A Note on the Existence of \{k, k\}-equivelar Polyhedral Maps

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Abstract. A polyhedral map is called \(\{p, q\}\)-equivelar if each face has \(p\) edges and each vertex belongs to \(q\) faces. In [12], it was shown that there exist infinitely many geometrically realizable \(\{p, q\}\)-equivelar polyhedral maps if \(q > p = 4, p > q = 4\) or \(q - 3 > p = 3\). It was shown in [6] that there exist infinitely many \(\{4, 4\}\)- and \(\{3, 6\}\)-equivelar polyhedral maps. In [1], it was shown that \(\{5, 5\}\)- and \(\{6, 6\}\)-equivelar polyhedral maps exist. In this note, examples are constructed, to show that infinitely many self dual \(\{k, k\}\)-equivelar polyhedral maps exist for each \(k \geq 5\). Also vertex-minimal non-singular \(\{p, p\}\)-patterns are constructed for all odd primes \(p\).

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1. Introduction and results

A polyhedral complex (of dimension 2) is collection of cycles (finite connected 2-regular graphs) together with the edges and the vertices in the cycles such that the intersection of any two cycles is empty, a vertex or an edge. The cycles are called the faces of the polyhedral complex. For a polyhedral complex \(K\), \(V(K)\) denotes its vertex-set and \(EG(K)\) denotes its edge-graph or 1-skeleton. We say \(K\) finite if \(V(K)\) is finite. If \(EG(K)\) is connected then \(K\) is said to be connected.

A polyhedral complex is called a polyhedral 2-manifold (or an abstract polyhedron) if for each vertex \(v\) the faces containing \(v\) are of the form \(F_1, \ldots, F_m\), where \(F_1 \cap F_2, \ldots, F_{m-1} \cap F_m\)
$F_m, F_m \cap F_1$ are edges for some $m \geq 3$. A connected finite polyhedral 2-manifold is called a polyhedral map. A combinatorial 2-manifold is a polyhedral 2-manifold whose faces are 3-cycles. A polyhedral map is called $(p, q)$-equivelar if each face is a $p$-cycle and each vertex is in $q$ faces. A polyhedral map is called equivelar if it is $(p, q)$-equivelar for some $p, q$ (cf. [10, 3, 4, 11]).

To each polyhedral complex $K$, we associate a pure 2-dimensional simplicial complex $B(K)$ (called the barycentric subdivision of $K$) whose 2-faces are of the form $ueF$, where $(u, e, F)$ is a flag (i.e., $e$ is an edge of the face $F$ and $u$ is a vertex of $e$) in $K$. The geometric carrier of $B(K)$ is called the geometric carrier of $K$ and is denoted by $|K|$. Clearly, $K$ is a polyhedral 2-manifold if and only if $B(K)$ is a combinatorial 2-manifold (equivalently, $|K|$ is a 2-manifold). A polyhedral 2-manifold $K$ is called orientable if $|K|$ is orientable.

An isomorphism between two polyhedral complexes $K$ and $L$ is a bijection $\varphi : V(K) \rightarrow V(L)$ such that $(v_1, \ldots, v_m)$ is a face of $K$ if and only if $(\varphi(v_1), \ldots, \varphi(v_m))$ is a face of $L$. Two complexes are called isomorphic if there is a morphism between them. We identify two isomorphic polyhedral complexes. An isomorphism from $K$ to itself is called an automorphism of $K$. The set $\Gamma(K)$ of automorphisms of $K$ forms a group. A polyhedral 2-manifold $K$ is called combinatorially regular if $\Gamma(K)$ is transitive on flags (cf. [10]).

For a polyhedral 2-manifold $K$, consider the polyhedral complex $\tilde{K}$ whose vertices are the faces of $K$ and $(F_1, \ldots, F_m)$ is a face of $\tilde{K}$ if $F_1, \ldots, F_m$ have a common vertex and $F_1 \cap F_2, \ldots, F_{m-1} \cap F_m, F_m \cap F_1$ are edges. Then $\tilde{K}$ is a polyhedral 2-manifold and called the dual of $K$. If $K$ is isomorphic to $\tilde{K}$ then $K$ is called self dual.

A pattern is an ordered pair $(M, G)$, where $M$ is a connected closed surface in some Euclidean space and $G$ is a finite graph on $M$ such that each component of $M \setminus G$ is simply connected. The closure of each component of $M \setminus G$ is called a face of $(M, G)$. For a face $F$, the closed path (in $G$) consisting of all the edges and the vertices in $F$ is called the boundary of $F$. A pattern $(M, G)$ is said to be non-singular if the boundary of each face is a cycle. A non-singular pattern is said to be a polyhedral pattern if the intersection of any two faces is empty, a vertex or an edge. A pattern $(M, G)$ is called a $(p, q)$-pattern if each face contains $p$ edges and the degree of each vertex in $G$ is $q$ (cf. [7]).

