Biharmonic Curves in the Generalized Heisenberg Group

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Abstract. We find the conditions under which a curve in the generalized Heisenberg group is biharmonic and non-harmonic. We give some existence and non-existence examples of such curves.

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1. Introduction

First we should recall some notions and results related to the harmonic and the biharmonic maps between Riemannian manifolds, as they are presented in [2], [9] and in [5].

Let \( f : M \to N \) be a smooth map between two Riemannian manifolds \( (M, g) \) and \( (N, h) \). Let \( f^{-1}(TN) \) be the induced bundle over \( M \) of the tangent bundle, \( TN \), defined as follows. Denote by \( \pi : TN \to N \) the projection. Then

\[
f^{-1}(TN) = \{(x, u) \in M \times TN, \pi(u) = f(x), x \in M\} = \bigcup_{x \in M} T_{f(x)}N.
\]

The set of all \( C^\infty \)-sections of \( f^{-1}(TN) \), denoted by \( \Gamma(f^{-1}(TN)) \), is \( \Gamma(f^{-1}(TN)) = \{V : M \to TN, C^\infty\text{-map}, V(x) \in T_{f(x)}N, x \in M\} \). Denote by \( \nabla^M, \nabla^N \), the Levi-Civita connections on \( (M, g) \) and \( (N, h) \) respectively. For a smooth map \( f \) between \( (M, g) \) and \( (N, h) \), we define the induced connection \( \nabla \) on the induced bundle \( f^{-1}(TN) \) as follows. For \( X \in \chi(M), V \in \Gamma(f^{-1}(TN)) \), define \( \nabla_XV \in \Gamma(f^{-1}(TN)) \) by \( \nabla_XV = \nabla^N_{f_*X}V \).

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The differential of the smooth map $f$ can be viewed as a section of the bundle $\Lambda^1(f^{-1}(TN)) = T^*M \otimes f^{-1}(TN)$, and we denote by $|df|$ its norm at a point $x \in M$.

Suppose that $M$ is a compact manifold. Define the energy density of $f$ by $e(f) = \frac{1}{2} |df|^2$, and the energy of $f$ by $E(f) = \int_M e(f) \, *1$, where $*1$ is the volume form on $M$. The map $f$ is a harmonic map if it is a critical point of the energy, $E(f)$. In [9] it is proved that a map $f : M \to N$ is a harmonic map if and only if it satisfies the Euler-Lagrange equation

$$\tau(f) = \text{trace} \nabla df = \text{an element of } \Gamma(f^{-1}(TN))$$

called the tension field of $f$. The Laplacian acting on $\Gamma(f^{-1}(TN))$, induced by the connection $\nabla$, is given by the Weitzenböck formula

$$\Delta V = - \text{trace} \nabla^2 V,$$

for some $V \in \Gamma(f^{-1}(TN))$.

The bienergy of $f$ is defined by $E_2(f) = \frac{1}{2} \int_M |\tau(f)|^2 \, *1$. We say that $f$ is a biharmonic map if it is a critical point of the bienergy, $E_2(f)$. It is proved in [5] that a map $f : M \to N$ is a biharmonic map if and only if it satisfies the equation $\tau_2(f) = 0$, where

$$\tau_2(f) = - \Delta \tau(f) - \text{trace} R^N(df(\cdot), \tau(f)) df(\cdot), \quad (1.1)$$

where $R^N$ denotes the curvature tensor field on $(N, h)$.

Note that any harmonic map is a biharmonic map and, moreover, an absolute minimum of the bienergy functional.

2. Generalized Heisenberg group

Consider $\mathbb{R}^{2n+1}$ with the elements of the form $X = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n, z)$. Define the product on $\mathbb{R}^{2n+1}$ by

$$X \tilde{X} = (x_1 + \bar{x}_1, y_1 + \bar{y}_1, \ldots, x_n + \bar{x}_n, y_n + \bar{y}_n, z + \bar{z}) + \frac{1}{2} \sum_{i=1}^n (\bar{x}_i y_i - \bar{y}_i x_i),$$

where $X = (x_1, y_1, \ldots, x_n, y_n, z)$, $\tilde{X} = (\bar{x}_1, \bar{y}_1, \ldots, \bar{x}_n, \bar{y}_n, \bar{z})$.

