Abstract. A compact Riemann surface $X$ of genus $g > 1$ is said to be $p$-hyperelliptic if $X$ admits a conformal involution $\rho$, called a $p$-hyperelliptic involution, for which $X/\rho$ is an orbifold of genus $p$. Here we give a new proof of the well known fact that for $g > 4p + 1$, $\rho$ is unique and central in the group of all automorphisms of $X$. Moreover we prove that every two $p$-hyperelliptic involutions commute for $3p + 2 \leq g \leq 4p + 1$ and $X$ admits at most two such involutions if $g > 3p + 2$. We also find some bounds for the number of commuting $p$-hyperelliptic involutions and general bound for the number of central $p$-hyperelliptic involutions.

Keywords: $p$-hyperelliptic Riemann surfaces, automorphisms of Riemann surfaces, fixed points of automorphisms.

1. Introduction

A Riemann surface $X = \mathcal{H}/\Gamma$ of genus $g \geq 2$ is said to be $p$-hyperelliptic if $X$ admits a conformal involution $\rho$, called a $p$-hyperelliptic involution, such that $X/\rho$ is an orbifold of genus $p$. This notion has been introduced by H. Farkas and I. Kra in [1] where they also proved that for $g > 4p + 1$, $p$-hyperelliptic involution is unique and central in the group of all automorphisms of $X$. We prove these facts in a combinatorial way using the Hurwitz-Riemann formula and certain theorem of Macbeath [2] about fixed points of an automorphism of $X$; the Hurwitz-Riemann formula asserts that a $p$-hyperelliptic involution has $2g + 2 - 4p$ fixed points. The advantage of our approach is that it allows us to study of $p$-hyperelliptic involutions in case $g \leq 4p + 1$ also. First we show that for $g$ in range $3p + 2 \leq g \leq 4p + 1$,
every two \( p \)-hyperelliptic involutions commute and afterwards we argue that \( X \) admits at most two such involutions for \( 3p + 2 < g \leq 4p + 1 \) and at most 6 for \( g = 3p + 2 \). Finally we find some bounds for the number of commuting \( p \)-hyperelliptic involutions and general bound for the number of central \( p \)-hyperelliptic involutions.

2. Preliminaries

We shall approach the problem using Riemann uniformization theorem by which each compact Riemann surface \( X \) of genus \( g \geq 2 \) can be represented as the orbit space of the hyperbolic plane \( \mathcal{H} \) under the action of some Fuchsian surface group \( \Gamma \). Furthermore a group \( G \) of automorphisms of a surface \( X = \mathcal{H}/\Gamma \) can be represented as \( G = \Lambda/\Gamma \) for another Fuchsian group \( \Lambda \). Each Fuchsian group \( \Lambda \) is given a signature \( \sigma(\Lambda) = (g; m_1, \ldots, m_r) \), where \( g, m_i \) are integers verifying \( g \geq 0 \), \( m_i \geq 2 \). The signature determines the presentation of \( \Lambda \):

- generators: \( x_1, \ldots, x_r, a_1, b_1, \ldots, a_g, b_g \),
- relations: \( x_1^{m_1} = \cdots = x_r^{m_r} = x_1 \cdots x_r[a_1, b_1] \cdots [a_g, b_g] = 1 \).

Such set of generators is called the canonical set of generators and often, by abuse of language, the set of canonical generators. Geometrically \( x_i \) are elliptic elements which correspond to hyperbolic rotations and the remaining generators are hyperbolic translations. The integers \( m_1, m_2, \ldots, m_r \) are called the periods of \( \Lambda \) and \( g \) is the genus of the orbit space \( \mathcal{H}/\Lambda \). Fuchsian groups with signatures \( (g; -) \) are called surface groups and they are characterized among Fuchsian groups as these ones which are torsion free.

