Adjacency Preserving Mappings of Rectangular Matrices

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Abstract. Let $D$ be a division ring and let $m, n$ be integers $\geq 2$. Let $M_{m \times n}(D)$ be the space of $m \times n$ matrices. In the fundamental theorem of the geometry of rectangular matrices all bijective mappings $\varphi$ of $M_{m \times n}(D)$ are determined such that both $\varphi$ and $\varphi^{-1}$ preserve adjacency. We show that if a bijective map $\varphi$ of $M_{m \times n}(D)$ preserves the adjacency then also $\varphi^{-1}$ preserves the adjacency. Thus the supposition that $\varphi^{-1}$ preserves adjacency may be omitted in the fundamental theorem.

MSC 2000: 15A99, 51D20
Keywords: Geometry of matrices, rectangular matrices, mappings preserving adjacency, distance preserving mappings

1. Introduction

L.K. Hua initiated the study of the geometry of matrices in the mid fourties of the last century (cf. [3]). In this geometry, the points are a certain kind of matrices of a given size. The four kinds of matrices studied by Hua are rectangular matrices, symmetric matrices, skew-symmetric matrices and hermitian matrices. To each such space there is associated a group of motions. It is the aim to characterize the group of motions in the space by as few
geometric invariants as possible. Two rectangular, symmetric, or hermitian matrices $A, B$ are called adjacent, if $A - B$ has rank 1. Two skew-symmetric matrices $A, B$ are called adjacent if $A - B$ has rank 2. Hua discovered that the invariant “adjacency” is sufficient to characterize the group of motions. He and his followers determined all bijections $\varphi$ of the set of points which satisfy

$$A, B \text{ are adjacent} \iff A^\varphi, B^\varphi \text{ are adjacent.} \quad (1.1)$$

This result is known as the fundamental theorem of the geometry of matrices.

In view of the fundamental theorems of affine and projective geometry, where all bijections of affine resp. projective spaces are determined which take lines to lines, it is a natural and important question posed in [4], whether in the geometries of matrices it is possible to replace the condition (1.1) by

$$A, B \text{ are adjacent} \implies A^\varphi, B^\varphi \text{ are adjacent.} \quad (1.2)$$

It is shown in [2] that this is possible in the case of symmetric and hermitian matrices. In the present paper we answer this question for the space $M_{m \times n}(D)$ of $m \times n$ rectangular matrices over a division ring $D$.

**Theorem 1.1.** Let $D$ be a division ring. Let $m, n$ be integers $\geq 2$. If a bijective map $\varphi$ from $M_{m \times n}(D)$ to itself preserves the adjacency in $M_{m \times n}(D)$ then also $\varphi^{-1}$ preserves the adjacency.

According to the fundamental theorem of the geometry of rectangular matrices [1, 5], when $m \neq n$, then $\varphi$ is of the form

$$X^\varphi = PX^\sigma Q + R \quad \text{for all } X \in M_{m \times n}(D), \quad (1.3)$$

where $P \in \text{GL}_m(D)$, $Q \in \text{GL}_n(D)$, $R \in M_{m \times n}(D)$, and $\sigma$ is an automorphism of $D$. When $m = n$, then in addition to (1.3), $\varphi$ might also be a mapping of the form

$$X^\varphi = P^t(X^\tau)Q + R \quad \text{for all } X \in M_{m \times n}(D), \quad (1.4)$$

where $\tau$ is an anti-automorphism of $D$.

The space $M_{m \times n}(D)$ can be treated as a graph. We call the points of $M_{m \times n}(D)$ vertices and define two vertices $A, B$ to be adjacent if rank$(A - B) = 1$. Then we obtain the graph of $m \times n$ matrices over $D$, denoted by $\Gamma(M_{m \times n}(D))$. If $D$ is infinite, then $\Gamma(M_{m \times n}(D))$ is an infinite graph. For finite graphs, a bijection which satisfies (1.2) is an automorphism. But there are counterexamples in the infinite case. For the graph $\Gamma(M_{m \times n}(D))$, the above theorem can be interpreted as follows.

**Theorem 1.2.** Let $D$ be a division ring. Let $m, n$ be integers $\geq 2$ and $\Gamma(M_{m \times n}(D))$ be the graph of $m \times n$ rectangular matrices over $D$. If $\varphi$ is a bijective map from $\Gamma(M_{m \times n}(D))$ to itself for which

$$A, B \text{ are adjacent} \implies A^\varphi, B^\varphi \text{ are adjacent,}$$

is satisfied for any two vertices $A, B$ of $\Gamma(M_{m \times n}(D))$ then $\varphi$ is a graph automorphism of $\Gamma(M_{m \times n}(D))$. 
2. Preliminaries

In this section we mention some definitions and propositions which are also contained in Wan’s book [3]. We then define the notion of covering radius of a subset of $M_{m \times n}(D)$ and show some of its properties.

**Definition 2.1.** (Points, motions) Let $D$ be a division ring. Let $m, n$ be integers $\geq 2$. Denote by $M_{m \times n}(D)$ the space of $m \times n$ matrices over $D$. We call elements of $M_{m \times n}(D)$ the points of the space $M_{m \times n}(D)$. The group $G_{m \times n}(D)$ of motions of $M_{m \times n}(D)$ consists of transformations of the form

$$X \mapsto PXQ + R \quad \text{for all } X \in M_{m \times n}(D),$$

where $P \in \text{GL}_m(D)$, $Q \in \text{GL}_n(D)$, $R \in M_{m \times n}(D)$.