Let $\mathbb{H}_{2n+1} = (\mathbb{R}^{2n+1}, g)$ be the generalized Heisenberg group endowed with the Riemannian metric $g$ which is defined by

$$g = \sum_{i=1}^n (dx_i^2 + dy_i^2) + \left[ dz + \frac{1}{2} \sum_{i=1}^n (y_i dx_i - x_i dy_i) \right]^2. \quad (2.1)$$

Note that the metric $g$ is left invariant.

We can define a global orthonormal frame field in $\mathbb{H}_{2n+1}$ by

$$E_{2i-1} = \frac{\partial}{\partial x_i} - \frac{x_i}{2} \frac{\partial}{\partial z}, \quad E_{2i} = \frac{\partial}{\partial y_i} + \frac{x_i}{2} \frac{\partial}{\partial z}, \quad E_{2n+1} = \frac{\partial}{\partial z},$$

for $i = 1, \ldots, n$. The Levi-Civita connection of the metric $g$ is given by, (see [7] for the
3-dimensional case),
\[
\begin{align*}
\nabla_{E_{2i-1}} E_{2j-1} &= 0, & \nabla_{E_{2i}} E_{2j} &= \frac{1}{2} \delta_{ij} E_{2n+1}, \\
\nabla_{E_{2i}} E_{2j} &= 0, & \nabla_{E_{2i-1}} E_{2j-1} &= -\frac{1}{2} \delta_{ij} E_{2n+1}, \\
\nabla_{E_{2i-1}} E_{2j-1} &= -\frac{1}{2} \delta_{ij} E_{2n+1}, & \nabla_{E_{2i}} E_{2n+1} &= \frac{1}{2} E_{2i-1},
\end{align*}
\]

for \(i, j = 1, \ldots, n\). We have too
\[
\begin{align*}
[E_{2i-1}, E_{2j-1}] &= 0, & [E_{2i}, E_{2j}] &= 0, \\
[E_{2i-1}, E_{2n+1}] &= 0, & [E_{2i}, E_{2n+1}] &= 0, \\
[E_{2i-1}, E_{2j}] &= \delta_{ij} E_{2n+1}.
\end{align*}
\]

The curvature tensor field of \(\nabla\) is
\[R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,\]
and Riemann-Christoffel tensor field is
\[R(X, Y, Z, W) = g(R(X, Y)W, Z),\]
where \(X, Y, Z, W \in \chi(\mathbb{R}^{2n+1})\). We will use the notations
\[R_{abc} = R(E_a, E_b)E_c, \quad R_{abcd} = R(E_a, E_b, E_c, E_d),\]
where \(a, b, c, d = 1, \ldots, 2n + 1\). Then the non-zero components of the curvature tensor field and of the Riemann-Christoffel tensor field are, respectively
\[
\begin{align*}
R_{(2i-1)(2j-1)(2k)} &= -\frac{1}{4} \delta_{jk} E_{2i} + \frac{1}{2} \delta_{ik} E_{2j}, \\
R_{(2i-1)(2j)(2k-1)} &= \frac{1}{4} \delta_{jk} E_{2i} + \frac{1}{2} \delta_{ij} E_{2k}, \\
R_{(2i-1)(2j)(2k)} &= -\frac{1}{4} \delta_{ik} E_{2j-1} - \frac{1}{2} \delta_{ij} E_{2k-1}, \\
R_{(2i-1)(2n+1)(2j-1)} &= -\frac{1}{4} \delta_{ij} E_{2n+1}, \\
R_{(2i-1)(2n+1)(2n+1)} &= \frac{1}{4} \delta_{ij} E_{2i-1}, \\
R_{(2i)(2j)(2k-1)} &= -\frac{1}{4} \delta_{jk} E_{2i-1} + \frac{1}{2} \delta_{ik} E_{2j-1}, \\
R_{(2i)(2n+1)(2j)} &= -\frac{1}{4} \delta_{ij} E_{2n+1}, \\
R_{(2i)(2n+1)(2n+1)} &= \frac{1}{4} \delta_{ij} E_{2i},
\end{align*}
\]

