \( h \in H \) and \( a, b \in R \). We denote by \( R\#H \) the associated smash product. Both \( R \) and \( H \) are naturally embedded in \( R\#H \). The multiplication in \( R\#H \) is defined by the rule \( (a\#h)(b\#g) = \sum (h_i) a(h_1, b) \# h_2 g \). For further information on Hopf algebras and the ring \( R\#H \), the reader is referred to [1, 8 and 10]. We denote by \( P(H) \) the space of primitive elements of \( H \). We say that \( H \) is cocommutative if \( \Delta = \tau \circ \Delta \) where \( \tau \) is the usual twist map \( \tau(a \otimes b) = b \otimes a \). By [8, Corollary 1.5,12], the antipode of a cocommutative Hopf algebra is involutive. We say that \( H \) is irreducible if any two nonzero subcoalgebras of \( H \) have nonzero intersection.

If \( H \) is irreducible cocommutative, then so is any subHopfalgebra of \( H \); if the characteristic of \( k \) is 0, then \( H \) is the enveloping algebra of \( P(H) \).

Let \( X \) be an element of \( P(H) \). A sequence of divided powers over \( X \) of maximum length \( l \) possibly infinite is a sequence \( X^{(0)} = 1, X^{(1)} = X, \ldots, X^{(l)} \) such that \( X^{(i)} X^{(j)} = \left( \begin{array}{c} i + j \\ i \end{array} \right) X^{(i+j)} \) and \( \Delta(X^{(j)}) = \sum j=0 \ X^{(j')} \otimes X^{j-j'} \) for each \( i, j \leq l \). It follows routinely from the counitary property that \( \epsilon(X^{(l)}) = 0 \) for \( l > 0 \). If \( k \) has characteristic 0, then \( X^{(n)} = X^n / n! \).

If \( k \) is perfect and if \( H \) is irreducible with \( P(H) \) finite-dimensional \( n \), then by [11, Theorems 2, 3] and [12], \( H \) has a basis consisting of ordered monomials \( X_1^{(i_1)} X_2^{(i_2)} \cdots X_n^{(i_n)} ; i_j \in \mathbb{N} \); where \( (X_1, X_2, \ldots, X_n) \) is a basis for \( P(H) \).

**Examples 1.1.** (1) Let \( k \) be of characteristic 0, \( g \) a finite-dimensional \( k \)-Lie algebra of dimension \( n \) and \( H = U(g) \). Then \( H \) is an irreducible cocommutative Hopf algebra and \( P(H) = g \). Furthermore \( H \) has a basis consisting of ordered monomials \( X_1^{(i_1)} X_2^{(i_2)} \cdots X_n^{(i_n)} ; i_j \in \mathbb{N} \) as above and the sequences of divided powers are all infinite.

(2) Let \( k \) be perfect, \( G \) an affine algebraic group over \( k \) of dimension \( n \) and \( H = \text{hyp}(G) \) the hyperalgebra of \( G \). Then \( H \) is an irreducible cocommutative Hopf algebra and \( P(H) \) is the Lie algebra of \( G \). Furthermore \( H \) has a basis consisting of ordered monomials \( X_1^{(i_1)} X_2^{(i_2)} \cdots X_n^{(i_n)} ; i_j \in \mathbb{N} \) as above and the sequences of divided powers are all infinite.

This paper accomplishes the following: Let \( k \) be perfect, \( H \) irreducible cocommutative with \( P(H) \) finite-dimensional \( n \) and \( R H \)-locally finite. If the sequences of divided powers are all infinite and if \( H \) is almost solvable, then \( GK(R\#H) = GK(R) + n \).

**2. The main result**

We consider \( H \) as a left \( H \)-module by the left adjoint action, that is \( h.h' = \sum (h_i) h_1 h's(h_2) \). We say that a subHopfalgebra \( N \) of \( H \) is normal in \( H \) if \( h.n \in N \) for all \( h \in H, n \in N \). Let \( N \) be a normal subHopfalgebra of \( H \). There is a natural action of \( H \) on \( R\#N \) defined by \( h.(rn) = \sum (h_i) (h_1, r)(h_2, n) \).

The bracket product in \( H \) is defined by \( [x, y] = \sum x_1 y_1 s(x_2) s(y_2) \) for \( x, y \in H \).
If \( I, J \) are subHopf algebras of \( H \), \([I, J]\) denotes the subalgebra of \( H \) generated by the elements \([x, y]\) with \( x \in I \) and \( y \in J \); if \( H \) is cocommutative, this is a subbialgebra of \( H \).