**Proposition 2.1.** ([3], Proposition 3.1) The group $G_{m \times n}(D)$ acts transitively on $M_{m \times n}(D)$.

**Definition 2.2.** (Adjacency) Two points $A, B \in M_{m \times n}(D)$ are said to be adjacent if

$$\text{rank}(A - B) = 1.$$

**Definition 2.3.** (Maximal set) A maximal set $\mathcal{M}$ in $M_{m \times n}(D)$ is a subset of $M_{m \times n}(D)$ with the property that any two points in $\mathcal{M}$ are adjacent and there is no point in $M_{m \times n}(D) \setminus \mathcal{M}$ which is adjacent to any point in $\mathcal{M}$.

**Proposition 2.2.** (cf. [3], Proposition 3.9) There are two types of maximal sets of adjacent matrices,

Type 1: \[ \left\{ P \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} Q + R \mid x_{11}, \ldots, x_{1n} \in D \right\}, \]

Type 2: \[ \left\{ P \begin{pmatrix} y_{11} & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} Q + R \mid y_{11}, \ldots, y_{m1} \in D \right\}, \]

where $P \in \text{GL}_m(D)$, $Q \in \text{GL}_n(D)$, and $R \in M_{m \times n}(D)$. Any maximal set belongs to only one type. A maximal set of type 1 cannot be carried to a maximal set of type 2 under the group of motions $G_{m \times n}(D)$.

**Proposition 2.3.** ([3], Corollary 3.10) For any pair of adjacent points $A, B \in M_{m \times n}(D)$ there are exactly one maximal set of type 1 and exactly one maximal set of type 2 containing both $A$ and $B$.

**Proposition 2.4.** Let $M_1, M_2$ be two distinct maximal sets with $M_1 \cap M_2 \neq \emptyset$. Then

$$|M_1 \cap M_2| \begin{cases} = 1 & \text{when } M_1 \text{ and } M_2 \text{ are of the same type}, \\ > 1 & \text{when } M_1 \text{ and } M_2 \text{ are not of the same type}. \end{cases}$$

In the second case we call $M_1 \cap M_2$ a line.
Proposition 2.5. ([3], Corollary 3.13) The parametric equation of a line in $M_{m \times n}(D)$ is
\[
\{tp + R \mid x \in D\},
\]
where $p$ is a nonzero $m$-dimensional row vector over $D$, $q$ is a nonzero $n$-dimensional row vector over $D$, and $R \in M_{m \times n}(D)$.

Proposition 2.6. ([3], Corollary 3.11 and Proposition 3.14) Two maximal sets which have only one point in common can be carried simultaneously under the group $G_{m \times n}(D)$ to
\[
\begin{align*}
\left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \mid x_{11}, \ldots, x_{1n} \in D \right\}, & \quad (2.1) \\
\left\{ \begin{pmatrix} 0 & \cdots & 0 \\ x_{21} & \cdots & x_{2n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \mid x_{21}, \ldots, x_{2n} \in D \right\}, & \quad (2.2)
\end{align*}
\]
or to
\[
\begin{align*}
\left\{ \begin{pmatrix} y_{11} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & y_{m1} \\ y_{m1} & 0 & \cdots & 0 \end{pmatrix} \mid y_{11}, \ldots, y_{m1} \in D \right\}, & \quad (2.3) \\
\left\{ \begin{pmatrix} 0 & y_{12} & 0 & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & y_{m2} \\ 0 & \cdots & 0 & y_{m2} \end{pmatrix} \mid y_{12}, \ldots, y_{m2} \in D \right\}. & \quad (2.4)
\end{align*}
\]
Two intersecting maximal sets of different type 1 and 2 can be carried simultaneously under the group $G_{m \times n}(D)$ to (2.1) and (2.3).

Proposition 2.7. ([3], Proposition 3.20) Any maximal set $M$ of type 1 has the structure of an $n$-dimensional left affine space, and any maximal set of type 2 has the structure of an $m$-dimensional right affine space, where the points and lines are defined above.

Definition 2.4. (Distance) The distance $d(A, B)$ between two distinct points $A, B \in M_{m \times n}(D)$ is defined to be the smallest nonnegative integer $k$ with the property that there exists a sequence of consecutively adjacent points $A = A_0, A_1, \ldots, A_k = B$. When $A = B$, we define $d(A, B) = 0$.

We have $d(A, B) = d(B, A)$ and $d(A, B) = 0$ if, and only if, $A = B$. Furthermore, the distance satisfies the triangle inequality
\[
d(A, C) \leq d(A, B) + d(B, C) \quad \text{for all } A, B, C \in M_{m \times n}(D),
\]
so $(M_{m \times n}(D), d)$ is a metric space. It was proved in [3] that for any two points $A, B \in M_{m \times n}(D)$,
\[
d(A, B) = \text{rank}(A - B).
\]
Hence $0 \leq d(A, B) \leq \min\{m, n\}$.
Lemma 2.1. For any maximal sets $M$ and $M'$ of type 1 resp. type 2, we can find a positive integer $k$ and a sequence $M = M_0, \ldots, M_k = M'$ of maximal sets of type 1 resp. type 2 satisfying $M_i \cap M_{i+1} \neq \emptyset$, $i = 0, \ldots, k - 1$.