If $(M, G)$ is a polyhedral pattern then clearly the boundaries of the faces of $(M, G)$ form a polyhedral map. Conversely, for a polyhedral map $K$, let $M = |K|$ and $G = EG(K)$. Then $(M, G)$ is a polyhedral pattern and the faces of $K$ are the boundaries of the faces of $(M, G)$. This pattern $(M, G)$ is called a geometric realization of $K$. A geometric realization $(M, G)$ (in some $\mathbb{R}^n$) is called linear if each face of $M$ is a convex polygon and no two adjacent faces (i.e., faces which share a common edge) lie in the same plane. If a polyhedral map has a linear geometric realization in $\mathbb{R}^3$ then it is called geometrically realizable.

If $f_0(K), f_1(K)$ and $f_2(K)$ are the number of vertices, edges and faces respectively of a polyhedral complex $K$ then the number $\chi(K) := f_0(K) - f_1(K) + f_2(K)$ is called the Euler characteristic of $K$. Observe that $\chi(B(K)) = \chi(K)$. If $u$ and $v$ are vertices of a face $F$ and $uv$ is not an edge of $F$ then $uv$ is called a diagonal. Clearly, if $d(K)$ is the number of diagonals of a polyhedral complex $K$ then $d(K) + f_1(K) \leq \left(\frac{f_0(K)}{2}\right)$ and in the case of equality each pair of vertices belongs to a face. A polyhedral map $K$ is called a weakly neighbourly polyhedral map (in short, wnp map) if each pair of vertices belongs to a common face.

We know (cf. [6]) that there exists a unique $(p, q)$-equivelar polyhedral map if $(p, q) = (3, 3)$.
(3, 4) or (4, 3) and there are exactly two \( \{p, q\} \)-equivelar polyhedral maps if \( (p, q) = (3, 5) \) or \( (5, 3) \). In [12], McMullen et al. constructed infinitely many geometrically realizable \( \{p, q\} \)-equivelar polyhedral maps for each \( (p, q) \in \{(r, 4) : r \geq 5\} \cup \{(4, s) : s \geq 5\} \cup \{(3, k) : k \geq 7\} \). In [6], it was shown that there exist infinitely many \( \{4, 4\} \)- and \( \{3, 6\} \)-equivelar polyhedral maps. It was also shown that there are exactly two neighbourly \( \{3, 8\} \)-equivelar polyhedral maps and there are exactly 14 neighbourly \( \{3, 9\} \)-equivelar polyhedral maps.

In [5], Coxeter constructed a geometrically realizable combinatorially regular infinite polyhedral 2-manifold whose faces are hexagons and each vertex is in six faces (namely, \( \{6, 6 | 3\} \)). In [9], Grünbaum constructed another combinatorially regular infinite polyhedral 2-manifold of type \( \{6, 6\} \) (namely, \( \{6, 6 | 4\} \) cf. [10]). In [8], Gott constructed a geometrically realizable infinite polyhedral 2-manifold whose faces are pentagons and each vertex is in five faces. If \( K \) is a \( \{p, q\} \)-equivelar polyhedral map on \( n \) vertices then \( d(K) = nq(p - 3)/2 \) and \( f_1(K) = nq/2 \). Therefore, if \( K \) is an \( n \)-vertex \( \{p, p\} \)-equivelar polyhedral map then \( np(p - 3)/2 + np/2 \leq n(n - 1)/2 \) and hence \( n \geq (p - 1)^2 \). Clearly, equality holds if and only if \( K \) is a wnp map. Let \( \alpha(p) \) denote the smallest \( n \) such that there exists an \( n \)-vertex \( \{p, p\} \)-equivelar polyhedral map. Clearly, the 4-vertex 2-sphere (the boundary of a 3-simplex) is the unique \( \{3, 3\} \)-equivelar wnp map. In [1], Brehm proved that there exist exactly three \( \{4, 4\} \)-equivelar wnp maps and constructed the 16-vertex \( \{5, 5\} \)-equivelar polyhedral map \( M_{5, 16} \) (of Example 1). It was shown in [2] that \( M_{5, 16} \) is the unique \( \{5, 5\} \)-equivelar polyhedral map on 16 vertices. So, \( \alpha(k) = (k - 1)^2 \) for \( k \leq 5 \). In [1], Brehm also constructed the 26-vertex \( \{6, 6\} \)-equivelar polyhedral map \( M_{6, 26} \) (of Example 1). Here we show:

**Theorem 1.** For each \( m \geq 3 \) and \( n \geq 0 \), there exist a \( 2(3^{m - 1} + 2n - 1) \)-vertex self dual \( \{2m - 1, 2m - 1\} \)-equivelar polyhedral map and a \( (3^m + 2n - 1) \)-vertex self dual \( \{2m, 2m\} \)-equivelar polyhedral map.