We will say that \( I \) is central in \( H \) if \([H, I] = k \). Clearly, \( I \) is central in \( H \) if and only if \([x, y] = \epsilon(x)\epsilon(y) \) for all \( x \in H \) and \( y \in I \). If \( I \) is central in \( H \), then \( I \) is normal in \( H \).

Let \( G \) be a connected abelian algebraic group, then \( G \) is central in \( G \); so by [14, Corollary 3.4.15], \( \text{hyp}(G) \) is central in \( \text{hyp}(G) \); i.e., \( \text{hyp}(G) \) is a commutative Hopf algebra.

An ideal \( I \) of \( R \) is \( H \)-invariant if \( h.I \subseteq I \) for all \( h \in H \). Any ideal of \( R\#H \) is \( H \)-invariant.

We say that \( R \) is \( H \)-simple, if the only \( H \)-invariant ideals of \( R \) are \((0) \) and \( R \).

A proper \( H \)-invariant ideal \( Q \) of \( R \) is \( H \)-prime if, whenever \( I \) and \( J \) are \( H \)-invariant ideals of \( R \) with \( IJ \subseteq Q \) then either \( I \subseteq Q \) or \( J \subseteq Q \).

Any \( H \)-invariant prime ideal of \( R \) is \( H \)-prime. Let \( I \subseteq Q \) be \( H \)-invariant ideals of \( R \). If \( Q \) is \( H \)-prime, then \( Q/I \) is an \( H \)-prime ideal of \( R/I \). We say that the ring \( R \) is \( H \)-prime if the ideal \((0) \) is \( H \)-prime.

If \( Q \) is an \( H \)-prime ideal of \( R \), then \( R/Q \) is an \( H \)-prime ring. Any \( H \)-simple ring is \( H \)-prime. The \( H \)-invariant prime ideals of \( R\#H \) are precisely its \( H \)-prime ideals. If \( P \) is a prime ideal of \( R\#H \) then \( P \cap R \) is an \( H \)-prime ideal of \( R \) (see [4, Lemma 1.2]).

We say that \( R \) is \( H \)-locally finite if every element of \( R \) is contained in a finite-dimensional \( H \)-stable subspace of \( R \). If \( H \) acts trivially on \( R \) then \( R \) is \( H \)-locally finite; in particular, if \( H \) is commutative, \( H \) is \( H \)-locally finite. If \( R \) and \( H \) are \( H \)-locally finite, then \( R\#H \) is \( H \)-locally finite. By [13, page 259], if \( p > 0 \) and if \( H \) is irreducible cocommutative with \( P(H) \) finite-dimensional, then \( H \) is the union of its finite-dimensional normal subHopf algebras; so \( H \) is \( H \)-locally finite; hence any normal subHopf algebra of \( H \) is \( H \)-locally finite. Clearly, \( R \) is \( g \)-locally finite as in [5, section 1] if and only if \( R \) is \( U(g) \)-locally finite.

**Lemma 2.1.** Let \( G \) be a connected algebraic group acting rationally on \( R \) and \( H = \text{hyp}(G) \) the hyperalgebra of \( G \). Then \( R \) is \( H \)-locally finite.

**Proof.** Let \( a \in R \). Since \( R \) is a rational \( G \)-module, there exists a finite dimensional \( G \)-stable subspace \( V \) of \( R \) such that \( a \in V \). By [14, Corollary 3.4.17], \( V \) is also \( H \)-stable. \( \square \)

From now on \( k \) is perfect and \( H \) is irreducible cocommutative with \( P(H) \) finite-dimensional \( n \). So \( H \) has a basis consisting of ordered monomials \( X_1^{(i_1)}X_2^{(i_2)}\cdots X_n^{(i_n)} \); \( i_j \in \mathbb{N} \); where \((X_1, X_2, \ldots X_n) \) is a basis for \( P(H) \). This basis will be fixed in the remainder of the paper.

We will say that \( H \) is almost solvable if there exists a chain of subHopf algebras

\[ k = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n = H \]

of \( H \) such that for each \( i \leq n \), \( H_{i-1} \) is normal in \( H_i \) and the monomials \( X_1^{(j_1)}X_2^{(j_2)}\cdots X_i^{(j_i)} \); \( j_i \in \mathbb{N} \) form a basis for \( H_i \).

Thus \( H \) commutative implies \( H \) almost solvable; in particular, if \( \text{dim}_k(P(H)) = 1 \), then \( H \) is almost solvable. Let \( g \) be as in Examples 0.1 (1), then \( U(g) \) is almost solvable if \( g \) is solvable in the usual sense. Let \( G \) be a connected affine algebraic group, then \( \text{hyp}(G) \) is almost solvable.