Proof. Choose $X \in M$, $Y \in M'$, $X \neq Y$, $k - 1 := d(X,Y)$. Then there is a sequence of consecutively adjacent points $X = X_0, \ldots, X_{k-1} = Y$. For $i = 1, \ldots, k - 1$ define $M_i$ to be the maximal set of type 1 resp. type 2 which contains $X_{i-1}$ and $X_i$. Let $M_0 := M$ and $M_k := M'$. Then we have $M_i \cap M_{i+1} \neq \emptyset$, $i = 0, \ldots, k - 1$. □

Definition 2.5. (Covering radius) Let $A \in M_{m \times n}(D)$ and let $M \subseteq M_{m \times n}(D)$. The distance between $A$ and $M$ is $d(A, M) := \min \{d(A,B) \mid B \in M\}$. The covering radius of $M$ is

$$\rho(M) := \max \{d(A, M) \mid A \in M_{m \times n}(D)\}.$$

The covering radius of $M \subseteq M_{m \times n}(D)$ is the smallest positive integer $\rho$ with the property that the union of all balls

$$\bigcup_{A \in M} \{X \in M_{m \times n}(D) \mid d(A,X) \leq \rho\}$$

covers $M_{m \times n}(D)$. Clearly, for any two subsets $M, M' \subseteq M_{m \times n}(D)$, if there is an element $\psi \in G_{m \times n}(D)$ such that $M^\psi = M'$ then $\rho(M) = \rho(M')$.

Lemma 2.2. Let $M = \left\{ \begin{pmatrix} \delta \\ \vdots \\ \delta \end{pmatrix} \mid x \in D^n \right\}$ and $P = \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} \in M_{m \times n}(D)$ with rank $\begin{pmatrix} p_2 \\ \vdots \\ p_m \end{pmatrix} = k$. Then for all $X = \begin{pmatrix} x \\ \vdots \\ 0 \end{pmatrix} \in M$ we have

$$d(X, P) \in \{k, k+1\} \quad \text{and} \quad d(X, P) = k \Leftrightarrow x \in p_1 + \langle p_2, \ldots, p_m \rangle \subset D^n,$$

where $\langle p_2, \ldots, p_m \rangle$ denotes the subspace of $D^n$ which is spanned by $\{p_2, \ldots, p_m\}$.

Proof. We have $d(X, P) = \text{rank} \begin{pmatrix} x-p_1 \\ -p_2 \\ \vdots \\ -p_m \end{pmatrix} \geq \text{rank} \begin{pmatrix} p_2 \\ \vdots \\ p_m \end{pmatrix}$. Furthermore,

$$\text{rank} \begin{pmatrix} x-p_1 \\ -p_2 \\ \vdots \\ -p_m \end{pmatrix} = \text{rank} \begin{pmatrix} p_2 \\ \vdots \\ p_m \end{pmatrix} \Leftrightarrow x = p_1 + \langle p_2, \ldots, p_m \rangle,$$

$$\Leftrightarrow x \in p_1 + \langle p_2, \ldots, p_m \rangle. \quad \Box$$

Corollary 2.1. a) In the case $m \leq n$ let $M$ be a maximal set of type 1. Let $P \in M_{m \times n}(D)$ with $d(P,M) = m - 1$. Then $\{X \in M \mid d(X, P) = m - 1\}$ is an affine $(m - 1)$-flat of $M$, if we consider $M$ as an $n$-dimensional affine space.

b) In the case $n \leq m$ let $M$ be a maximal set of type 2. Let $P \in M_{m \times n}(D)$ with $d(P,M) = n - 1$. Then $\{X \in M \mid d(X, P) = n - 1\}$ is an affine $(n - 1)$-flat of $M$, if we consider $M$ as an $m$-dimensional affine space.
Lemma 2.3. Let $M$ be a maximal set of $M_{m\times n}(D)$. If $M$ is of type 1,

$$\rho(M) = \begin{cases} m - 1 & \text{when } m \leq n, \\ n & \text{when } m > n. \end{cases}$$

If $M$ is of type 2,

$$\rho(M) = \begin{cases} n - 1 & \text{when } m \geq n, \\ m & \text{when } m < n. \end{cases}$$

**Proof.** We only prove the case that $M$ is of type 1. The case that $M$ is of type 2 can be proved similarly. Since the covering radius is invariant under the group $G_{m\times n}(D)$, we can assume without loss of generality that $M$ is of the form (2.1). Let $A = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in M_{m\times n}(D)$ be any point. In the case $m \leq n$ from Lemma 2.2 we have $d(A,M) = \text{rank}(\begin{pmatrix} a_2 \\ \vdots \\ a_m \end{pmatrix}) \leq m - 1$. Thus we have $\rho(M) = m - 1$. In the case $m > n$, for any point $A$ we have $d(A,M) = \text{rank}(\begin{pmatrix} a_2 \\ \vdots \\ a_m \end{pmatrix}) \leq n$ and $d(A,M) = n$ if $\text{rank}(\begin{pmatrix} a_2 \\ \vdots \\ a_m \end{pmatrix}) = n$. □

Lemma 2.4. a) Let $m \leq n$. Let $M$ be a maximal set of type 1. Consider $M$ as an $n$-dimensional left affine space, then for any hyperplane $H$ of $M$ we have $\rho(H) = m$.

b) Let $n \leq m$. Let $M$ be a maximal set of type 2. Consider $M$ as an $m$-dimensional right affine space, then for any hyperplane $H$ of $M$ we have $\rho(H) = n$.