and
\[
\begin{align*}
R_{(2i-1)(2j)(2k)} &= -\frac{1}{4} \delta_{jk} + \frac{1}{2} \delta_{ik} \delta_{ij}, \\
R_{(2i-1)(2j)(2k-1)} &= \frac{1}{4} \delta_{jk} + \frac{1}{2} \delta_{ik} \delta_{ij}, \\
R_{(2i-1)(2j)(2k-1)} &= \frac{1}{2} \delta_{ij} + \frac{1}{4} \delta_{ik} \delta_{jk}, \\
R_{(2i)(2j-1)(2k-1)} &= \frac{1}{4} \delta_{ij} - \frac{1}{4} \delta_{ik} \delta_{ij}, \\
R_{(2i-1)(2n+1)(2j-1)} &= -\frac{1}{4} \delta_{ij}, \\
R_{(2i)(2n+1)(2n+1)} &= -\frac{1}{4} \delta_{ij},
\end{align*}
\]

for \(i, j, k = 1, \ldots, n\).
3. Biharmonic curves in $\mathbb{H}^{2n+1}$

Let $\gamma : I \rightarrow \mathbb{H}^{2n+1}$ be a non-inflexionar curve, parametrized by its arc length. Let $\{T, N_1, \ldots, N_{2n}\}$ be the Frenet frame in $\mathbb{H}^{2n+1}$ defined along $\gamma$, where $T = \gamma'$ is the unit tangent vector field of $\gamma$, $N_1$ is the unit normal vector field of $\gamma$, with the same direction as $\nabla_T T$ and the vectors $N_1, \ldots, N_{2n}$ are the unit vectors obtained from the following Frenet equations for $\gamma$.

\[
\begin{align*}
\nabla_T T &= \chi_1 N_1, \\
\nabla_T N_1 &= -\chi_1 T + \chi_2 N_2, \\
&\vdots &\vdots \\
\nabla_T N_{2n-1} &= -\chi_{2n-2} N_{2n-2} + \chi_{2n-1} N_{2n}, \\
\nabla_T N_{2n} &= -\chi_{2n-1} N_{2n-1},
\end{align*}
\]

where $\chi_1 = \|\nabla_T T\| = \|\tau(\gamma)\|$, and $\chi_2, \ldots, \chi_{2n}$ are real valued functions, where $s$ is the arc length of $\gamma$. If $\chi_k \in \mathbb{R}$, $k = 1, \ldots, 2n+1$ we say that $\gamma$ is a helix.

The biharmonic equation of $\gamma$ is

\[
\tau_2(\gamma) = \nabla_T^2 T - R(T, \nabla_T T)T = 0.
\]

Using the Frenet equations one obtains

\[
\nabla_T^2 T = (-3(\chi_1')^2)T + (\chi_1'' - \chi_1^3 - \chi_1^3 \chi_2) N_1 + (2\chi_1' \chi_2 + \chi_1 \chi_2^2) N_2 + \chi_1 \chi_2 \chi_3 N_3.
\]