The group \( \Lambda \) has associated to it a fundamental region whose area \( \mu(\Lambda) \), called the area of the group, is:

\[
\mu(\Lambda) = 2\pi \left( 2g - 2 + \sum_{i=1}^{r} (1 - 1/m_i) \right).
\]  

If \( \Gamma \) is a subgroup of finite index in \( \Lambda \), then we have the Riemann-Hurwitz formula which says that

\[
[\Lambda : \Gamma] = \frac{\mu(\Gamma)}{\mu(\Lambda)}. \tag{2}
\]

The points of \( \mathcal{H} \) with non-trivial stabilizers in \( \Lambda \) fall into \( r \) \( \Lambda \)-orbits \( o_1, \ldots, o_r \) such that every point belonging to \( o_i \) has a stabilizer which is a cyclic group of order \( m_i \). The points of \( X \) with non-trivial stabilizers fall into \( r \) \( G \)-orbits \( O_1, \ldots, O_r \), where \( O_i = \pi(o_i) \) and \( \pi : \mathcal{H} \to X \) is a projection map. Furthermore a homomorphism \( \theta : \Lambda \to G \) induces an isomorphism between stabilizers and so the stabilizer of \( y \in O_i \) is cyclic of order \( m_i \). The number of fixed points of an automorphism of \( X \) can be calculated by the following theorem of Macbeath [2].

**Theorem 2.1.** Let \( X = \mathcal{H}/\Gamma \) be a Riemann surface with the automorphism group \( G = \Lambda/\Gamma \) and let \( x_1, \ldots, x_r \) be elliptic canonical generators of \( \Lambda \) with periods \( m_1, \ldots, m_r \) respectively. Let \( \theta : \Lambda \to G \) be the canonical epimorphism and for \( 1 \neq g \in G \) let \( \varepsilon_i(g) \) be 1 or 0 according as \( g \) is or is not conjugate to a power of \( \theta(x_i) \). Then the number \( F(g) \) of points of \( X \) fixed by \( g \) is given by the formula

\[
F(g) = |N_G(\langle g \rangle)| \sum_{i=1}^{r} \varepsilon_i(g)/m_i. \tag{3}
\]
3. On $p$-hyperelliptic involutions of Riemann surfaces

Here we deal with the number of $p$-hyperelliptic involutions which a Riemann surface can admit. Along the chapter $X$ is a $p$-hyperelliptic Riemann surface of genus $g \geq 2$ and we call its $p$-hyperelliptic involutions briefly by $p$-involutions. First we give a new proof of the well known result of H. Farkas and I. Kra.

**Theorem 3.1.** A $p$-involution of a surface $X$ of genus $g > 4p + 1$ is unique and central in the full automorphism group of $X$.

**Proof.** Suppose that a Riemann surface $X = \mathcal{H}/\Gamma$ admits two distinct $p$-involutions $\rho$ and $\rho'$. Then they generate a dihedral group $G$, say of order $2n$ and there exist a Fuchsian group $\Lambda$ and an epimorphism $\theta : \Lambda \to G$ with the kernel $\Gamma$. If $x_i$ is a canonical elliptic generator of $\Lambda$ corresponding to some period $m_i > 2$ then $\theta(x_i) \in \langle \rho \rho' \rangle$. But none conjugation of $\rho$ nor of $\rho'$ belongs to $\langle \rho \rho' \rangle$ and so in terms of Macbeath’s theorem $\varepsilon_i(\rho) = \varepsilon_i(\rho') = 0$.

Now if $n$ is odd then $|N_G(\langle \rho \rangle)| = 2$ and $F(\rho) = 2g + 2 - 4p$ implies that $\Lambda$ has $2g + 2 - 4p$ periods equal to 2. If $n$ is even then $|N_G(\langle \rho \rangle)| = 4$ and so $g + 1 - 2p$ canonical elliptic generators are mapped by $\theta$ onto conjugates of $\rho$. Similarly another $g + 1 - 2p$ canonical elliptic generators are mapped by $\theta$ onto conjugates of $\rho'$. So in both cases $\sigma(\Lambda) = (\gamma; 2, \ldots, 2, m_{s+1}, \ldots, m_r)$, for $s = 2g + 2 - 4p$ and some integer $r \geq s$. Now applying the Hurwitz-Riemann formula for $(\Lambda, \Gamma)$, we obtain $2g - 2 = 2n(2 \gamma - 2 + g + 1 - 2p + \sum_{i=s+1}^{r} (1 - 1/m_i))$ which implies

$$g - 1 \geq n(g - 1 - 2p). \quad (4)$$

Since $n \geq 2$, it follows that $g \leq 4p + 1$. Thus for $g > 4p + 1$ a $p$-involution is unique.

Now given $g \in G$, $g g p^{-1}$ has the same number of fixed points as $\rho$. So by the Hurwitz-Riemann formula it is also a $p$-involution which implies that $g g p^{-1} = \rho$ for $g > 4p + 1$. 