**Proof.** We prove a). Let $M$ be a maximal set of type 1. Without loss of generality let $0 \in M_1 \cap M_2$. Then for any $A \in M_{m\times n}(D)$ with $\text{rank}(A) = m$ we have $d(A,M) = m - 1 = d(A,M_1 \cup M_2)$. □

Lemma 2.5. Let $M_1$, $M_2$ be two distinct maximal sets with $M_1 \cap M_2 \neq \emptyset$.

a) Let $m \leq n$. If $M_1$ and $M_2$ are of type 1, or $m = n$ and $M_1$, $M_2$ are of different type, then $\rho(M_1 \cup M_2) = m - 1$.

b) Let $n \leq m$. If $M_1$ and $M_2$ are of type 2, or $m = n$ and $M_1$, $M_2$ are of different type, then $\rho(M_1 \cup M_2) = n - 1$.

**Proof.** a) Let $m \leq n$. Without loss of generality we can assume that $0 \in M_1 \cap M_2$. Then for any $A \in M_{m\times n}(D)$ with $\text{rank}(A) = m$ we have $d(A,M_1) = m - 1 = d(A,M_2)$. Thus $d(A,M_1 \cup M_2) = m - 1$ and $\rho(M_1 \cup M_2) = m - 1$. The case b) can be proved similarly. □
Lemma 2.6. a) Let $m \leq n$,
\[
M_1 := \left\{ \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \\ \end{pmatrix} \mid x_1 \in D^n \right\}, \quad M_2 := \left\{ \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ 0 \\ \end{pmatrix} \mid x_2 \in D^n \right\},
\]
\[
M_{12} := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ 0 \\ \end{pmatrix} \mid \text{rank} \left( \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ 0 \\ \end{pmatrix} \right) = 2 \right\}
\]
and $S := M_1 \cup M_2 \cup M_{12}$. Then $\rho(S) = m - 2$ if $m > 2$ and $\rho(S) = m - 1$ if $m = 2$.

b) Let $n \leq m$,
\[
M_1 := \left\{ \begin{pmatrix} y_{11} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & 0 & \cdots & 0 \\ \end{pmatrix} \mid y_{11}, \ldots, y_{m1} \in D \right\}, \quad M_2 := \left\{ \begin{pmatrix} 0 & y_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & y_{m2} & \cdots & 0 \\ \end{pmatrix} \mid y_{12}, \ldots, y_{m2} \in D \right\},
\]
\[
M_{12} := \left\{ \begin{pmatrix} y_{11} & y_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & 0 \\ \end{pmatrix} \mid \text{rank} \left( \begin{pmatrix} y_{11} & y_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & 0 \\ \end{pmatrix} \right) = 2 \right\}
\]
and $S := M_1 \cup M_2 \cup M_{12}$. Then $\rho(S) = n - 2$ if $n > 2$ and $\rho(S) = n - 1$ if $n = 2$.

Proof. Let $m \leq n$. Since $M_1 \subset S$ and $d(X,S) = m - 2$ for all $X \in M_{m \times n}(D)$ with $\text{rank}(X) = m$, we have $m - 2 \leq \rho(S) \leq \rho(M_1) = m - 1$. For any $X \in M_{m \times n}(D)$ we have $d(X,S) \leq d(X,0) = \text{rank}(X)$. Thus $d(X,S) = m - 1$ implies $\text{rank}(X) \in \{m, m - 1\}$. In the case $m > 2$ let $x_i$ denote the $i^{th}$ row vector of $X$. If $\text{rank} \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = 2$, let $Y = \begin{pmatrix} x_2 \\ \vdots \\ 0 \end{pmatrix} \in S$, then $d(X,Y) = \text{rank}(X - Y) = \text{rank} \left( \begin{pmatrix} \vdots \\ x_m \end{pmatrix} \right) \leq m - 2$. Now let $\text{rank}(X) = m - 1$ and $\text{rank} \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) = 1$. There exists $v \in \langle x_3, \ldots, x_m \rangle \setminus \langle x_1, x_2 \rangle$. Let $Y = \begin{pmatrix} x_1 - v \\ \vdots \\ 0 \end{pmatrix} \in S$, then $d(X,Y) = \text{rank} \left( \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \right) = m - 2$. Thus $\rho(S) = m - 2$ for $m > 2$. In the case $m = 2$, for any $X \in M_{m \times n}(D) \setminus S$ we have $d(X,S) = 1$, thus $\rho(S) = 1 = m - 1$.

Lemma 2.7. Let $M_1$, $M_2$ and $M_{12}$ be defined as in Lemma 2.6. Then for any $X \in M_{12}$ there is a maximal set $M_3$ of the same type as $M_1$, which contains $X$ and satisfies $M_3 \cap M_1 \neq \emptyset \neq M_3 \cap M_2$.

Proof. Let $m \leq n$. Let $X \in M_{12}$. For any $Y_i \in M_i$ which are adjacent to $X$, $i = 1, 2$, we have
\[
Y_1 = \begin{pmatrix} x_1 + \lambda_1 x_2 \\ \vdots \\ 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} \lambda_2 x_1 + x_2 \\ \vdots \\ 0 \end{pmatrix},
\]
and $Y_1, Y_2$ are adjacent if, and only if, $\lambda_1 \lambda_2 = 1$. Choose $\lambda_1 \neq 0$, $\lambda_2 = \lambda_1^{-1}$ and let $M_3$ be a maximal set which contains $X$, $Y_1$ and $Y_2$. Then $M_3 \cap M_1 = \{Y_1\}$, $|M_3 \cap M_1| = 1$, and $M_3$ is of type 1.