Using (2.3) we get

\[
R(T, \nabla_T T)T = \sum_{i=1}^{n} (\xi_{2i-1} E_{2i-1} + \xi_{2i} E_{2i}) + \xi_{2n+1} E_{2n+1},
\]

with

\[
\begin{align*}
\xi_{2i-1} &= \frac{3}{4} T_{2i} \sum_{j=1}^{n} (-T_{2j-1} N_{1j}^2 + T_{2j} N_{1j}^2 - 1) + \frac{1}{4} T_{2i-1} T_{2n+1} N_{12n+1} + \frac{1}{4} T_{2i} T_{2n+1} N_{12n+1}, \\
\xi_{2i} &= \frac{3}{4} T_{2i} \sum_{j=1}^{n} (T_{2j-1} N_{1j}^2 - T_{2j} N_{1j}^2) + \frac{1}{4} T_{2i} T_{2n+1} N_{12n+1} + \frac{1}{4} T_{2i} T_{2n+1} N_{12n+1}, \\
\xi_{2n+1} &= \frac{1}{4} \sum_{j=1}^{n} (-T_{2j-1} N_{1j}^2 - T_{2j} N_{1j}^2 - 1) + T_{2j-1} T_{2n+1} N_{12n+1} + T_{2j} T_{2n+1} N_{12n+1},
\end{align*}
\]

where $T = \sum_{a=1}^{2n+1} T_a E_a$ and $N_1 = \sum_{a=1}^{2n+1} N_a E_a$. After a straightforward computation, we have

\[
R(T, \nabla_T T)T = \sum_{k=1}^{2n} \eta_k N_k,
\]

with

\[
\eta_1 = \frac{3}{4} \left[ \sum_{i=1}^{n} (T_{2i} N_{1i}^2 - T_{2i-1} N_{1i}^2)^2 - \frac{1}{4} T_{2n+1}^2 - \frac{1}{4} (N_{12n+1})^2 \right],
\]

\[
(3.1)
\]
\[ \eta_k = \frac{3}{4} \left[ \sum_{i=1}^{n} (T_{2i} N_{1i}^{2i-1} - T_{2i-1} N_{1i}^{2i}) \right] \left[ \sum_{i=1}^{n} (T_{2i} N_{k}^{2i-1} - T_{2i-1} N_{k}^{2i}) \right] - \frac{1}{4} N_{1}^{2n+1} N_{k}^{2n+1}, \tag{3.6} \]

where \( N_{k} = \sum_{a=1}^{2n+1} N_{k}^{a}E_{a} \).

From (3.2), (3.3) and (3.4) it follows that the biharmonic equation of \( \gamma \) is

\[ \tau_{2}(\gamma) = \nabla^{2}T - R(T, \nabla T)T = (-3\chi_{1}' \chi_{1}')T + (\chi_{1}'' - \chi_{1}^{3} - \chi_{1} \chi_{2} - \chi_{1} \eta_{1})N_{1} + \\
(2\chi_{1}' \chi_{2} + \chi_{1} \chi_{2} - \chi_{1} \eta_{2})N_{2} + (\chi_{1} \chi_{2} \chi_{3} - \chi_{1} \eta_{3})N_{3} - \chi_{1} \sum_{i=4}^{2n} \eta_{k} N_{k}, \]

where \( \eta_{a}, a = 1, \ldots, 2n \) are given by (3.5) and (3.6). Hence

**Theorem 3.1.** Let \( \gamma : I \to \mathbb{H}_{2n+1} \) be a curve, parametrized by its arc length. Then \( \gamma \) is a biharmonic and non-harmonic curve if and only if

\[
\begin{align*}
\chi_{1} & \in \mathbb{R} \setminus \{0\}, \\
\chi_{1}^{2} + \chi_{2}^{2} & = -\eta_{1}, \\
\chi_{2} & = \eta_{2}, \\
\chi_{2} \chi_{3} & = \eta_{3}, \\
\eta_{k} & = 0, \ k = 4, \ldots, 2n,
\end{align*}
\tag{3.7}
\]

where \( \eta_{k}, \ k = 1, \ldots, 2n \), are given by (3.5) and (3.6).