Thus \( (2m - 2)^2 \leq \alpha(2m - 1) \leq 2(3^{m - 1} - 1) \) and \( (2m - 1)^2 \leq \alpha(2m) \leq 3^m - 1 \) for all \( m \geq 3 \). In [13], using a computer, Nilakantan has shown that there does not exist any 25-vertex \( \{6, 6\} \)-equivelar polyhedral map. So, \( \alpha(6) = 26 \) and hence there does not exist any \( \{6, 6\} \)-equivelar wnp map. We believe the following is true:

**Conjecture 1.** There does not exist any \( \{k, k\} \)-equivelar wnp map for \( k \geq 7 \).

For the existence of an \( n \)-vertex \( \{k, k\} \)-pattern \( n \) must be at least \( k + 1 \). Here we show:

**Theorem 2.** There exists a \( (p + 1) \)-vertex non-singular \( \{p, p\} \)-pattern for each odd prime \( p \).

2. Examples and proofs of the results

We first construct infinitely many \( \{k, k\} \)-equivelar polyhedral maps. We need these to prove our results. We identify a polyhedral complex with the set of faces in it.

**Example 1.** For \( m \geq 3 \) and \( n \geq 0 \), let

\[
M_{2m - 1, 2(3^{m - 1} + 2n - 1)} = \{F_{i, 2m - 1} : 1 \leq i \leq 2(3^{m - 1} + 2n - 1)\},
\]

\[
M_{2m, 3^m + 2n - 1} = \{F_{i, 2m} : 1 \leq i \leq 3^m + 2n - 1\},
\]
where $b_{2l-1} = 3^{l-1} - 1$, $b_{2l} = 2 \times 3^{l-1} - 1$, for $l \geq 1$ and

\[
F_{1,2m-1} = (i + b_1, i + b_2, \ldots, i + b_{2m-3}, i + b_{2m-2} + n, i + b_{2m-1} + 2n),
\]

\[
F_{1,2m} = (i + b_1, i + b_2, \ldots, i + b_{2m-2}, i + b_{2m-1}, i + b_{2m} + n)
\]

are cycles ($(2m - 1)$-cycles and $(2m)$-cycles respectively) with vertices from $Z_{2(3^{m-1}+2n-1)}$ and $Z_{3^{m}+2n-1}$ respectively. Clearly, there are $2m-1$ faces through each vertex in $M_{2m-1,2(3^{m-1}+2n-1)}$ and there are $2m$ faces through each vertex in $M_{2m,3^{m}+2n-1}$ respectively. So, $f_1(M_{2m-1,2(3^{m-1}+2n-1)}) = (3^{m-1}+2n-1)(2m-1)$ and $f_1(M_{2m,3^{m}+2n-1}) = (3^{m}+2n-1)m$. Thus, $\chi(M_{2m-1,2(3^{m-1}+2n-1)}) = (3^{m-1}+2n-1)(5-2m)$ and $\chi(M_{2m,3^{m}+2n-1}) = (3^{m}+2n-1)(2-m)$. By Lemma 2 below, $M_{2m-1,2(3^{m-1}+2n-1)}$ and $M_{2m,3^{m}+2n-1}$ are polyhedral maps. But, by Lemma 4, none of these polyhedral maps are combinatorially regular.

**Lemma 1.** For a collection $\mathcal{C}$ of cycles, let $\bar{\mathcal{C}}$ be the 2-dimensional pure simplicial complex whose 2-faces are of the form $xyF$, where $F \in \mathcal{C}$ and $xy$ is an edge in $F$. If $B(\mathcal{C})$ is as defined earlier then the following three are equivalent.

(i) $B(\mathcal{C})$ is a combinatorial 2-manifold.

(ii) $\bar{\mathcal{C}}$ is a combinatorial 2-manifold.

