An Invariance Property of the Tridens Curve in the Isotropic Plane

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Abstract. The tridens curves of third order and their generalizations in the isotropic plane over \( \mathbb{R} \) were studied by D. Palman [1] and H. Sachs [2,3]. For additional properties see [6,7]. In this paper we prove that for every such tridens curve \( T \) of third order there exists an inscribed triangle \( \Delta \) with the property: \( T \) remains invariant under the correspondence of opposite angle points of \( \Delta \).

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1. The equation of every irreducible tridens curve \( T \) of third order in the isotropic plane \( I_2(\mathbb{R}) \) can be written in the form (see [6, Lemma, part (a)])

\[
T(x, y) \equiv \frac{1}{R} \{y(x - a) - Rx(x - a)(x - A)\}, \quad \text{with } a, a, A, R \in \mathbb{R}
\]

and

\[
\alpha a A(\alpha - a)(\alpha - A)(a - A) \neq 0 \quad \text{and} \quad (2\alpha - a)(2\alpha - A)(2\alpha - a - A) \neq 0.
\]

In the above selected affine \( x, y \)-coordinate system the absolute point of \( I_2(\mathbb{R}) \) is supposed to have the homogeneous coordinates \( 0 : 0 : 1 \). Using the definitions

\[
2\lambda R(2\alpha - a - A) = 1 \quad \text{and} \quad \lambda b := 2\alpha - A
\]

of numbers \( \lambda \) and \( b \) we get instead of (1) with (2)

\[
T(x, y) \equiv [x(x - a) - \lambda(\lambda b - a)y](a - x) + \lambda(\lambda b - a)(x - \lambda b)y = 0
\]

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with

\[(A + \lambda b)aA(A + \lambda b - 2a)(\lambda b - A)(a - A) \neq 0 \quad \text{and} \quad (A + \lambda b - a)\lambda b(\lambda b - a) \neq 0.\]

Hence the triangle \(\Delta\) with the vertices

\[(A_1 := (0, 0), \ A_2 := (a, 0), \ A_3 := (\lambda b, b)\]

is an \textit{inscribed triangle} of the tridens curve \(T\) with the equation (4) with (5).

2. The correspondence of opposite angle points for an admissible triangle \(\Delta = \Delta(A_1A_2A_3)\) (see [4, p.22]) of \(I_2(\mathbb{R})\) is explained as follows. Let us denote with \(\sigma_i\) the line determined by the side of \(\Delta\) which does not contain the vertex \(A_i\) and with \(\omega_i\) the isotropic bisectrix of the straight lines \(\sigma_{i+1}\) and \(\sigma_{i+2}\) (in this order). For a point \(P(x, y)\) we regard the line \(P \lor A_i\) and its image line \(r_i\) under reflection at \(\omega_i\) in the sense of the isotropic metric. The lines \(r_1, r_2, r_3\) have a common point \(P^* (x^*, y^*)\), the so called \textit{opposite angle point} of \(P(x, y)\) with respect to \(\Delta\). Basic properties of this involutory, quadratic correspondence were studied by K. Strubecker (see [4, p.528f]).

Referring us to the triangle \(\Delta\) with the vertices (6) we have for the coordinates of the opposite angle points \(P\) and \(P^*\) the analytical expressions (see [5, p.158])

\[x = \lambda x^* \frac{\sigma_1(x^*, y^*)}{\kappa(x^*, y^*)}, \quad y = (x^* - \lambda y^*) \frac{\sigma_1(x^*, y^*)}{\kappa(x^*, y^*)}.\]

Hereby we have

\[\kappa(x, y) \equiv x(x - a) - \lambda(\lambda b - a)y = 0\]

as the isotropic circumcircle of \(\Delta\) and

\[\sigma_1(x, y) \equiv b(x - a) - (\lambda b - a)y = 0\]

as the line determined by that side of \(\Delta\) which is opposite to the vertex \(A_1\). Using (7), a simple calculation leads to

\[\kappa(x^*, y^*)T(x, y) = \kappa(x, y)T(x^*, y^*).\]

\textbf{Theorem.} For every irreducible tridens curve \(T\) of third order in the isotropic plane over \(\mathbb{R}\) exists an inscribed triangle \(\Delta\) with the property: \(T\) remains invariant under the correspondence of opposite angle points with respect to \(\Delta\).
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References

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