$\blacksquare$

**Theorem 3.2.** Every two $p$-involutions of a Riemann surface $X$ of genus $3p + 2 \leq g \leq 4p + 1$ commute. Moreover for $3p + 2 < g \leq 4p + 1$, $X$ can admit two and no more such involutions.

**Proof.** Let $X$ be a Riemann surface of genus $3p + 2 \leq g \leq 4p + 1$. If $X$ admits two $p$-involutions then they generate the group $D_n = \Lambda/\Gamma$ for some $n$ satisfying the inequality (4), which implies

$$n \leq 1 + \frac{2p}{g - 1 - 2p}. \quad (5)$$

Thus $n = 2$ and so every two $p$-involutions of $X$ commute. Moreover their product cannot be a $p$-involution. Otherwise, by Theorem 2.1, $\Lambda$ would have the signature $(\gamma; 2, 3(g + 1 - 2p), 2)$ and applying the Hurwitz-Riemann formula for $(\Lambda, \Gamma)$ we would obtain $2\gamma = 3p - g$ and consequently $g \leq 3p$, a contradiction. So if $X$ admits three $p$-involutions $\rho_1, \rho_2, \rho_3$ then they generate the group $G = Z_2 \oplus Z_2 \oplus Z_2$ which can be identified with $\Delta/\Gamma$ for some Fuchsian group $\Delta$ with a signature $(\delta; 2, \ldots, 2)$. Let $\theta : \Delta \to G$ be the canonical epimorphism and let $s_k$ denote the number of elliptic generators of $\Delta$ which are transformed by $\theta$ onto $\rho_k$, for $k = 1, 2, 3$. Then by Theorem 2.1, $s_k = (g + 1 - 2p)/2$ for $k = 1, 2, 3$ and so applying the Hurwitz-Riemann formula for $(\Delta, \Gamma)$ we obtain $2g - 2 = 8(2\delta - 2 + 3(g + 1 - 2p)/4 + t/2)$,
Let $p$ be a positive integer. By Proposition 3.3, the number of pairwise commuting involutions of a Riemann surface $X$ of genus $3p + 2 < g \leq 4p + 1$ admits at most two $p$-involutions.

Now we shall prove that Riemann surfaces of such genera with two $p$-involutions actually exist. For, let $\Delta$ be a Fuchsian group with the signature $(0; 2, \ldots, 2)$, where $r = g + 3$ and let us define an epimorphism $\theta : \Delta \to Z_2 \oplus Z_2 = (\rho) \oplus (\rho')$ by the assignment $\theta(x_1) = \cdots = \theta(x_s) = \rho, \theta(x_{s+1}) = \cdots = \theta(x_{2s}) = \rho', \theta(x_{2s+1}) = \cdots = \theta(x_r) = \rho\rho'$, where $s = g + 1 - 2p$. Since $s$ and $r - 2s$ have the same parities, it follows that the relation $\theta(x_1) \cdots \theta(x_r) = 1$ holds. Moreover by Theorem 2.1, $F(\rho) = F(\rho') = 2g + 2 - 4p$ and so by the Hurwitz-Riemann formula, $\rho$ and $\rho'$ are two commuting $p$-involutions. \hfill $\Box$

**Proposition 3.3.** Let $\rho_1, \ldots, \rho_l$ be pairwise commuting $p$-involutions of a surface $X$ of genus $g$ and let they generate the group $G_k = Z_2 \oplus \ldots \oplus Z_2$, where $l \geq k$. Then

(i) $g \equiv 1(2^{k-2})$ and $p \equiv 1(2^{k-3})$,

(ii) the integers $k$ and $l$ are limited in the following cases:

\begin{align*}
    k &\leq 2 \quad \text{and} \quad l \leq 3 \quad \text{if} \quad g \equiv 0(2) \\
    k &\leq 3 \quad \text{and} \quad l \leq 4 \quad \text{if} \quad p \equiv 0(2) \\
    k &\leq 3 \quad \text{and} \quad l \leq 7 \quad \text{if} \quad g \equiv 3(4) \\
    k &\leq 4 \quad \text{and} \quad l \leq 15 \quad \text{if} \quad p \equiv 3(4).
\end{align*}

**Proof.** (i) Suppose that pairwise commuting $p$-involutions of a Riemann surface $X$ generate a group $G_k = Z_2 \oplus \ldots \oplus Z_2$. Then $G_k$ can be identified with $\Delta/\Gamma$ for a Fuchsian group $\Delta$ with the signature $(\gamma; 2, \ldots, 2)$. Applying the Hurwitz-Riemann formula for $(\Delta, \Gamma)$ we obtain $g - 1 = 2^{k-2}(4\gamma - 4 + r)$ which implies that $g \equiv 1(2^{k-2})$. Furthermore, by Theorem 2.1, a $p$-involution $\rho \in G_k$ admits fixed points in $(g + 1 - 2p)/2^{k-2}$ orbits and so in particular $g + 1 - 2p \equiv 0(2^{k-2})$. Consequently $p \equiv 1(2^{k-3})$.