(iii) For any vertex $v$, the cycles containing $v$ are of the form $F_1 = (v, v_{1,1}, \ldots, v_{1,n_1})$, $F_m = (v, v_{m,1}, \ldots, v_{m,n_m})$ such that $v_{1,n_1} = v_{2,1}, \ldots, v_{m-1,n_{m-1}} = v_{m,1}$, $v_{m,n_m} = v_{1,1}$ for some $m \geq 2$.

**Proof.** Clearly, $B(\mathcal{C})$ is a subdivision of $\bar{\mathcal{C}}$. Therefore, (i) and (ii) are equivalent.

For a 2-dimensional pure simplicial complex $X$, the link of a vertex $v$ is the graph $\text{lk}_X(v)$ whose vertex-set is $\{u \in V(X) : uv \in X\}$ and edge-set is $\{xy : xyv \in X\}$. Clearly, $X$ is a combinatorial 2-manifold if and only if $\text{lk}_X(v)$ is a cycle for each $v \in V(X)$.

Let $v$ be a vertex of $\bar{\mathcal{C}}$. If $v = F \in \mathcal{C}$ then $\text{lk}_F(v)$ is $F$ itself. Let $v$ be a vertex of $\bar{\mathcal{C}}$ which is not a cycle in $\mathcal{C}$. If the cycles containing $v$ are of the form $F_1 = (v, v_{1,1}, \ldots, v_{1,n_1}), F_m = (v, v_{m,1}, \ldots, v_{m,n_m})$ such that $v_{1,n_1} = v_{2,1}, \ldots, v_{m-1,n_{m-1}} = v_{m,1}$, $v_{m,n_m} = v_{1,1}$ for some $m \geq 2$ then $\text{lk}_F(v)$ is the cycle $v_{1,1}F_1v_{2,1}F_2 \cdots v_{m,1}F_m$. Conversely, if $\text{lk}_F(v)$ is a cycle then, from the definition of $\bar{\mathcal{C}}$, $\text{lk}_F(v)$ must be of the form $v_{1,1}F_1v_{2,1}F_2 \cdots v_{m,1}F_m$, where $F_1 = (v, v_{1,1}, \ldots, v_{1,n_1}), F_m = (v, v_{m,1}, \ldots, v_{m,n_m})$ such that $v_{1,n_1} = v_{2,1}, \ldots, v_{m-1,n_{m-1}} = v_{m,1}$, $v_{m,n_m} = v_{1,1}$. This proves that (ii) and (iii) are equivalent. □

**Lemma 2.** $M_{2m-1,2(3^{m-1}+2n-1)}$ and $M_{2m,3^{m}+2n-1}$ are polyhedral maps for $m \geq 3$, $n \geq 0$. 

![Diagram](attachment:image.png)
Proof. Since \(\{i, i+1\}\) is an edge in \(M_{2m-1,2(3^{m-1}+2n-1)}\) for each \(i\), \(EG(M_{2m-1,2(3^{m-1}+2n-1)})\) is connected. Similarly, \(EG(M_{2m,3^{m}+2n-1})\) is connected.

Observe that the faces in \(M_{2m-1,2(3^{m-1}+2n-1)}\) containing \(i\) are \(F_i, F_{i-b_2}, F_{i-b_3}, F_{i-b_4}, \ldots, F_{i-b_{2m-3}}, F_{i-b_{2m-2}}-n, F_{i-b_{2m-2}}-2n\), where \(F_i = F_{i,2m-1} = (i + b_1, i + b_2, \ldots, i + b_{2m-3}, i + b_{2m-2} + n, i + b_{2m-1} + 2n)\). Clearly, \(F_i \cap F_{i-b_3} = \cdots = F_i \cap F_{i-b_{2m-2}}-2n = \cdots = F_{i-b_{2m-2}}-2n \cap F_{i-b_{2m-3}} = \{i\}\).

Since \(b_{2j+1} = 2b_{2j} - b_{2j-1}\) for all \(j\), \(F_{i-b_{2j-1}} \cap F_{i-b_{2j}}\) is the edge \(\{i, i + b_{2l} - b_{2l-1}\}\), \(F_{i-b_{2l}} \cap F_{i-b_{2l+1}}\) is the edge \(\{i + b_{2l} - b_{2l+1}, i\}\) for \(1 \leq l \leq m - 2\), \(F_{i-b_{2m-3}} \cap F_{i-b_{2m-2}}-2n\) is the edge \(\{i, i + b_{2m-1} - b_{2m-2} + n\}\) and \(F_{i-b_{2m-2}}-2n \cap F_{i-b_{2m-1}}-2n\) is the edge \(\{i + b_{2m-3} - b_{2m-2} - n, i\}\). Again, since \(2b_{2m-1} + 2n \equiv 0\) (mod \(3^{m-1} + 2n - 1\)), \(F_{i-b_{2m-1}}-2n \cap F_i\) is the edge \(\{i, i + b_{2m-1} + 2n\}\).