**Lemma 2.8.** Let $m = n$. Let $M_1$ be a maximal set of type 1 and $M_2$ a maximal set of type 2 such that $M_1 \cap M_2 \neq \emptyset$. Then for any $A \in M_1 \cap M_2$ there exists $Q \in M_{m \times n}(D)$ such that $d(Q, M_1) = d(Q, M_2) = m - 1$, $d(Q, A) = m - 1$ and $H_1 \cap H_2 = \{A\} = H_1 \cap M_2 = H_2 \cap M_1$ where

$$H_1 = \{X \in M_1 \mid d(Q, X) = m - 1\}, \quad H_2 = \{Y \in M_2 \mid d(Q, Y) = m - 1\}.$$ 

**Proof.** Without loss of generality we can assume that $M_1$ is of the form (2.1) and $M_2$ is of the form (2.3). For any $A \in M_1 \cap M_2$, $A = \left( \begin{array}{c} a_{11} \ 0 \ 0 \ \ddots \ 0 \\ \vdots \ 0 \ 0 \ \ddots \ \ddots \ 0 \\ \vdots \ \vdots \ \ddots \ \ddots \ \ddots \ \ddots \ 0 \end{array} \right)$, let $Q := \left( \begin{array}{c} a_{11} \ 0 \ 0 \ \ddots \ 0 \\ \vdots \ 0 \ 0 \ \ddots \ \ddots \ 0 \\ \vdots \ \vdots \ \ddots \ \ddots \ \ddots \ \ddots \ 0 \end{array} \right) \in M_{m \times n}(D)$. Then $d(Q, M_1) = d(Q, M_2) = m - 1$ and $d(Q, A) = m - 1$. Let $H_1 = \{X \in M_1 \mid d(Q, X) = m - 1\}$ and $H_2 = \{Y \in M_2 \mid d(Q, Y) = m - 1\}$, then we have $H_1 = \{X \in M_1 \mid x_{11} = a_{11}\}$, $H_2 = \{Y \in M_2 \mid y_{11} = a_{11}\}$ and $H_1 \cap H_2 = \{A\} = H_1 \cap M_2 = H_2 \cap M_1$. 

**Lemma 2.9.** Let $n \geq 2$. Let $V, V'$ be $n$-dimensional left or right affine spaces over a division ring $D$, $D \neq \mathbb{F}_2$. Let $f : V \to V'$ be an injective mapping which takes collinear points to collinear points, and let $f(V)$ not be contained in any affine hyperplane of $V'$. Then $f$ induces injective mappings $f_t : V(t) \to V'(t)$ where $V(t)$, $V'(t)$ denote the sets of all affine $t$-flats of $V$ resp. $V'$, $0 \leq t \leq n$. Furthermore, assume there exists an integer $k$, $0 < k < n$ such that $f_k : V(k) \to V'(k)$ is bijective. Then $f$ is bijective and $f$ takes lines to lines.

**Proof.** Since $f$ takes collinear points to collinear points, we have

$$f(\langle a_0, a_1, \ldots, a_t \rangle) \subseteq (f(a_0), f(a_1), \ldots, f(a_t))$$

for all points $a_0, \ldots, a_t \in V$, where $\langle a_0, \ldots, a_t \rangle$ denotes the affine flat spanned by $a_0, \ldots, a_t$. $f(V)$ is not contained in any affine hyperplane of $V'$, thus $f$ takes any affine basis of $V$ to an affine basis of $V'$. This implies $\dim(v + U) = \dim(\{f(v + U)\})$ for any affine flats $v + U$ of $V$. Let $V(t)$, $V'(t)$ denote the sets of all affine $t$-flats of $V$ resp. $V'$, $0 \leq t \leq n$. Then $f$ induces injective mappings $f_t : V(t) \to V'(t)$ for all $0 \leq t \leq n$ defined by $f_t(v + U) := \langle f(v + U) \rangle \subset V'(t)$ for any $t$-dimensional affine flat $v + U \subset V(t)$. Now let $k$ be an integer, $0 < k < n$ such that $f_k : V(k) \to V'(k)$ is bijective. We prove by induction that $f_t$ is bijective for all $0 \leq t \leq k$. This is the assumption on $f$ in the case $t = k$. Let $f_t$ be bijective for some $0 \leq t < k$. Let $s + T$ be an affine $(t-1)$-flat of $V'$. Let $v'_1 + U'_1, v'_2 + U'_2$ be two distinct affine $t$-flats of $V'$ with $(v'_1 + U'_1) \cap (v'_2 + U'_2) = s + T$. Since $f_1 : V(t) \to V'(t)$ is bijective, there are $v_1 + U_1, v_2 + U_2 \subset V(t)$ with $f_1(v_1 + U_1) = v'_1 + U'_1$, $i = 1, 2$. Since $v'_1 + U'_1$ and $v'_2 + U'_2$ are contained in an affine $(t-1)$-flat, also $v_1 + U_1$ and $v_2 + U_2$ are contained in an affine $(t-1)$-flat. Suppose $(v_1 + U_1) \cap (v_2 + U_2) = \emptyset$, i.e., $(v_1 + U_1) \parallel (v_2 + U_2)$. For any point $x \in v_1 + U_1$, its image $f(x) \not\in (v'_1 + U'_1) \cap (v'_2 + U'_2)$ since otherwise the join $\{x\} \cup (v_2 + U_2)$ would be contained in $v'_2 + U'_2$, and $f(V)$ would be contained in an affine hyperplane of $V'$. Let $v' + U'$ be any affine $t$-flat of $V'$ such that $v' + U'$ is contained in the affine $(t+1)$-flat $(v'_1 + U'_1) \cup (v'_2 + U'_2)$.
and \((w'+U') \parallel (w_2'+U_2')\). Then \((v+U) \parallel (v_2+U_2)\) implies \((v+U) \parallel (v_1+U_1)\) where \(v+U := f_i(v'+U')^{-1}\). Analogously we have \(f(x) \notin (w'+U') \cap (w_1'+U_1')\) for all \(x \in (v_1+U_1)\). Thus
\[
f(x) \notin v_1'+U_1' = \bigcup_{v'\in(v_1'+U_1') \cap (v_2'+U_2')} ((v'+U') \cap (v_1'+U_1')) \quad \forall x \in v_1+U_1,
\]
a contradiction to \(f_i(v_1+U_1) = v_1'+U_1'\). So we have \((v_1+U_1) \cap (v_2+U_2) \neq \emptyset\) and \(\dim((v_1+U_1) \cap (v_2+U_2)) = t-1\). Hence \(f = f_0\) is bijective. Let \(l\) be any line of \(V\), then \(f_i(l)\) is a line in \(V'\). Let \(Q\) be any point of \(f_i(l)\). Since \(f\) is bijective there is a point \(P\) with \(f(P) = Q\). Assume \(P \notin l\). Then the plane spanned by \(P\) and \(l\) is mapped by \(f\) to a subset of the line \(f_i(l)\), a contradiction. So \(f(l) = f_i(l)\) for any line \(l \subset V\). \(\square\)

