Configurations of Lines and General Hyperplane Sections

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Abstract. Let $Y \subset \mathbb{P}^n$ be a finite union of lines and $H \subset \mathbb{P}^n$ a general hyperplane. Here we study the linearly general position of the finite set $Y \cap H$.

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1. Introduction

Let $S \subset \mathbb{P}^N$ be a finite set. $S$ is said to be in linearly general position if for any $S' \subseteq S$ the linear span $\langle S' \rangle$ of $S$ spans a linear subspace of dimension $\min\{N, \text{card}(S') - 1\}$. It is well-known that if $C \subset \mathbb{P}^n$ is an integral non-degenerate curve and the algebraically closed base field $K$ has characteristic zero, then a general hyperplane section of $C$ is in linearly general position ([5], Lemma 1.1). Obviously, this result is not true for an arbitrary reducible curve and the first aim of this paper is to classify exactly when it is true when $C$ is a union of lines. We will also work over an arbitrary algebraically closed field $K$. We recall that in general the corresponding result for irreducible curve is not true in positive characteristic ([5]). In Section 2 we will prove the following result.

Theorem 1. Let $X \subset \mathbb{P}^n$ be a finite union of lines. Assume that a general hyperplane section of $X$ is not in linearly general position and let $s$ be the first integer such that $1 \leq s \leq n - 2$ and for a general hyperplane $H$ the set $X \cap H$ contains a set of at least $s + 2$ points spanning a linear space of dimension $s$. Let $A_1, \ldots, A_x$, $x \geq 1$, be the $s$-dimensional linear subspaces of $H$ containing at least $s + 2$ points of $X \cap H$. Set $S_i := A_i \cap X$. Then there are $(s+1)$-dimensional

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linear subspaces $M_i \subset \mathbb{P}^n$ and subcurves $X_i \subset M_i \cap X$ with $\deg(X_i) = \text{card}(S_i) \geq s + 2$. Furthermore, $S_i = X_i \cap H$ for a general hyperplane $H$.

Theorem 1 is a quantitative and precise version of the following immediate corollary of it.

**Corollary 1.** Let $X \subset \mathbb{P}^n$ be a finite union of lines. A general hyperplane section of $X$ is in linearly general position if and only if there is no positive integer $m \leq n - 2$ such that at least $m + 2$ lines of $X$ are contained in an $m$-dimensional linear subspace of $\mathbb{P}^n$.

We will say that a reduced curve $X \subset \mathbb{P}^n$ is a dismantled curve or a configuration of lines if each irreducible component of $X$ is a line.

**Remark 1.** Let $X \subset \mathbb{P}^n$ be a non-degenerate dismantled curve. In general, it is not true that $X \cap H$ spans $H$ for a general hyperplane $H$. For instance take $n = 3$ and $X$ the union of two disjoint lines. However, by Theorem 1 if $X \cap H$ spans a linear subspace of dimension $s \leq n - 2$ and $X$ contains no subcurve of degree at least $s + 2$ contained in a linear space of dimension at most $s + 1$, then $\deg(X) = s + 1$.

**Remark 2.** Fix integers $n$, $d$ with $n \geq 3$ and $d \geq n$. Set $m := [(d - 1)/(n - 1)]$, $\epsilon := d - 1 - m(n - 1)$ and $\pi(n, d) := m(m - 1)(n - 1)/2 + m \epsilon$. Let $X \subset \mathbb{P}^n$ be a degree $d$ non-degenerate reduced curve such that the general hyperplane section of $X$ is in linearly general position and spans the corresponding hyperplane. By Castelnuovo theory (see e.g. [4], Ch. 3, or [3], p. 252) we have $p_a(X) \leq \pi(n, d)$. Fix a hyperplane $H$ not containing any irreducible component of $X$. From the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_X(1) \to \mathcal{I}_{X \cap H,H}(1) \to 0$$

we see that $X \cap H$ spans $H$ if $X$ is connected.

Now we consider the postulation of the subsets of a generic hyperplane section of a configuration of lines with respect to the homogeneous forms of degree $s \geq 2$.

**Theorem 2.** Fix integers $n \geq 3$, $s \geq 2$ and $b > (2n^2)/n^2 / (n - 1)$ and a configuration $Y \subset \mathbb{P}^3$ of lines. Assume that for a general hyperplane $H$ the following conditions are satisfied:

(a) for all integers $t$ with $1 \leq t < s$ and all $A \subseteq Y \cap H$ we have $h^0(\mathbb{P}^n, \mathcal{I}_A(t)) = \max\{0, (\binom{n + t}{n} - \text{card}(A))\};$

(b) there is $S \subseteq Y \cap H$ such that $\text{card}(S) = b$ and $h^0(\mathbb{P}^3, \mathcal{I}_S(s)) \neq 0$.