**Corollary 3.2.** If \( \chi_{1} \in \mathbb{R} \setminus \{0\} \) and \( \chi_{2} = 0 \) for a curve \( \gamma : I \to \mathbb{H}_{2n+1} \), parametrized by its arc length, then \( \gamma \) is a biharmonic and non-harmonic curve if and only if \( \chi_{1}^{2} = -\eta_{1} \) and \( \eta_{k} = 0, \ k = 2, \ldots, 2n \).

**Corollary 3.3.** Let \( \gamma : I \to \mathbb{H}_{2n+1} \) be a curve, parametrized by its arc length. If \( \eta_{1} \geq 0 \) then \( \gamma \) cannot be a biharmonic and non-harmonic curve.

In [7] the following two results for the usual Heisenberg group, \( \mathbb{H}_{3} \), are proved.

**Theorem 3.4.** Let \( \gamma \) be the helix given by

\[ \gamma(s) = (r \cos(as), r \sin(as), c a s), \]

where \( r > 0, \ \frac{1}{a_{2}} = r^{2}(1 + \frac{1}{4}r^{2}) \). Then \( \gamma \) is a biharmonic and non-geodesic curve.

**Remark 3.5.** In the case above if \( r = \sqrt{1+\frac{\chi_{2}^{2}}{2}} \), then \( \gamma \) is a biharmonic and non-harmonic curve with \( \chi_{2} = 0 \).

In the case of the higher dimensions, we find a similar example related to Theorem 3.1. We consider a curve in \( \mathbb{R}^{2n+1} \), given by

\[ \gamma(s) = (c_{1} \cos(a_{1}s), c_{1} \sin(a_{1}s), \ldots, c_{n} \cos(a_{n}s), c_{n} \sin(a_{n}s), cs), \]
where \(c_i > 0\), \(a_i \neq 0\), \(c \neq 0\), \(i = 1, \ldots, n\). Then one obtains
\[
T(s) = \gamma'(s) = \sum_{i=1}^{n} [-c_i a_i \sin(a_i s)E_{2i-1} + c_i a_i \cos(a_i s)E_{2i}] + A E_{2n+1},
\]
where \(A = c - \frac{1}{2} \sum_{i=1}^{n} c_i^2 a_i\). From \(\|T(s)\| = 1\) we have \(A^2 + \sum_{i=1}^{n} c_i^2 a_i^2 = 1\). After a straightforward computation, using (2.2), one obtains
\[
\nabla_T T = \sum_{i=1}^{n} [c_i a_i (A - a_i) \cos(a_i s)E_{2i-1} + c_i a_i (A - a_i) \sin(a_i s)E_{2i}].
\]
From the first equation in (3.1) and from \(\|N_1\| = 1\), we have
\[
\chi_1 = \left[ \sum_{i=1}^{n} c_i^2 a_i^2 (A - a_i)^2 \right]^{1/2} \in \mathbb{R},
\]
and
\[
N_1 = \sum_{i=1}^{n} \frac{c_i a_i (A - a_i)}{\left[ \sum_{j=1}^{n} c_j^2 a_j^2 (A - a_j)^2 \right]^{1/2} } \left[ \cos(a_i s)E_{2i-1} + \sin(a_i s)E_{2i} \right].
\]
Note that \(N_1^{2n+1} = 0\). Next, one obtains
\[
\nabla_T N_1 + \chi_1 T = \frac{1}{2\chi_1} \sum_{i=1}^{n} \left\{ c_i a_i [ (A - a_i)(A - 2a_i) - 2\chi_1^2 ] \right\} \left[ \cos(a_i s)E_{2i-1} - \chi_1^2 \right] \left[ 2\chi_1^2 A - \sum_{j=1}^{n} c_j^2 a_j^2 (A - a_j) \right] E_{2n+1}.
\]
From (3.1), using \(\|N_2\| = 1\) we have \(\chi_2^2 = \|N_2\|^2\).