Thus, any pair of faces containing \(i\) intersects in either at \(i\) or on an edge through \(i\) and the faces containing \(i\) form a single cycle of adjacent faces (sharing a common edge). Therefore, \(M_{2m-1,2(3^{m-1}+2n-1)}\) is a polyhedral map.

The faces in \(M_{2m,3^{m}+2n-1}\) containing \(i\) are \(C_i, C_{i-b_2}, C_{i-b_3}, \ldots, C_{i-b_{2m-1}}, C_{i-b_{2m-2}}, C_{i-b_{2m-3}}, \ldots\), where \(C_i = F_{i,2m} = (i + b_1, i + b_2, \ldots, i + b_{2m-1}, i + b_{2m} + n)\) and \(C_i \cap C_{i-b_3} = \cdots = C_i \cap C_{i-b_{2m-1}} = \cdots = C_{i-b_{2m-2}} \cap C_{i-b_{2m-3}} = \cdots = C_{i-b_{2m-2}} \cap C_{i-b_{2m-3}} = \{i\}\). Also, since \(2b_{2m-2} - b_{2m-1} + 2n \equiv 0\) (mod \(3^{m} + 2n - 1\)), \(C_{i-b_{2m-1}} \cap C_{i-b_{2m-2}}\) is the edge \(\{i, i + b_{2m-1} - b_{2m-2}\}\), \(C_{i-b_{2m-1}} \cap C_{i-b_{2m-2}}\) is the edge \(\{i + b_{2m-2} - b_{2m-1}, i\}\), \(C_{i-b_{2m-1}} \cap C_{i-b_{2m-2}}\) is the edge \(\{i + b_{2m-2} - b_{2m-1}, i\}\) for \(1 \leq l \leq m - 1\), \(C_{i-b_{2m-1}} \cap C_{i-b_{2m-2}}\) is the edge \(\{i, i - b_{2m-2}, i\}\) and \(C_{i-b_{2m-1}} \cap C_{i-b_{2m-2}}\) is the edge \(\{i + b_{2m} + n\}\). Thus, any pair of faces containing \(i\) intersects in either at \(i\) or on an edge through \(i\) and the faces containing \(i\) form a single cycle of adjacent faces. Therefore, \(M_{2m,3^{m}+2n-1}\) is a polyhedral map.

From the uniqueness of 16-vertex \(\{5,5\}\)-equivelar polyhedral map it follows that \(M_{5,16}\) is self dual. Here we prove.

**Lemma 3.** \(M_{2m-1,2(3^{m-1}+2n-1)}\) and \(M_{2m,3^{m}+2n-1}\) are self dual for \(m \geq 3\) and \(n \geq 0\).

**Proof.** Let \(\varphi: M_{2m-1,2(3^{m-1}+2n-1)} \rightarrow M_{2m-1,2(3^{m-1}+2n-1)}\) be the mapping given by \(\varphi(i) = F_i := F_{i,2m-1}\). Clearly \(\varphi\) is a bijection. Consider the face \(F_i = (i + b_1, \ldots, i + b_{2m-3}, i + b_{2m-2} + n, i + b_{2m-1} + 2n)\). Now, \((\varphi(i + b_1), \ldots, \varphi(i + b_{2m-3}), \varphi(i + b_{2m-2} + n), \varphi(i + b_{2m-2} + 2n)) = (F_{i-b_1}, \ldots, F_{i-b_{2m-3}}, F_{i-b_{2m-2}}, F_{i-b_{2m-2}}-n) = F_{i,bm}\) (say). From the proof of Lemma 2, \(F_{i,bm}\) is a cycle of adjacent faces (sharing a common edge) containing the common vertex \(-i\). Therefore, by the definition, \(F_{i,bm}\) is a face of \(M_{2m-1,2(3^{m-1}+2n-1)}\). This implies that \(M_{2m-1,2(3^{m-1}+2n-1)}\) is isomorphic to \(M_{2m-1,2(3^{m-1}+2n-1)}\). Similarly, \(\psi: M_{2m,3^{m}+2n-1} \rightarrow M_{2m,3^{m}+2n-1}\) given by \(\psi(i) = F_{i,2m}\) defines an isomorphism.

Clearly, \(\Gamma(M_{2m-1,2(3^{m-1}+2n-1)})\) and \(\Gamma(M_{2m,3^{m}+2n-1})\) are transitive on the vertices and the faces. Here we prove.