Then there is a variety $F \subset \mathbb{P}^n$ such that at least $b - (2n^2)/n^2 / (n - 1)$ lines of $Y$ are contained in $F$, $F$ is a union of lines, $2 \leq \dim(F) \leq n - 1$, and $h^0(\mathbb{P}^n, \mathcal{I}_F(s)) \neq 0$.

The integer $(2n^2)/n^2 / (n - 1)$ in the statement of Theorem 2 is the degree of the Grassmannian $G(2, n + 1)$ of all lines of $\mathbb{P}^n$ with respect to the Plücker embedding of $G(2, n + 2)$ induced by the positive generator $\mathcal{O}_{G(2,n+1)}(1)$ of Pic$(G(2,n+1))$ ([2], Example 14.7.11). Since $\dim(G(2, n + 1)) = 2n - 2$, the integer $(2n^2)/n^2 / (n - 1)$ is the top self-intersection of $\mathcal{O}_{G(2,n+1)}(1)$. This observation explains the lower bound for $b$ appearing in the statement of Theorem 2. The thesis of Theorem 2 has two parts:
(a) several lines of $Y$ are contained in a degree $s$ hypersurface $G$ lifting the hypersurface of the generic hyperplane $H$ of $\mathbb{P}^n$ containing many points of $Y \cap H$;

(b) $G$ contains infinitely many lines; in particular if $n = 3$, then $G$ is a ruled surface.

In Section 4 we will consider the linear general position of a general section of a double structure $Z$ on a configuration of lines $Y$. We will allow unreduced curves $Z$ which are reduced at a general point of some of the lines contained in $Y$ (see Theorem 3).

2. Proof of Theorem 1

Proof of Theorem 1. For every $P \in S_1$ let $L_P \subseteq X$ be the line such that $\{P\} = L_P \cap H$. Set $X_i := \cup_{P \in S_1}L_P$ and $V_i := \langle X_i \rangle$, $1 \leq i \leq x$. Since the set of all hyperplanes of $\mathbb{P}^n$ is irreducible, while $X$ has only finitely many irreducible components, the integer $x$ and the subcurves $X_i$, $1 \leq i \leq x$, of $X$ do not depend upon the choice of the sufficiently general hyperplane $H$. Hence the linear spaces $M_i$, $1 \leq i \leq x$, do not depend upon the choice of the sufficiently general hyperplane $H$. Notice that $X_i$ is the maximal subcurve of $X$ such that $X_i \cap H$ is contained in $M_i$. By the minimality of $s$ for every $S' \subseteq S_i$ with $\text{card}(S') = s + 1$ we have $A_i = \langle S' \rangle$, i.e. each $S_i$ is in linearly general position in its linear span. To prove Theorem 1 it is sufficient to show that $\dim(V_i) = s + 1$ for every $i$. We assume that Theorem 1 fails for some dismantled curve and we take $n$ minimal with this property. If $n > s + 2$ a general projection of $Y$ into $\mathbb{P}^{s+2}$ gives a counterexample to Theorem 1. Hence by the minimality of $n$ and $s$ we obtain $n = s + 2$ and $\dim(M_i) = s + 2$ for some index $i$, say for $i = 1$. The dismantled curve $X_1$ gives a counterexample to Theorem 1. Hence, we reduced to the case $X = X_1$, $n = s + 2$. Let $Y$ be a degree $s + 1$ subcurve of $X_1$. By the minimality of $s$ either $\dim(\langle Y \rangle) = s + 1$ or $\dim(\langle Y \rangle) = s + 2$. We saw that $Y \cap H$ spans $A_1$. For every line $T \subseteq Y$ choose a general $P \in T$ and call $B$ the union of these $s + 1$ points. Since $\dim(\langle Y \rangle) \geq s + 1$ and the points of $B$ are sufficiently general, we have $\dim(\langle B \rangle) = s$. By the generality of $B$ a general hyperplane $H$ of $\mathbb{P}^{s+2}$ containing $B$ may be considered as a general hyperplane. Fix any general hyperplane $H$ containing $B$ and any line $D$ of $X_1$ with $D$ not in $Y$. By the definition of $A_1$ and $X_1$ and the generality of $H$ we have $H \cap D \subseteq \langle B \rangle$. Now move $H$ among all hyperplanes containing $B$. We obtain $D \subseteq \langle B \rangle$. Thus $X_1 \setminus Y \subseteq \langle B \rangle$. Moving each point of $B$ in the corresponding line of $Y$ we easily obtain a contradiction.

