The Problem of Polygons with Hidden Vertices

Joseph M. Ling

University of Calgary, Calgary, Alberta, Canada T2N 1N4

Abstract. G. Ewald proved that it is possible for a polygon(al path) in $\mathbb{R}^3$ to hide all its vertices behind its edges from the sight of a point $M$ not on the polygon. Ewald also stated that it takes at least 8 vertices to do the job and constructed an example with 14 vertices. It was then suggested that the least number of vertices $n_{\text{min}}$ for such a configuration is closer to 14 than to 8. In this paper, we shall prove that $11 \leq n_{\text{min}} \leq 12$.

Let $P = [P_1, P_2, \ldots, P_n]$ be a polygon without self intersection in $\mathbb{R}^3$. So, $P_1, \ldots, P_n$ are distinct points, and $P$ is the union of the (closed) line segments $[P_1, P_2], [P_2, P_3], \ldots, [P_{n-1}, P_n]$ such that for all $i \neq j$, $[P_{j-1}, P_j] \cap (P_{i-1}, P_i) = \emptyset$, where $(P_{i-1}, P_i)$ denotes the relative interior of of $[P_{i-1}, P_i]$. The points $P_1, \ldots, P_n$ are called the vertices of the polygon, and the line segments $[P_{i-1}, P_i]$ the edges.

Ewald [1] proved that there is a polygon and a point $M$ (not on the polygon) such that all vertices are hidden from the sight of $M$, i.e., for all $i$, there exists $j$ such that the line segments $[M, P_i]$ and $[P_{j-1}, P_j]$ intersect at a point in $(P_{j-1}, P_j)$. More specifically, Ewald [1] gave an example of such a configuration with $n = 14$. Ewald asked for the smallest number of vertices $n_{\text{min}}$ for which such a configuration exists, argued that $8 \leq n_{\text{min}} \leq 14$, and suggested that it should be closer to 14 than to 8. We shall prove that $11 \leq n_{\text{min}} \leq 12$. Here, let us first mention an example with 12 vertices:

$$M = (0, 0, 0)$$
$$P_1 = \left(19, \frac{17}{2}, 5\right) = 5P_5 + \frac{7}{2}P_6$$
\[ P_2 = (-19, -8, -5) = 11P_6 + 3P_7 \]
\[ P_3 = (0, 0, 1) = P_7 + \frac{19}{3}P_8 \]
\[ P_4 = (0, 1, 0) = 2P_1 + 2P_2 \]
\[ P_5 = (1, 0, 0) = 4P_{10} + P_{11} \]
\[ P_6 = (0, 1, 1) = P_3 + P_4 \]
\[ P_7 = \left( -\frac{19}{3}, -\frac{19}{3}, -\frac{16}{3} \right) = \frac{4}{3}P_{11} + \frac{1}{3}P_{12} \]
\[ P_8 = (1, 1, 1) = P_5 + P_6 \]
\[ P_9 = (4, 1, 0) = P_4 + 4P_3 \]
\[ P_{10} = (5, 2, 1) = P_8 + P_9 \]
\[ P_{11} = (-19, -8, -4) = P_2 + P_3 \]
\[ P_{12} = (57, 13, 0) = 13P_4 + 57P_5 \]

Thus, \( n_{\min} \leq 12 \). In what follows, we shall prove that \( n_{\min} \geq 11 \).

To facilitate our discussion, we shall also denote the vertices by \( Q_1, \ldots, Q_n \), without any implication that the endpoints of the edges \([Q_i, Q_j]\) should have consecutive integral indices. Thus, \( Q_1, \ldots, Q_n \) is a permutation of \( P_1, \ldots, P_n \), and \([Q_i, Q_j]\) is an edge of the polygon \([P_1, \ldots, P_n]\) if and only if there is some \( k \) such that \( \{Q_i, Q_j\} = \{P_{k-1}, P_k\} \), but it is not necessary that \(|i - j| = 1\). However, we note that as \( P_1, \ldots, P_n \) are distinct, so are \( Q_1, \ldots, Q_n \).