**Lemma 4.** \(M_{2m-1,2(3^{m-1}+2n-1)}\) and \(M_{2m,3^{m}+2n-1}\) are not combinatorially regular for all \(m \geq 3\) and \(n \geq 0\).

**Proof.** Let \(\mu = 2(3^{m-1} + 2n - 1)\). If \(m > 3\) then consider the flags \(F_1 = (0, \{0, b_{m-1}, F_{b_{m-1}}\})\) and \(F_2 = (0, \{0, b_{m+2} - b_{m+1}\}, F_{b_{m+1}})\) in \(M_{2m-1,\mu}\). If possible let there exist \(\varphi \in \Gamma(M_{2m-1,\mu})\) such that \(\varphi(F_1) = F_2\). Then \(\varphi(0) = 0, \varphi(F_{b_{m+1}}) = F_{b_{m+1}}\) and hence
\( \varphi(1 - b_{m+1}) = -b_{m+1} \) and \( \varphi(1) = 1 \). If \( m > 5 \) then, by considering the faces containing 1, \( \varphi(F_{1-b_{m+2}}) = F_{1-b_{m+2}}, \varphi(F_{1-b_{m+1}}) = F_{1-b_{m+3}} \). These imply \( 1 + b_4 - b_{m+3} = \varphi(1 - b_{m+1}) = -b_{m+1} \) in \( \mathbb{Z}_\mu \), a contradiction. If \( m = 5 \) then \( \varphi(F_{1-b_5}) = F_{1-b_5-n} \) and hence \( 1 + b_4 - b_8 - n = \varphi(1 - b_6) = -b_6 \) in \( \mathbb{Z}_\mu \). This is not possible. If \( m = 4 \) then \( \varphi(F_{1-b_5}) = F_{1-b_7-2n} \) and hence \( 1 + b_4 - b_7 - 2n = \varphi(1 - b_5) = -b_5 \) in \( \mathbb{Z}_\mu \), a contradiction.

For \( m = 3 \), if \( \psi \in \Gamma(M_{5,n}) \) such that \( \psi((0, \{3 + n\}, F_{b_4-n})) = (0, \{3 + 3n\}, F_{b_4-n}) \) then \( \psi(12 + 3n) = 11 + 3n \) and \( \psi(F_{b_4-n}) = F_1 \) and hence \( 3 = \psi(12 + 3n) = 11 + 3n \) in \( \mathbb{Z}_\mu \). This is also not possible.

Thus, \( M_{2m-1,2}(3^{m-1}+2n-1) \) always has a pair of flags \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) such that \( \varphi(\mathcal{F}_1) \neq \mathcal{F}_2 \) for all \( \varphi \in \Gamma(M_{2m-1,2}(3^{m-1}+2n-1)) \). So, \( M_{2m-1,2}(3^{m-1}+2n-1) \) is not combinatorially regular.

Let \( \eta = 3^m + 2n - 1 \) and \( C_i = F_{i,2m} \). Consider the flags \( C_1 = (0, \{0, (-1)^m(b_{m+2} - b_{m+1})\}, C_{b_{m+1}}) \) and \( C_2 = (0, \{0, (-1)^m(b_{m+2} - b_{m+1})\}, C_{b_{m+2}}) \) in \( M_{2m,n} \). If \( \varphi \in \Gamma(M_{2m,n}) \) such that \( \varphi(C_1) = C_2 \), then \( \varphi(C_2) = C_1, \varphi(1 - b_{m+2}) = -b_{m+1} \) and \( \varphi(1) = 1 \). If \( m > 3 \) then \( \varphi(C_{1-b_{m+2}}) = C_{1-b_{m+3}} \) and hence \( 1 + b_2 - b_{m+3} = \varphi(1 - b_{m+2}) = -b_{m+1} \) in \( \mathbb{Z}_\mu \), a contradiction. If \( m = 3 \) then \( \varphi(C_{1-b_5}) = C_{1-b_6} \) and hence \( 1 + b_2 - b_6 + n = \varphi(1 - b_5) = -b_5 \) in \( \mathbb{Z}_\mu \). This is not possible. Therefore, by similar argument as before, \( M_{2m,3^m+2n-1} \) is not combinatorially regular.

\( \square \)

**Example 2.** Let \( C_4 \) be the collection of 4-cycles of the complete graph \( K_5 \) on the vertex set \( \mathbb{Z}_4 \cup \{u\} \) given by \( C_4 = \{(0,1,2,3), (u,i,i+1,i+3) : i \in \mathbb{Z}_4\} \). Then \( |C_4| \) is the torus and hence \( (|C_4|, K_5) \) is a non-singular \( \{4,4\} \)-pattern.