**Lemma 2.2.** Let \( G \) be a connected affine algebraic group and \( H = \text{hyp}(G) \). If \( G \) is unipotent then \( H \) is almost solvable.
Proof. It is well known that $G$ has a composition series

$$1 = G_0 \subset G_1 \cdots \subset G_{n-1} \subset G_n = G$$

where each $G_i$ is normal in $G$ and each $G_i/G_{i-1}$ is isomorphic to $G_a$, the one-dimensional additive group. Set $H_i = hyp(G_i)$, then $H_0 = k$ and $H_n = H$. By [14, Corollary 3.4.15], each $H_i$ is a normal subHopf algebra of $H$. Since $P(H)$ is nilpotent, there exists an element $X_i \in P(H_i) - P(H_{i-1})$ such that $(X_1, X_2, \ldots, X_{i-1}, X_i)$ is a basis for $P(H_i)$. By [11, Theorems 2, 3] and [12], the monomials $X_1^{(j_1)}X_2^{(j_2)} \cdots X_i^{(j_i)}; \ j_i \in \mathbb{N}$ form a basis for $H_i$, where the $X_i^{(j_i)}$ are infinite sequences of divided powers over $X_i$. □

We are now ready to prove the main result of the paper.

**Theorem 2.3.** Let $k$ be a perfect field, $H$ a irreducible cocommutative almost solvable Hopf algebra with $P(H)$ finite-dimensional $n$ and $R$ an $H$-locally finite $H$-module algebra. Assume that the sequences of divided powers are all infinite. Then

$$GK(R\#H) = GK(R) + n.$$  

Proof. Suppose that $n = 1$ and set $g = P(H)$. So $H$ has a basis consisting of ordered monomials $X^{(i)}$, where $X$ is a $k$-basis of $g$. Note that $R$ is $g$-locally finite. By [7], $GK(R\#U(g)) = GK(R) + 1$. So $GK(R\#H) \geq GK(R) + 1$, since $R\#U(g)$ is a subalgebra of $R\#H$. For the reverse inequality, let $V$ be a finite-dimensional subspace of $R\#H$. Using the fact that $R$ is $H$-locally finite, we see that

$$V \subseteq W + WX^{(1)} + WX^{(2)} + \cdots + WX^{(m)}$$

for some $m$ and some finite-dimensional $H$-invariant subspace $W$ of $R$. It is not difficult to show that

$$V^n \subseteq W^n + W^nX + W^nX^2 + \cdots + W^nX^n + W^nX^{(2)} + W^nX^{(3)} + \cdots + W^nX^{(m)}.$$  

So $dim_k V^n \leq (n + nm)(dim_k W^n)$ and we get

$$log_n(dim_k V^n) \leq log_n(dim_k W^n) + log_n(n + nm) = log_n(dim_k W^n) + 1 + log_n(1 + m).$$

This yields the reverse inequality $GK(R\#H) \leq GK(R) + 1$.

For the general case, let

$$k = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = H$$

be a chain of subHopf algebras of $H$ such that for each $i \leq n$, $H_{i-1}$ is normal in $H_i$ and the monomials $X_1^{(j_1)}X_2^{(j_2)} \cdots X_i^{(j_i)}; \ j_i \in \mathbb{N}$ form a basis for $H_i$. Set $R_i = R\#H_i$; so $R_0 = R$ and $R_n = R\#H$. Clearly, $R_{i+1} = R_i\#(k < X_{i+1} >)$ for each $i \leq n - 1$, where $k < X_{i+1} >$ is the divided power Hopf algebra spanned by the monomials $X_i^{(j_i)}$, this is a subHopf algebra of $H_{i+1}$. Now each $R_i$ is $k < X_{i+1} >$-locally finite, since each $R_i$ is $H_{i+1}$-locally finite. On the other hand, the space of primitive elements of $k < X_{i+1} >$ is the $k$-vector subspace $kX_{i+1}$ of $H_{i+1}$. By the previous paragraph, $GK(R_{i+1}) = GK(R_i) + 1$ and the result follows. □
Theorem 1.3 may be applied in the following circumstances:
- $k$ is of characteristic 0, $g$ is a finite-dimensional solvable $k$-Lie algebra, $H$ is the enveloping algebra of $g$ and $R$ is a $g$-locally finite $U(g)$-module algebra.
- $k$ is perfect, $G$ is a connected unipotent affine algebraic group acting rationally on $R$ and $H$ is the hyperalgebra of $G$.
- $k$ is perfect, $G$ is a connected abelian affine algebraic group acting rationally on $R$ and $H$ is the hyperalgebra of $G$.
- $k$ is perfect, $H$ is a divided powers Hopf algebra (with $dim P(H) = 1$) acting on $R$ such that $R$ is an $H$-locally finite $H$-module algebra.