We say that \([Q_i, Q_j]\) blocks \( Q_k \) if the line segment \([M, Q_k]\) intersects \([Q_i, Q_j]\) at a point in \((Q_i, Q_j)\), and that \( Q_i \) helps block \( Q_k \) if for some vertex \( Q_j \), \([Q_i, Q_j]\) is an edge of the polygon which blocks \( Q_k \). We denote the point of intersection of \([M, Q_k]\) and \([Q_i, Q_j]\) by \( R_k \) in case the values of \( i \) and \( j \) are clear from the context. (See Figure 1) Sometimes, we find it convenient to say that a set \( S \) of points blocks or does not block a point \( Q \) according as whether or not there is some line segment with endpoints in \( S \) that blocks \( Q \).

![Figure 1](image-url)

The plane that passes through three noncollinear points \( A, B, C \) will be denoted by \( \pi_{ABC} \). Thus, \( \pi_{ABC} = \pi_{ACB} = \pi_{BCA} \), etc. If \( A \) happens to be some \( Q_j \), we also write \( \pi_{ABC} \) as \( \pi_{jBC} \), etc.
etc. Thus, $\pi_{ijk}$ is the plane that passes through $Q_i, Q_j, Q_k$, and $\pi_{Mij}$ is the plane that passes through $M, Q_i, Q_j$. We find it convenient to write $\pi_{Mijk}$ for the plane, when exists, that passes through $M, Q_i, Q_j, Q_k, \ldots$.

Each plane $\pi$ determines two closed half-spaces in $\mathbb{R}^3$, denoted by $\pi^\geq$ and $\pi^\leq$, with common boundary $\pi$. We denote the corresponding open half-spaces by $\pi^\gt$ and $\pi^\lt$, respectively. We note that the choice of designations $<$ and $>$ for the pair of half-spaces is arbitrary, and the same applies to $\leq$ and $\geq$.

The following facts will be used frequently. We omit their easy proofs.

1. If $(P, Q)$ intersects $\pi$, then $P \in \pi^\gt \Rightarrow Q \in \pi^\lt$ (and $P \in \pi^\lt \Rightarrow Q \in \pi^\gt$). We also describe this scenario by saying that $(P, Q)$ pierces through $\pi$.

2. If $Q_i$ helps block $Q_j$, then $Q_j$ does not help block $Q_i$.

3. If $M \in \pi$ and $Q \in \pi^\gt$, then $Q$ is not blocked by any line segments with endpoints in $\pi^\leq$.

4. If $M \in \pi$ and if $Q_k$ is blocked by $[Q_i, Q_j]$, where $(Q_i, Q_j) \subseteq \pi^\gt$, then $Q_k \in \pi^\gt$.

5. If all the vertices of a polygon are hidden from sight of $M$, then $M$ must lie in the interior of the convex hull of the polygon. (This was proved in [1].)

Below, we shall assume that $[P_1 \ldots P_n]$ is a polygon whose vertices are hidden from the sight of the point $M$.

To begin, it is quite easy to see that $n \geq 10$.

**Lemma 1.** If $\pi$ is any plane through $M$, then there are at least three vertices in $\pi^\gt$ and at least three vertices in $\pi^\lt$.

**Proof.** By [1] Theorem 1, not all vertices lie on the same plane. So, there is some vertex $P \in \pi^\gt$, say. But then, since $M$ lies in the interior of the convex hull of the polygon, there must be some vertex $Q$ in $\pi^\lt$. Since $P$ is blocked from $M$, and since points in $\pi^\leq$ are not sufficient to block $P$, there must be some $P'$ in $\pi^\gt$ which helps block $P$. Now, as $P'$ helps block $P$, $P'$ cannot help block $P''$. Thus, there must be another vertex $P''$ in $\pi^\gt$ which helps block $P'$. Thus, $\pi^\gt$ contains at least three vertices. The same argument applies to $Q \in \pi^\lt$. □

**Theorem 1.** $n_{\text{min}} \geq 10$.

**Proof.** Since the polygon $[P_1 \ldots P_n]$ has $n$ vertices hidden behind $n - 1$ edges, at least one of the edges must block more than one vertex. Suppose $Q_1, Q_2$ are two vertices that are blocked by $[Q_3, Q_4]$. Then $M, Q_1, Q_2, Q_3, Q_4$ all lie on the same plane $\pi$. By Lemma 1, there are at least three vertices in $\pi^\gt$ and at least three vertices in $\pi^\lt$. So, the total number of vertices is at least $4 + 3 + 3 = 10$. □

We need another lemma to establish $n_{\text{min}} \geq 11$. 