**Lemma 5.** Suppose \( C(\pi_p) = \{(0,1,\ldots,p-1), (u,i + \pi_p(1), \ldots, i + \pi_p(p-1)) : i \in \mathbb{Z}_p \} \) is a collection of cycles of the complete graph \( K_{p+1} \) on the vertex set \( \mathbb{Z}_p \cup \{u\} \), where \( p \) is an odd prime and \( \pi_p \) is a permutation of \( \mathbb{Z}_p \setminus \{0\} = \{1, \ldots, p-1\} \). If

1. \( \pi_p(i) + \pi_p(p-i) = p \) for \( 1 \leq i \leq p-1 \),
2. \( \pi_p(p^{p-1}) = \frac{p^{p-1}}{2} \) and
3. exactly one of \( j, -j \) is in \( \{\pi_p(2) - \pi_p(1), \pi_p(3) - \pi_p(2), \ldots, \pi_p(p^{p-1}) - \pi_p(p-1)\} \)

then \( \mathcal{C}(\pi_p) \) is a connected combinatorial 2-manifold.

**Proof.** Since edges of cycles of \( C(\pi_p) \) form a connected graph, \( \text{EG}(\mathcal{C}(\pi_p)) \) is connected.

Let \( a_i = \pi_p(i+1) - \pi_p(i) \) for \( 1 \leq i \leq p-2 \). Then, by (pp1), \( a_i = a_{i-1} - 1 \). Let \( r = \frac{p^{p-2} - 2}{2} \).

Then, by (pp1), (pp2), \( a_{r+1} = 1 \) and, by (pp3), \( \{a_1, \ldots, a_{r+1}, -a_1, \ldots, -a_{r+1}\} = \mathbb{Z}_p \setminus \{0\} \).

If \( r \) is even then the cycles containing \( i \) are \( (i, u, \ldots, i + a_1), \ (i, i + a_1, \ldots, i + a_2), \ (i, i - a_2, \ldots, i + a_3), \ (i, i + a_1, a_r, \ldots, i + a_r), \ (i, i - a_r, \ldots, i + a_1), \ (i, i + 1, i + 2, \ldots, i + p - 1), \ (i, i + p - 1, \ldots, i + a_r), \ (i, i + 2, \ldots, i + a_{r+1}), \ (i, i + a_r, \ldots, i - a_{r+1}), \ (i, i - a_{r+1}, \ldots, u) \).

If \( r \) is odd then the cycles containing \( i \) are \( (i, u, i + a_1), \ (i, i + a_1, \ldots, i + a_2), \ (i, i - a_2, \ldots, i + a_3), \ (i, i - a_r, \ldots, i + a_{r+1}), \ (i, i + a_{r+1}, \ldots, i + a_r), \ (i, i + p - 1, \ldots, i + 2, i + 1), \ (i, i + 1, \ldots, i + a_{r+1}), \ (i, i + a_r, \ldots, i - a_{r+1}), \ (i, i - a_{r+1}, \ldots, u) \).

The cycles containing \( u \) are \( (u, \pi_p(1), \ldots, \pi_p(p-1)), \ (u, 1 + \pi_p(1), \ldots, 1 + \pi_p(p-1)), \ (u, p-1 + \pi_p(1), \ldots, p-1 + \pi_p(p-1)) \). Since \( \{\pi_p(p-1), 1 + \pi_p(p-1), \ldots, p-1 + \pi_p(p-1)\} = \mathbb{Z}_p \), the cycles containing \( u \) can be arranged as \( (u, \pi_p(i_1), \ldots, \pi_p(j_1)), \ldots, (u, \pi_p(i_p), \ldots, \pi_p(j_p)) \), where \( j_1 = i_2, \ldots, j_{p-1} = i_p, j_p = i_1 \). The lemma now follows by Lemma 1. \( \square \)
Clearly, \( \pi_3 \) is the identity permutation and \( C(\pi_3) \) is the 4-vertex 2-sphere \( S_4^2 \). Also, \( \chi(\bar{C}(\pi_p)) = 2(p+1) - \left( \left( \frac{p+1}{2} \right) + p(p+1) \right) - (p+1) = (p+1)(4-p)/2 \). So, if \( p = 4k+1 \) for some \( k \geq 1 \) then \( \chi(\bar{C}(\pi_p)) \) is odd and hence \( \bar{C}(\pi_p) \) is non-orientable. Here we prove.