**Lemma 2.10.** Let \(n \geq 2\). Let \(V, V'\) be \(n\)-dimensional left or right affine spaces over a division ring \(D\). Let \(f : V \rightarrow V'\) be an injective mapping which takes any line onto a line, i.e., for any line \(l \in V\), its image \(f(l)\) is a line in \(V'\). Then \(f\) is a bijection.

**Proof.** The assertion is true when \(D\) is finite. Now let \(D\) be infinite. For any \(f(X) \neq f(Y) \in f(V)\) the line \(\langle f(X), f(Y) \rangle = f(\langle X, Y \rangle)\) is contained in \(f(V)\). Then \(f(V)\) is an affine subspace of \(V'\). \(V\) and \(f(V)\) are isomorphic, so we have \(n = \dim(V) = \dim(f(V))\). This implies \(V' = f(V)\). \(\square\)

3. Proof of Theorem 1.1

We will prove the theorem only in the case \(m \leq n\). We can prove the theorem in the case \(n < m\) analogously to the case \(m < n\) by replacing maximal sets of type 1 by maximal sets of type 2 and vice versa.

We prove the theorem in several steps.

(i) For any maximal set \(M\), there is a maximal set \(M'\) containing \(M^\varphi\).

**Proof.** This follows immediately from the fact that \(\varphi\) preserves adjacency. \(\square\)

(ii) Let \(A \in M_{m \times n}(D)\) and let \(M \subset M_{m \times n}(D)\). Then we have \(d(A, M) \geq d(A^\varphi, M^\varphi) \geq d(A^\varphi, M')\) and \(\rho(M) \geq \rho(M')\) for all \(M' \subset M_{m \times n}(D)\) with \(M^\varphi \subset M'\).

**Proof.** We prove that \(\rho(M) \leq \rho(M')\). Let \(X\) be a point with \(d(X, M') = \rho(M')\). Since \(\varphi\) is bijective, there is a point \(Y\) with \(Y^\varphi = X\) and we have \(d(Y, M) \geq d(X, M')\), thus \(\rho(M) \geq d(Y, M) \geq d(X, M') = \rho(M')\). \(\square\)

(iii) Let \(M\) be a maximal set. Then there exists exactly one maximal set \(M'\) with \(M' \supset M^\varphi\). If \(m < n\) and \(M\) is of type 1, then also \(M'\) is of type 1.

**Proof.** Assume there are two distinct maximal sets \(M'_1\) and \(M'_2\) with \(M^\varphi \subset M'_i\), \(i = 1, 2\). Since \(|M| > 1\) we have \(|M^\varphi| > 1\), which implies \(|M'_1 \cap M'_2| > 1\). By Proposition 2.3, \(M'_1\) and \(M'_2\) are not of the same type. By Proposition 2.6 we can assume that \(M'_1\) is of the form \((2.1)\) and \(M'_2\) is of the form \((2.3)\). Then \(M'_1 \cap M'_2 = \{x \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} : x \in D\}\).
Clearly, $\rho(M_1' \cap M_2') = m$. By Lemma 2.3, $\rho(M) = m - 1$. But $M^\varphi \subset M_1' \cap M_2'$. By (ii), $\rho(M_1' \cap M_2') \leq \rho(M)$, a contradiction. Therefore there is a unique maximal set $M' \supset M^\varphi$. Now let $m < n$. Since $M' \supset M^\varphi$, by (ii) we have $\rho(M') \leq \rho(M) = m - 1$. By Lemma 2.3, $M'$ is of type 1.

(iv) Let $M_1$, $M_2$ be two distinct maximal sets of type 1. Let $M_i'$ be two maximal sets with $M_i' \supset M_i^\varphi$. Then $M_1'$ and $M_2'$ are of the same type.