Lemma 2. If \( Q_i, Q_j, Q_k \) are the only vertices in \( \pi^\gtrdot \) for some plane \( \pi \) through \( M \), then \( Q_i, Q_j, Q_k \) will help block one another in a cyclical manner, i.e., either
\[
Q_i \text{ helps block } Q_k, \ Q_k \text{ helps block } Q_j, \ Q_j \text{ helps block } Q_i,
\]
or
\[
Q_i \text{ helps block } Q_j, \ Q_j \text{ helps block } Q_k, \ Q_k \text{ helps block } Q_i.
\]

Proof. One of \( Q_i \) and \( Q_j \) must help block \( Q_k \). If \( Q_i \) helps block \( Q_k \), then \( Q_k \) does not help block \( Q_i \), and so \( Q_j \) must help block \( Q_i \). Then \( Q_i \) does not help block \( Q_j \), and so, \( Q_k \) must help block \( Q_j \). We arrive at the first scenario. Similarly, if \( Q_j \) helps block \( Q_k \), we arrive at the second scenario. (We may imagine the vertices \( Q_i, Q_j, Q_k \) together with their blocking edges forming something like a tripod. See Figure 2.) \[\Box\]

Figure 2

Theorem 2. \( n_{\min} \geq 11. \)

Proof. As in the proof of Theorem 1, we suppose that \( Q_1, Q_2 \) are two vertices that are blocked by \([Q_3, Q_4] \), and we denote the plane \( \pi_{M1234} \) by \( \pi \). We look at various ways \( Q_3 \) and \( Q_4 \) may be blocked. The cases are separated according to whether the line segments blocking them lie on or pierce through the plane \( \pi \). In each case, we prove that it is impossible for the configuration to have just 10 vertices. In view of Lemma 1, this implies that no plane through \( M \) can contain more than 4 vertices.

CASE 1. \( Q_3 \) and \( Q_4 \) are blocked by two edges piercing through the plane \( \pi = \pi_{M1234} \), and that the two edges share a vertex. That is, there are points \( Q_5, Q_6, Q_7 \not\in \pi \) such that \([Q_5, Q_6]\) blocks \( Q_3 \) and \([Q_6, Q_7]\) blocks \( Q_4 \). (See Figure 3.) Since \((Q_5, Q_6)\) and \((Q_6, Q_7)\) pierce through \( \pi \), we may assume without loss of generality that \( Q_6 \in \pi^< \) and \( Q_5, Q_7 \in \pi^\gtrdot \). In view of Lemma 2, we have vertices \( Q_8, Q_{10} \in \pi^< \) and we may assume that
\[
Q_6 \text{ helps block } Q_8; \ Q_8 \text{ helps block } Q_{10}; \ Q_{10} \text{ helps block } Q_6.
\]
Also, there is a vertex \( Q_9 \in \pi^\gtrdot \). Suppose that \( Q_1, \ldots, Q_{10} \) are all the vertices. Since
[Q_5, Q_6] blocks Q_3, [Q_6, Q_7] blocks Q_4, and [Q_3, Q_4] blocks Q_1 and Q_2. \( \pi_{567} \) strictly separates Q_1, Q_2, Q_3, Q_4 from M, i.e., \( M \in \pi_{567} \) and Q_1, Q_2, Q_3, Q_4 \( \in \pi_{567}^\leq \). (See Figure 4.) Since Q_6 helps block Q_8 and the edges in the polygon that connect to Q_6 are [Q_5, Q_6] and [Q_6, Q_7], Q_8 must be blocked by one of [Q_5, Q_6] and [Q_6, Q_7]. In either case, the plane \( \pi_{567}^\leq \) strictly separates Q_8 from M. So, the vertices Q_1, \ldots , Q_8 are all in \( \pi_{567}^\leq \) whereas M \( \in \pi_{567}^\geq \). So, if \( \pi_{//} \) denotes the plane through M parallel to \( \pi_{567} \), then one of the half-spaces \( \pi_{//}^\geq \) and \( \pi_{//}^\leq \) contains no more than two vertices Q_9 and Q_10. This contradicts Lemma 1. This completes CASE 1.
CASE 2. $Q_3$ and $Q_4$ are blocked by two edges piercing through the plane $\pi = \pi_{M1234}$, and that they do not share a common vertex. That is, there are (distinct) points $Q_5, Q_6, Q_7, Q_8$ not on $\pi$ such that $[Q_5, Q_6]$ blocks $Q_3$ and $[Q_7, Q_8]$ blocks $Q_4$. (See Figure 5.)