**Lemma 6.** \( \bar{C}(\pi_p) \) is non-orientable for \( p > 3 \).

**Proof.** Let \( F = (0,1,\ldots,p-1) \) and \( F_i = (u,i+\pi_p(1),\ldots,i+\pi_p(p-1)) \) for \( 1 \leq i \leq p-1 \). We can choose a \( p \)-gonal disc (not necessarily convex) in the plane for each cycle in \( \bar{C}(\pi_p) \) so that the disc corresponding to \( F_i \) is attached with that for \( F \) along the common edge \( \{i + \pi_p(\frac{p-1}{2}), i + \pi_p(\frac{p+1}{2})\} \) for each \( i \) and there are no other intersections. This gives us a \( p(p-1) \)-gonal disc \( D(\pi_p) \). Then there are two edges in \( D(\pi_p) \) corresponding to an edge \( jk \) \( (j,k \in \mathbb{Z}_p, -1 \neq j-k \neq 1) \) in some cycle \( F_i \) and they appear in the same direction (clockwise or anti-clockwise). Since \( |\bar{C}(\pi_p)| \) is homeomorphic to the space obtained by identifying such pairs of edges (and some more) of \( D(\pi_p), |\bar{C}(\pi_p)| \) is non-orientable. \( \Box \)

**Lemma 7.** Let \( p > 3 \) be a prime.

(a) If \( p = 4k+3 \) for some \( k \geq 1 \) then the permutation \( \sigma_p = (2,4k+1)(4,4k-1)\cdots(2k,2k+3) \) \( \text{of } \mathbb{Z}_p \setminus \{0\} \) satisfies (pp1), (pp2) and (pp3) of Lemma 5.

(b) If \( p = 4l+1 \) for some \( l \geq 1 \) then the permutation \( \rho_p = (1,4l)(3,4l-2)\cdots(2l-1,2l+2) \) \( \text{of } \mathbb{Z}_p \setminus \{0\} \) satisfies (pp1), (pp2) and (pp3) of Lemma 5.

**Proof.** Clearly, \( \sigma_p \) and \( \rho_p \) satisfy hypothesis (pp1) and (pp2).

Now, \( \{\sigma_p(2) - \sigma_p(1), \ldots, \sigma_p(\frac{p+1}{2}) - \sigma_p(\frac{p-1}{2})\} = \{4k, -(4k-2), 4k-4, \ldots, 4, -2, 1\} \)
\( = \{-2, 4, -6, \ldots, -(4k-2), 4k, -(4k+2)\} \). Thus \( \sigma_p \) satisfies (pp3).

Again, \( \{\rho_p(2) - \rho_p(1), \ldots, \rho_p(\frac{p+1}{2}) - \rho_p(\frac{p-1}{2})\} = \{-4l-2, 4l-4, -(4l-6), \ldots, 4, -2, 1\} \)
\( = \{-2, 4, -6, \ldots, (4l-4), -(4l-2), -4l\} \). Thus \( \rho_p \) satisfies (pp3). \( \Box \)

**Proof of Theorem 1.** Let \( m \geq 3 \) and \( n \geq 0 \). By Lemma 2, \( M_{2m-1,2(3m-1+2n-1)} \) is a \( 2(3^{m-1} + 2n - 1) \)-vertex polyhedral map and hence a \( \{2m-1,2m-1\} \)-equivelar polyhedral map. Again, by Lemma 2, \( M_{2m,3^{m}+2n-1} \) is a \( (3^{m} + 2n - 1) \)-vertex polyhedral map and hence a \( \{2m,2m\} \)-equivelar polyhedral map. The theorem now follows from Lemma 3. \( \Box \)

**Proof of Theorem 2.** Let \( p > 3 \) be a prime and \( K_{p+1} \) be the complete graph on the vertex set \( \mathbb{Z}_p \cup \{u\} \). By Lemma 7, there exists a permutation \( \pi_p \) of \( \mathbb{Z}_p \setminus \{0\} \) which satisfies (pp1),
Let $C(\pi)$ be as in Lemma 5. Then, by Lemma 5, $\bar{C}(\pi)$ is a connected combinatorial 2-manifold. So, if $N_p := |\bar{C}(\pi)|$ then $(N_p, K_{p+1})$ is a non-singular \{p,p\}-pattern and the cycles in $\bar{C}(\pi)$ are the boundaries of the faces of $(N_p, K_{p+1})$. Finally, the 4-vertex 2-sphere $S^2_4$ gives a \{3,3\}-pattern. This completes the proof. \hfill \Box

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**References**


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