As an application of Theorem 1.3 we shall show some results concerning incomparability and prime length. In the remainder of this section, $R$ will be noetherian of infinite Gelfand-Kirillov dimension and all the smash products are noetherian. We denote by $dim$ the classical Krull dimension and by $H-dim$ its $H$-invariant version; i.e. the maximal length of a chain of $H$-prime ideals of $R$. We have $H-dim (R#H) = dim(R#H)$. If $R$ is $H$-locally finite, the $H$-prime ideals of $R$ are prime [2, Proposition 1.3]; so $H-dim (R) \leq dim(R)$.

**Corollary 2.4.** Let $k$ be a perfect field, $H$ a irreducible cocommutative almost solvable Hopf algebra with $P(H)$ finite-dimensional, $R$ an $H$-locally finite $H$-module algebra and $A = R#H$. Assume that the sequences of divided powers are all infinite. Let $P$ be a prime ideal of $A$ such that $P \cap R = 0$. Then $ht(P) \leq n$. If $R$ is $H$-simple, then $dim(A) \leq n$.

**Proof.** Since $R = R/(P \cap R)$ is a subalgebra of $A/P$, we have $ GK(R) \leq GK(A/P)$. Theorem 1.3 implies that $ GK(A) - GK(A/P) \leq n$. By [6, Proposition 3.16], $ht(P) \leq n$. If $R$ is $H$-simple, $ht(Q) \leq n$ for any prime ideal $Q$ of $A$. \□

The next result bounds $dim(R#H)$ in terms of $H-dim(R)$. Although, the bound is surely not sharp.

**Proposition 2.5.** Let $k$ be a perfect field, $H$ a irreducible cocommutative almost solvable Hopf algebra with $P(H)$ finite-dimensional, $R$ an $H$-locally finite $H$-module algebra and $A = R#H$. Assume that the sequences of divided powers are all infinite. Suppose that $P_0 \subset P_1 \subset \cdots \subset P_{n+1}$ is a strictly increasing chain of prime ideals of $A$, then $P_0 \cap R \subset P_{n+1} \cap R$ and $dim(A) \leq (n+1)(H-dim(R) + 1)$.

**Proof.** Suppose that $P_0 \cap R = P_{n+1} \cap R = I$. By [4, Lemma 1.2], $I$ is an $H$-prime ideal of $R$ and $IA = AI$ is an ideal of $A$. By [2, Proposition 1.3], $I$ is a prime ideal of $R$. One can show that $A/IA \simeq (R/I)#H$. Set $\bar{R} = R/I$ and $\bar{A} = A/IA$. In $\bar{A}$, we have a strictly increasing chain of prime ideals $\overline{P}_0 \subset \overline{P}_1 \subset \cdots \subset \overline{P}_{n+1}$ of length $n + 1$ such that $\overline{P}_0 \cap \bar{R} = \overline{P}_{n+1} \cap \bar{R} = \bar{I} = 0$; where $\overline{P}_i$'s denote the natural images of $P_i$'s in $\bar{A}$. It follows that $ht(\overline{P}_{n+1}) \geq n + 1$. By Corollary 1.4, $ht(\overline{P}_{n+1}) \leq n$ and we get a contradiction.

Let $P_0 \subset P_1 \subset \cdots \subset P_s$ be a strictly increasing chain of prime ideals of $A$. By the preceding paragraph,

$$P_0 \cap R \subset P_{n+1} \cap R \subset P_{2(n+1)} \cap R \subset P_{3(n+1)} \cap R \subset \cdots$$

is a strictly increasing chain of $H$-invariant prime ideals of $R$. Since this chain can contain at most $(1+H-dim(R))$ $H$-invariant prime ideals, we conclude that $s \leq (n+1)(H-dim(R) + 1)$. \□
Proposition 1.5 may be applied to the smash product $R\#U(g)$, where $k$ is of characteristic 0, $R$ is noetherian of finite Gelfand-Kirillov dimension and $g$ is a finite dimensional solvable $k$-Lie algebra. For related work, see [3] and [9, Corollary 4.4].

References


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