![Figure 5](image_url)

We may assume that $Q_5, Q_7 \in \pi >$ and $Q_6, Q_8 \in \pi <$. In view of Lemma 1, we have $Q_9$ in $\pi >$, $Q_{10}$ in $\pi <$, and we assume that

$Q_5$ helps block $Q_7$; $Q_7$ helps block $Q_9$; $Q_9$ helps block $Q_5$.

Now, suppose that these are all the vertices. We shall derive a contradiction.

Consider the plane $\pi_{M478}$. Since $Q_7 \notin \pi = \pi_{M34}$, we have $Q_3 \notin \pi_{M478}$. Suppose $Q_5 \in \pi <_{M478}$. Then $(Q_3, Q_4) \subseteq \pi <_{M478}$. It follows that $Q_1, Q_2 \in \pi <_{M478}$. Also, $R_3 = (M, Q_3) \cap [Q_5, Q_6] \in \pi <_{M478}$. This implies that at least one of $Q_5, Q_6$ is in $\pi <_{M478}$. But then, $Q_1, Q_2, Q_3, Q_4, Q_7, Q_8$, and at least one of $Q_5, Q_6$ are in $\pi <_{M478}$. It follows by Lemma 1 that $Q_9, Q_{10} \in \pi >_{M478}$, and exactly one of $Q_5$ and $Q_6$ must be in each of $\pi >_{M478}$ and $\pi <_{M478}$. As a summary,

$Q_1, Q_2, Q_3$, and one of $Q_5, Q_6 \in \pi <_{M478}$; $Q_9, Q_{10}$, and one of $Q_5, Q_6 \in \pi >_{M478}$.

By a similar argument,

$Q_1, Q_2, Q_4$, and one of $Q_7, Q_8 \in \pi <_{M356}$; $Q_9, Q_{10}$, and one of $Q_7, Q_8 \in \pi >_{M356}$.

Subcase 2(a). $Q_5 \in \pi <_{M478}$ (and so $Q_6 \in \pi >_{M478}$).

As $Q_5$ helps block $Q_7$, $Q_7$ must be blocked by $[Q_5, Q_4]$ for some $Q_4 \in \pi >_{M478}$. So, $i = 6, 9$, or 10. Since $Q_7$ helps block $Q_9$, $Q_9$ does not help block $Q_7$, and so, $i \neq 9$. Also, since $Q_5, Q_6 \in \pi_{M356}$ but $Q_7 \notin \pi_{M356}$, $[Q_5, Q_6]$ does not block $Q_7$. So, $i \neq 6$, and we conclude that $i = 10$, and so $[Q_5, Q_{10}]$ blocks $Q_7$. 
This in turn implies that
\[ Q_7 \in \pi^>_M 356, \] and so \[ Q_8 \in \pi^<_M 356. \]

Now, \( Q_9, Q_{10}, Q_7 \) are the only vertices in \( \pi^>_M 356 \), and as \( Q_{10} \) helps block \( Q_7 \), it follows from Lemma 2 that
\[ Q_7 \text{ helps block } Q_9, \] and \( Q_9 \text{ helps block } Q_{10}. \)

Then, as the only vertices in \( \pi^>_M 478 \) are \( Q_9, Q_{10}, Q_6 \), and as \( Q_9 \) helps block \( Q_{10} \), it follows from Lemma 2 again that
\[ Q_{10} \text{ helps block } Q_6, \] and \( Q_6 \text{ helps block } Q_9. \)

Next, as the only vertices in \( \pi^< \) are \( Q_6, Q_8, Q_{10} \), and as \( Q_{10} \) helps block \( Q_6 \), it follows once more from Lemma 2 that
\[ Q_6 \text{ helps block } Q_8, \] and \( Q_8 \text{ helps block } Q_{10}. \)

So, \( Q_6 \) helps block the vertices \( Q_3, Q_8, \) and \( Q_9 \). As there are only two edges connecting \( Q_6 \), one of them is going to block two of \( Q_3, Q_8, Q_9 \).

Sub-subcase 2(a)(i): \([Q_6, Q_7]\) blocks both \( Q_8 \) and \( Q_9 \) for some \( i \).