3. Proof of Theorem 2

Proof of Theorem 2. Let $G(2, n+1)$ be the Grassmanian of all lines of $\mathbb{P}^n$ and $\mathcal{O}_{G(2, n+1)}(1)$ the positive generator of $\text{Pic}(G(2, n+2))$, i.e. the line bundle inducing the Plücker embedding of $G(2, n+1)$. For any hyperplane $M \subseteq \mathbb{P}^n$ and any degree $s$ hypersurface $T$ of $M$, set $G(T, M) := \{D \in G(2, n+1): T \cap D \neq \emptyset\}$. $G(T, M)$ is the zero-locus of a non-zero section of $\mathcal{O}_{G(2, n+1)}(s)$. $G(2, n+1)$ has degree $\binom{2n-2}{n-2}/(n-1)$ with respect to the Plücker embedding and $\dim(G(2, n+1)) = 2n - 2$. Thus Bezout’s theorem implies that the intersection of at least $2n - 2$ hypersurfaces $G(T_i, M_i)$ either is infinite or it contains at most $\binom{2n-2}{n-2}s^{2n-2}/(n-1)$ points. Taking as $M_i$ all general hyperplanes of $\mathbb{P}^n$ and as $T_i$ the corresponding degree $s$ hypersurface containing at least $b$ points of $Y$, we obtain the existence of an irreducible variety $N \subseteq \mathbb{P}^n$ with $\dim(N) \geq 2$, $N$ union of lines, such that for a general hyperplane
H we have $h^0(H,\mathcal{I}_{N\cap H,H}(s)) \neq 0$ and $N \cap H$ contained in a degree $s$ hypersurface of $H$ containing $Y \cap H$. If $\text{card}(N \cap Y \cap H) \leq b - (\binom{2k-2}{n-2})s^{2n-2}/(n-1) - 1$, then we may iterate this construction. Since the intersection of all $G(T, M)$ has only finitely many irreducible components, after finitely many steps we find the ruled variety $F$ with all the properties claimed by Theorem 2.

\section{Double structures}

In this section we consider the linearly general position of general hyperplane sections of double structures on configurations of lines. The definition of linearly general position makes sense even for zero-dimensional subschemes of projective spaces ([1]).

**Theorem 3.** Let $Z \subset \mathbb{P}^n$ be a non-degenerate purely one-dimensional locally Cohen-Macaulay scheme such that $Y := Z_{\text{red}}$ is a configuration of lines and for each line $T \subseteq Y$ the multiplicity of $Z$ at a general point of $T$ is one or two. Assume that a general hyperplane section of $Y$ is in linearly general position in its linear span. Then the general hyperplane section of $Z$ is in linearly general position in its linear span.

**Proof.** Fix a general hyperplane $H$ and assume that the result is not true. Let $s$ be the minimal integer such that $1 \leq s < \dim(\langle Z \cap H \rangle)$ and there is an $s$-dimensional linear subspace $V$ of $H$ with $\text{length}(Z \cap V) \geq s + 2$. By the minimality of $s$ $V$ is spanned by $V \cap Z$. Let $Y' \subseteq Y$ be the union of all lines of $Y$ intersecting $V$ and $Z' \subseteq Z$ the maximal locally Cohen-Macaulay subscheme of $Z$ such that $Z'_{\text{red}} = Y'$. Since the general hyperplane section of $Y$ is in linearly general position, we have $Z' \cap V \neq Y' \cap V$. By the generality of $H$, $H$ is transversal to $Y$. Fix $P \in Y' \cap V$ such that $Z'$ is not reduced at $P$, i.e. it is unreduced at a general point of the line $T$ containing $P$, and the connected component of $Z \cap H$ supported by $P$ is contained in $V$. Let $A$ be the union of the connected components of the zero-dimensional scheme $Z \cap V$ supported by $Y \cap V \setminus \{P\}$. Hence $\text{length}(W) = \text{length}(Z \cap W) - 2$. Thus $\text{length}(W \cup \{P\}) = \text{length}(Z \cap V) - 1$. By the minimality of $s$ we have $\dim(\langle W \cup \{P\} \rangle) \geq s$ and hence $\langle W \cup \{P\} \rangle = V$. Now we move $H$ among all hyperplanes containing $M$ and call $H'$ a general such hyperplane. Since $H$ is general, the length two subscheme of $H' \cap Z$ supported by $P$ must be contained in $V$. Hence, varying $H'$ we see that $V$ contains $T$. Since $V \subset H$ and $Z \cap H$ is a zero-dimensional scheme, we have obtained a contradiction.

**References**


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