**Proof.** In the case $m < n$, by (iii) the sets $M_i'$ are of type 1.

In the case $m = n$ assume that $M_1'$ and $M_2'$ are not of the same type. Since $M_1$ and $M_2$ can be joined by a sequence of consecutively intersecting maximal sets of the same type, we may suppose that $M_1 \cap M_2 \neq \emptyset$. Let $\{X\} := M_1 \cap M_2$. By Proposition 2.6 we may assume without loss of generality that

$$X = 0, \quad M_1 = \left\{ \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mid v \in D^n \right\}, \quad M_2 = \left\{ \begin{pmatrix} 0 \\ w \\ \vdots \\ 0 \end{pmatrix} \mid w \in D^n \right\}.$$}

Let $Y := X^\varphi \in M_1' \cap M_2'$. From Lemma 2.8 there exists a point $Q$ with $d(Q, M_1') = m - 1 = d(Q, M_2')$ and $d(Q, Y) = m - 1$ and such that $H_i' \cap M_i' = \{Y\}$, $H_i' \cap M_i' = \{Q\}$, where $H_i' := \{ A \in M_i' \mid d(A, Q) = m - 1 \}, \ i = 1, 2$. Let $P \in M_{m \times n}(D)$ with $P^\varphi = Q$. Since $m - 1 = d(Q, M_i') \leq d(P, M_i) \leq m - 1$, $d(P, M_i) = m - 1$. Define $H_i := \{ A \in M_i \mid d(A, P) = m - 1 \}$. We write $P = \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix}$, then $p_1 \neq 0 \neq p_2$ and

$$H_1 = \left\{ \begin{pmatrix} v \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mid v \in p_1 + \langle p_2, \ldots, p_m \rangle \right\}, \quad H_2 = \left\{ \begin{pmatrix} 0 \\ w \\ \vdots \\ 0 \end{pmatrix} \mid w \in p_2 + \langle p_1, p_3, \ldots, p_m \rangle \right\}.$$}

If $p_1 = \lambda p_2$ for some $\lambda \in D^*$ then let $A := \begin{pmatrix} p_1 \\ \vdots \\ 0 \end{pmatrix} \in H_1 \setminus \{0\}$ and $B := \begin{pmatrix} 0 \\ p_2 \\ \vdots \\ 0 \end{pmatrix} \in H_2 \setminus \{0\}$, we have $d(A, B) = 1$. If $p_1, p_2$ are linearly independent, then for

$$A = \begin{pmatrix} p_1 + p_2 \\ \vdots \\ 0 \end{pmatrix} \in H_1 \setminus \{0\}, \quad B = \begin{pmatrix} 0 \\ p_2 + p_1 \\ \vdots \\ 0 \end{pmatrix} \in H_2 \setminus \{0\}$$}

we have $d(A, B) = 1$. Let $M$ be a maximal set containing 0, $A$ and $B$. Since $|M \cap M_i| \geq 2$, $M$ is of type 2. Let $M'$ be the maximal set containing $M^\varphi$. Then $|M' \cap M_i| \geq 2$, thus $M' = M_1'$ or $M' = M_2'$, and $B^\varphi \in M' \cap M_1' = M_i' \cap M_1'$ or $A^\varphi \in M_1' \cap M_1' = M_1' \cap M_2'$. But $B^\varphi \in H_2'$ and $A^\varphi \in H_1'$, so we have $B^\varphi = 0^\varphi$ or $A^\varphi = 0^\varphi$, a contradiction to the injectivity of $\varphi$. □

(v) For any two distinct maximal sets $M_1$ and $M_2$ of type 1 with $M_1 \cap M_2 \neq \emptyset$ there is no maximal set $M$ which contains $M_1^\varphi \cup M_2^\varphi$.

**Proof.** Suppose there is a maximal set $M$ which contains $M_1^\varphi \cup M_2^\varphi$. Without loss of generality let $M_1$, $M_2$, $M_{12}$, $S$ be defined as in Lemma 2.6 a). Then for any $X \in M_{12}$, there is a maximal
set $M_i$ of type 1 which contains $X$ and intersects $M_i$ with $M_3 \cap M_1 =: \{A\} \neq \{B\} := M_3 \cap M_2$. Then $A^\varphi, B^\varphi \in M_i^\varphi \cap M$, thus $|M_i^\varphi \cap M| \geq 2$. From (iv) we have that the maximal sets containing $M_i^\varphi$ and $M$ are of the same type. So $M_i^\varphi \subset M$, which implies $X^\varphi \in M$. Then $S^\varphi \subset M$. In the case $m > 2$, from Lemma 2.6 we have $\rho(S) = m - 2$. $S^\varphi \subset M$ implies 
$\rho(M) \leq \rho(S^\varphi) \leq \rho(S) = m - 2$, a contradiction to $\rho(M) = m - 1$. In the case $m = 2$ let $Q \in M_{m \times n}(D)$ with $P^\varphi = Q$ then 
$d(P, 0) \geq d(P^\varphi, 0^\varphi) = d(Q, 0^\varphi) = 2$. Thus rank($P$) = 2, i.e., $P \in S$, and $Q = P^\varphi \in S^\varphi \subset M$, a contradiction to $Q \notin M$.