Consider the plane \( \pi_{M_{689}} \). Certainly, \( i \neq 6, 8, 9 \). As \( Q_6 \in \pi^< \) and \( Q_9 \in \pi^> \), \( i \neq 1, 2, 3, 4, 10 \).
If \( i = 5 \), then \([Q_5, Q_6]\) blocks \( Q_3, Q_8, Q_9 \), and so \( \pi_{M_{56}} \) contains at least five vertices, a contradiction. If \( i = 7 \), then \([Q_6, Q_7]\) blocks \( Q_4, Q_8, Q_9 \) and we have five vertices on the same plane \( \pi_{M_{67}} \), a contradiction again.

Sub-subcase 2(a)(ii): \([Q_6, Q_i]\) blocks \( Q_3 \) and one of \( Q_8 \) and \( Q_9 \), for some \( i \).

Since \([Q_5, Q_6]\) blocks \( Q_5 \), we have \( Q_i \in \pi_{M_{56}} \), and so one of \( Q_8 \) or \( Q_9 \) must be on \( \pi_{M_{56}} \). But this contradicts the basic setup of Subcase 2(a), in which \( Q_8 \in \pi^<_M 356 \) and \( Q_9 \in \pi^>_M 356. \)

This completes Subcase 2(a).

Subcase 2(b). \( Q_5 \in \pi^>_M 478 \) (and so \( Q_6 \in \pi^<_M 478 \)).

Since \( Q_5, Q_9, Q_{10} \) are the only vertices in \( \pi^>_M 478 \), and as \( Q_9 \) helps block \( Q_5 \), it follows from Lemma 2 that \( Q_5 \) helps block \( Q_{10} \), and \( Q_{10} \) helps block \( Q_9 \). But then \( Q_5 \) helps block \( Q_3, Q_7, Q_{10}. \) As \( Q_5 \) is connected to at most two vertices, there must be some \( i \) such that \([Q_5, Q_i]\) blocks two of the vertices \( Q_3, Q_7, Q_{10}. \)

Sub-subcase 2(b)(i): \([Q_5, Q_i]\) blocks \( Q_7 \) and \( Q_{10}. \)

Certainly, \( i \neq 5, 7, 10. \) Since \( Q_5 \in \pi^> \) and \( Q_{10} \in \pi^<_\), it is easy to see that \( i \neq 1, 2, 3, 4, 9. \)
Since \( Q_5 \in \pi^>_M 478, [Q_5, Q_8] \) does not block \( Q_7. \) So, \( i \neq 8 \). Finally, if \([Q_5, Q_6]\) blocks \( Q_7, Q_{10}, \) then \( \pi_{M_{56}} \) contains five vertices, a contradiction. Thus, \( i \neq 6. \)

Sub-subcase 2(b)(ii): \([Q_5, Q_i]\) blocks \( Q_5 \) and one of \( Q_7, Q_{10}. \)

One of \( Q_7, Q_{10} \) will then be in \( \pi_{M_{56}} \). This contradicts the basic setup of Subcase 2(b) in which \( Q_7 \notin \pi_{M_{56}} \) and \( Q_{10} \notin \pi^>_M 356. \)

This completes Subcase 2(b), and hence CASE 2.

CASE 3. \( Q_3 \) and \( Q_4 \) are blocked by two edges at least one of which lies on the plane \( \pi = \pi_{M_{1234}} \).
The plane $\pi$ must contain a fifth vertex, a contradiction.

This completes CASE 3, and hence the proof of Theorem 2. \hfill \square

**Remarks.**

1. We conjecture that $n_{\text{min}} = 11$.

2. Our proof relies on the fact that the polygon does not close up onto itself: there are $n - 1$ edges connecting $n$ vertices. In the example given in Ewald [1], the line segment joining $P_1$ and $P_{14}$ does not intersect any of the edges $[P_{i-1}, P_i]$, $2 \leq i \leq 14$. Thus, we get a closed polygon with all vertices hidden from the sight of $M$. For such a configuration, a pigeon-hole argument does not apply to allow us to conclude that some edge blocks more than one vertex. The natural starting point is then with one edge blocking one vertex. The total number of vertices is at least $3 + 3 + 3 = 9$. The above argument may be repeated to conclude that the number of vertices is at least 10. On the other hand, in our example with 12 vertices, the polygon can also be closed up without introducing a self-intersection. Thus, for closed polygons, we may say $10 \leq n_{\text{min}} \leq 12$.

**Reference**


Received May 15, 2003