In order to find a curve which satisfies conditions of Corollary 3.2 we assume that \(\chi_2 = 0\) and \(\chi_1^2 = -\eta_1\). From this conditions, after a straightforward computation we get

**Proposition 3.6.** Let \(\gamma : I \rightarrow \mathbb{H}_{2n+1}\) be the curve defined by
\[
\gamma(s) = (c_1 \cos(a_1 s), c_1 \sin(a_1 s), \ldots, c_n \cos(a_n s), c_n \sin(a_n s), cs),
\]
where \(c_i > 0\), \(a_i \neq 0\), \(c \neq 0\). If \(a_i = a = A - \frac{1}{2A^2} \sum_{i=1}^{n} c_i^2 = \frac{4A^2(1-A^2)}{(2A^2-1)^2}\) and \(c = \frac{A^3}{2A^2-1}\), where
\[
A = \pm \left[ \frac{3n^2 - 1 + (9n^4 + 6n^2 + 5)1/2}{6n^2 + 2} \right]^{1/2},
\]
then \(\gamma\) is a biharmonic and non-harmonic curve in \(\mathbb{H}_{2n+1}\).

Note that, for such a curve and for \(k \neq 1\), one obtains
\[
\sum_{i=1}^{n} (T_{2i} N_{2i-1} - T_{2i-1} N_{2i}^2) = \frac{\chi_1}{|A - a|} \sum_{i=1}^{n} (N_{2i-1}^2 N_{2i-1}^2 + N_{2i}^2 N_{2i}^2) = - \frac{\chi_1}{|A - a|} N_{2n+1}^2 N_{2n+1} = 0.
\]
That is \(\eta_k = 0\), for any \(k = 2, \ldots, 2n\).

Also, note that the Proposition 3.6 is a generalization of the result in the Remark 3.5.
Proposition 3.7. Let \( \gamma : I \to \mathbb{H}_{2n+1} \) be the curve defined by
\[
\gamma(s) = (r \cos(as), r \sin(as), \ldots, r \cos(as), r \sin(as), cs),
\]
where \( r > 0, a_i \neq 0, c \neq 0 \). If \( \chi_2 = 0, \chi_1 \neq 0 \) and \( \gamma \) is biharmonic then \( n = 1 \).

In the following we obtain a class of biharmonic and non-harmonic curves for which the second curvature does not necessarily vanish, (see[1] for the similar result in 3-dimensional case).

Proposition 3.8. Let \( \gamma : I \to \mathbb{H}_{2n+1}, \gamma(s) = (x_1(s), y_1(s), \ldots, x_n(s), y_n(s), z(s)), \) be the curve with the parametric equations
\[
\begin{align*}
x_i(s) &= \frac{1}{\beta} \frac{\sin \alpha}{\sqrt{n}} \sin(\beta s + a_i) + b_i, \\
y_i(s) &= -\frac{1}{\beta} \frac{\sin \alpha}{\sqrt{n}} \cos(\beta s + a_i) + c_i, \\
z(s) &= \left( \frac{\cos \alpha + (\frac{\sin \alpha)^2}{2\beta}}{2\sqrt{n}} \right)s - \sum_{i=1}^{n} \frac{b_i}{2\beta \sqrt{n}} \sin \alpha \cos(\beta s + a_i) \\
&\quad - \sum_{i=1}^{n} \frac{c_i}{2\beta \sqrt{n}} \sin \alpha \cos(\beta s + a_i) + d,
\end{align*}
\]
with \( i = 1, \ldots, n \), where \( \beta = \frac{\cos(\pm \sqrt{5}(\cos \alpha)^2 - 4)}{2}, \alpha \in (0, \arccos(\frac{2\sqrt{5}}{3}) \cup \arccos(-\frac{2\sqrt{5}}{3}), \pi) \) and \( a_i, b_i, c_i, d \in \mathbb{R}. \) Then \( \gamma \) is a biharmonic and non-harmonic curve.