(vi) Let $M_1$ and $M_2$ be two maximal sets of type 1 and type 2 respectively, such that $M_1 \cap M_2$ is a line. Then $M_1^\varphi \cap M_2^\varphi$ is a line, where $M_i^\varphi$ is the maximal set with $M_i^\varphi \supset M_i^\varphi$, $i = 1, 2$. Thus $\varphi$ takes any line to the subset of a line.

Proof. Since $M_1^\varphi \cap M_2^\varphi \supset (M_1 \cap M_2)^\varphi$, $M_1^\varphi \cap M_2^\varphi$ is a line or $M_1^\varphi = M_2^\varphi$. Choose points $X \in M_1 \cap M_2$ and $Y \in M_2 \setminus M_1$. Let $M_3$ be the maximal set of type 1 with $X, Y \in M_3$. Then from (iv) and (v) we have that $M_3 \neq M_i^\varphi$ are of the same type where $M_3$ is the maximal set containing $M_i^\varphi$. Therefore $|M_1^\varphi \cap M_3^\varphi| = 1$. Since $X^\varphi, Y^\varphi \in M_1^\varphi \cap M_3^\varphi$, $X^\varphi \neq Y^\varphi$, we have $|M_2^\varphi \cap M_3^\varphi| \geq 2$ and thus $M_2^\varphi \neq M_i^\varphi$. It follows that $M_1^\varphi \cap M_2^\varphi$ is a line.

(vii) Let $M$ be a maximal set of type 1 and let $M'$ be the maximal set containing $M^\varphi$. Consider $M$ and $M'$ as affine spaces. Then $M^\varphi$ is not contained in any affine hyperplane of $M'$.

Proof. Assume $M^\varphi$ is contained in a hyperplane $H$ of $M'$. Then by Lemma 2.4 and (ii), we have $m = \rho(H) \leq \rho(M^\varphi) \leq \rho(M) = m - 1$, a contradiction.

(viii) Let $M$ be a maximal set of type 1 and let $M'$ be the maximal set containing $M^\varphi$. Then $\varphi : M \to M'$ is bijective and takes lines to lines.

Proof. The assertion is true when $D$ is finite. Now let $D$ be infinite. Consider $M, M'$ as affine spaces. Then $\varphi : M \to M'$ takes collinear points to collinear points, and by (vii), $M^\varphi$ is not contained in a hyperplane of $M'$. By Lemma 2.9, $\varphi$ induces an injective mapping $\varphi_{m-1} : M(m-1) \to M'(m-1)$, where $M(m-1), M'(m-1)$ denote the sets of all affine $(m-1)$-flats of $M$ resp. $M'$. Now let $U'$ be an arbitrary affine $(m-1)$-flat of $M'$. There is a point $P^\varphi$ of $M_{m \times n}(D)$ such that $d(P^\varphi, X) = m - 1 \ \forall X \in U'$, $d(P^\varphi, X) = m \ \forall X \in M' \setminus U'$.

By (ii) and Lemma 2.3, we have $m - 1 = d(P^\varphi, M') \leq d(P, M) \leq \rho(M) = m - 1$ and $d(P, M) = m - 1$. Let $U := \{X \in M \mid d(P, X) = m - 1\}$. By Corollary 2.1, $U$ is an $(m-1)$-flat of $M$, and $U^\varphi \subset U'$, this implies that $\varphi_{m-1}$ is bijective. By Lemma 2.9, $\varphi : M \to M'$ is bijective and takes lines to lines.

(ix) Let $l$ be any line of $M_{m \times n}(D)$. Then $l^\varphi$ is a line of $M_{m \times n}(D)$.

Proof. The assertion is true when $D$ is finite. Now let $D$ be infinite. Let $M$ be a maximal set of type 1 containing $l$. Let $M'$ be the maximal set containing $M^\varphi$. Consider $M, M'$ as affine spaces. Then by (vii) $\varphi : M \to M'$ takes lines to lines.

(x) Two points $A, B \in M_{m \times n}(D)$ are adjacent if $A^\varphi, B^\varphi$ are adjacent.
Proof. Choose maximal sets $M_1$, $M_2$ of type 1 with $M_1 \ni A$, $M_2 \ni B$. Let $M'_i$ be the unique maximal set containing $M_i^v$, $i = 1, 2$. By (iv), $M'_1$ and $M'_2$ are of the same type. Let $M'$ be a maximal set containing $A^\varphi$ and $B^\varphi$, which is not of the same type as $M'_i$. Then $A^\varphi \in M'_1 \cap M'$, $B^\varphi \in M'_2 \cap M'$, and $M'_i \cap M'$ is a line, $i = 1, 2$. There exist lines $g_1 \ni A$ and $g_2 \ni B$ in $M_1$ resp. $M_2$ with $g_i^\varphi = M_i' \cap M'$, $i = 1, 2$. Choose two maximal sets $S_i$ of type 2 with $S_i \cap M_i = g_i$. Then $S_i^\varphi \subset M'$. Consider $S_i$ as $m$-dimensional right affine space over $D$. If $m < n$ then $M'_i$ is of type 1 and $M'$ is of type 2. Then $M'$ is also an $m$-dimensional right affine space over $D$. In the case $m = n$, if $M'$ is of type 1 then $M'$ can be considered as $m$-dimensional left affine space over $D$. The restriction $\varphi|_{S_i} : S_i \to M'$ is injective and takes lines to lines by (ix). Thus by Lemma 2.10, $S_i^\varphi = M'$ for $i = 1, 2$. This implies that $S_1 = S_2$ and $A, B$ are adjacent. □

References


Received July 27, 2003