Proof. The covariant derivative of the unit tangent vector field, \( T \), of \( \gamma \), is
\[
\nabla_T T = \sum_{i=1}^{n} \left( (T'_{2i-1} + T_{2i}T_{2n+1})E_{2i-1} + (T'_{2i} - T_{2i-1}T_{2n+1})E_{2i} \right) + T'_{2n+1}E_{2n+1},
\]
and \( T \) is given by
\[
T(s) = \gamma'(s) = \frac{\sin \alpha}{\sqrt{n}} \sum_{i=1}^{n} \left[ \cos(\beta s + a_i)E_{2i-1} + \sin(\beta s + a_i)E_{2i} \right] + \cos \alpha E_{2n+1}.
\]
Taking into account the first Frenet equation one obtains
\[
\chi_1 = |\sin \alpha(\cos \alpha - \beta)|
\]
and, since we can assume, without loss of generality, that \( \sin \alpha(\cos \alpha - \beta) > 0 \), we have
\[
N_1 = \sum_{i=1}^{n} \left( \frac{\sin(\beta s + a_i)}{\sqrt{n}} E_{2i-1} - \frac{\cos(\beta s + a_i)}{\sqrt{n}} E_{2i} \right).
\]
After a straightforward computation one obtains that \( \eta_1 = (\sin \alpha)^2 - \frac{1}{4} \) and \( \eta_k = 0 \), for any \( k = 2, \ldots, 2n \).
In order to find \( \chi_2 \) we obtain, using the equations (2.2),
\[
\nabla_T N_1 + \chi_1 T = \sum_{i=1}^{n} \left\{ \frac{\cos(\beta s + a_i)}{\sqrt{n}} \left[ \beta - \frac{1}{2} \cos \alpha + (\cos \alpha - \beta)(\sin \alpha)^2 \right] E_{2i-1} \right\}
\]
Proof. We have
\[
\sum_{j=1}^{2n+1} c_j E_j, \quad \|T\| = 1, \quad \nabla_T T = c_{2n+1} \sum_{i=1}^{n} \left( c_{2i-1} E_{2i-1} - c_{2i-1} E_{2i} \right).
\]

It follows that \( \chi_1 = c_{2n+1} \sqrt{\sum_{i=1}^{n} (c_{2i-1}^2 + c_{2i}^2)} \), and
\[
N_1 = \sum_{i=1}^{n} \left[ \frac{c_{2i}}{\sqrt{\sum_{j=1}^{n} (c_{2j-1}^2 + c_{2j}^2)}} E_{2i-1} - \frac{c_{2i-1}}{\sqrt{\sum_{j=1}^{n} (c_{2j-1}^2 + c_{2j}^2)}} E_{2i} \right].
\]

One obtains
\[
\nabla_T N_1 + \chi_1 T = \sum_{k=1}^{n} \left( c_{2k-1}^2 + c_{2k}^2 \right) - \frac{c_{2n+1}^2}{2} \left[ \sum_{i=1}^{n} \left( \frac{c_{2i-1} c_{2n+1}}{\sqrt{\sum_{j=1}^{n} (c_{2j-1}^2 + c_{2j}^2)}} E_{2i-1} + \right. \right. \]
\[
\left. \left. \frac{c_{2i-1} c_{2n+1}}{\sqrt{\sum_{j=1}^{n} (c_{2j-1}^2 + c_{2j}^2)}} E_{2i} \right) \right].
\]

Then, using Frenet equations, we have
\[
\chi_2^2 = \|\nabla_T N_1 + \chi_1 T\|^2 = \beta^2 - \beta \cos \alpha + \frac{1}{4} (\sin \alpha)^2 (\cos \alpha - \beta)^2.
\]

Hence \( \chi_2 \) is a constant and, from hypothesis, one obtains
\[
\chi_1^2 + \chi_2^2 = \frac{1}{4} (\sin \alpha)^2 = -\eta_1.
\]

Since \( \chi_2 N_2 = \nabla_T N_1 + \chi_1 T \) and \( \chi_3^2 = \|\nabla_T N_2 + \chi_2 N_1\|^2 \), we obtain, after a straightforward computation, that \( \chi_3 = 0 \).

Hence, all conditions from Theorem 3.1 are verified by \( \gamma \) and then \( \gamma \) is a biharmonic and non-harmonic curve.

**Remark 3.9.** In the same way as above it is easy to see that all biharmonic and non-harmonic curves in \( \mathbb{H}_{2n+1} \) with constant second curvature and with the unit tangent vector field, \( T \), of the form
\[
T(s) = \frac{\sin \alpha}{\sqrt{n}} \sum_{i=1}^{n} \left[ \cos(f_i(s)) E_{2i-1} + \sin(f_i(s)) E_{2i} \right] + \cos \alpha E_{2n+1},
\]

where \( f_i \) are some smooth functions of the arc length, such that \( f_i' = f_j' \), for any \( i, j = 1, \ldots, n \), and \( \alpha \in \mathbb{R} \), are given by Proposition 3.8.

Finally, we have

**Proposition 3.10.** Let \( \gamma : I \rightarrow \mathbb{H}_{2n+1} \) be the curve defined by
\[
\gamma(s) = (c_1 s, c_2 s, \ldots, c_{2n+1} s)
\]

with \( \sum_{j=1}^{2n+1} c_j^2 = 1 \). Then \( \gamma \) is biharmonic if and only if is harmonic.

**Proof.** We have
\[
T(s) = \gamma'(s) = \sum_{j=1}^{2n+1} c_j E_j, \quad \|T\| = 1, \quad \nabla_T T = c_{2n+1} \sum_{i=1}^{n} \left( c_{2i-1} E_{2i-1} - c_{2i-1} E_{2i} \right).
\]

It follows that \( \chi_1 = c_{2n+1} \sqrt{\sum_{i=1}^{n} (c_{2i-1}^2 + c_{2i}^2)} \), and
\[
N_1 = \sum_{i=1}^{n} \left[ \frac{c_{2i}}{\sqrt{\sum_{j=1}^{n} (c_{2j-1}^2 + c_{2j}^2)}} E_{2i-1} - \frac{c_{2i-1}}{\sqrt{\sum_{j=1}^{n} (c_{2j-1}^2 + c_{2j}^2)}} E_{2i} \right].
\]

One obtains
\[
\nabla_T N_1 + \chi_1 T = \sum_{k=1}^{n} \left( c_{2k-1}^2 + c_{2k}^2 \right) - \frac{c_{2n+1}^2}{2} \left[ \sum_{i=1}^{n} \left( \frac{c_{2i-1} c_{2n+1}}{\sqrt{\sum_{j=1}^{n} (c_{2j-1}^2 + c_{2j}^2)}} E_{2i-1} + \right. \right. \]
\[
\left. \left. \frac{c_{2i-1} c_{2n+1}}{\sqrt{\sum_{j=1}^{n} (c_{2j-1}^2 + c_{2j}^2)}} E_{2i} \right) \right].
\]
\[
\frac{c_{2n}c_{2n+1}}{\sqrt{\sum_{j=1}^{n}(c_{2j-1}^2 + c_{2j}^2)}} E_{2i} - \sqrt{\sum_{j=1}^{n}(c_{2j-1}^2 + c_{2j}^2) E_{2n+1}}.
\]

That means
\[
\chi_2 = \frac{\sum_{k=1}^{n}(c_{2k-1}^2 + c_{2k}^2) - c_{2n+1}^2}{2}.
\]

Thus \(\chi_1^2 + \chi_2^2 = \frac{1}{4}\). But, one obtains that \(\eta_1 = \frac{3}{4} - c_{2n+1}^2\), and from (3.7) we have that if \(\gamma\) is biharmonic then \(\chi_1^2 + \chi_2^2 = -\eta_1 = -\frac{3}{4} + c_{2n+1}^2\). Thus if \(\gamma\) is biharmonic then \(\chi_1 = 0\), and then \(\gamma\) is a harmonic curve.

References


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