(ii) The restrictions for $k$ are direct consequence of the conditions from (i). We need only to show that for even $p$, the group $G_3$ can admit at most 4 $p$-involutions. For, let us suppose that the product of two $p$-involution $\rho_1, \rho_2 \in G_3$ is a $p$-involution. Then they generate the group $G_2$ isomorphic with $\Delta/\Gamma$, where $\Delta$ is a Fuchsian group with the signature $(\delta; 2, 3(g+1-2p), 2)$. Thus $\delta = (3p - g)/2$ and so $3p - g \equiv 0(2)$. However $p$ is even and $g$ is odd which implies that $3p - g$ is odd, a contradiction. Consequently in this case $G_3$ may admit only one more $p$-involution, namely $\rho_1\rho_2\rho_3$ and so $l \leq 4$. \hfill $\Box$

By Proposition 3.3, the number of pairwise commuting $p$-involutions corresponding to given $p$ is limited for $p \equiv 0(2)$ or $p \equiv 3(4)$. The next proposition give a bound for such number for $p \equiv 1(4)$.

**Proposition 3.4.** Let $p = 1 + 2^n\alpha$, where $\alpha$ is odd and $m \geq 2$. Then the number of pairwise commuting $p$-involutions of a Riemann surface $X$ of genus $g \neq 2p - 1$ does not exceed $2^n\alpha + 5$, where $n$ is the least integer in range $0 \leq n \leq m + 2$ such that $2^n\alpha \geq m - n - 1$.

**Proof.** Given such $p$, let $X$ be a Riemann surface whose pairwise commuting $p$-involutions generate $G_k = Z_2 \oplus \ldots \oplus Z_2$. Then by Proposition 3.3, $k \leq m + 3$. So let us write $k = m + 3 - n$ for some integer $n$ in range $0 \leq n \leq m + 2$ and let $G_k = \Delta/\Gamma$ for a Fuchsian group $\Delta$ with a signature $(\gamma; 2, \ldots, 2)$. Since no single $G_k$-orbit contains fixed points of two different $p$-involutions, it follows that $r \geq ks$, where $s$ is the number of $G_k$-orbits containing fixed points
of a single $p$-involution. In order to check the greatest value of $k$, we consider the minimum value of $s$ and the maximum value of $r$. Thus we take $s = 1$ and $\gamma = 0$. By Theorem 2.1, $s = (g + 1 - 2p)/2k-2$ and so $s = 1$ for $g = 1 + 2^{n+1-n} + 2^{m+1}\alpha$. But the Hurwitz-Riemann formula for such $g$ and $\gamma = 0$ gives $r = 2^n\alpha + 5$ which clearly limits the number of $p$-involutions in $G_k$. Since for $s = 1$, the epimorphism $\theta : \Delta \to G_k$ cannot be defined for $r < k + 1$, it follows that $n$ is the least integer satisfying the inequality $2^n\alpha \geq m - n - 1$. 

**Proposition 3.5.** Let $X$ be a $p$-hyperelliptic Riemann surface of genus $g = 3p + 2$. Then $X$ admits at most 2 $p$-involutions if $p \equiv 0 \pmod{2}$ or $p \equiv 3 \pmod{4}$ and at most 3 if $p \equiv 1 \pmod{4}$ and $p > 5$. For $p = 1$ or $p = 5$, $X$ can admit 5 or 6 and no more $p$-involutions respectively.

**Proof.** By Theorem 3.2, all $p$-involutions of a Riemann surface of genus $g = 3p + 2$ commute one to each other and so they generate the group $G_k = Z_2 \oplus \ldots \oplus Z_2$ for some $k$. Let $G_k = \Delta/\Gamma$ for some Fuchsian group $\Delta$, say with a signature $(\gamma; 2, \ldots, 2)$. Denote by $s_k$ the number of $G_k$-orbits containing the fixed points of a single $p$-involution from $G_k$. By Theorem 2.1, $s_k = (g + 1 - 2p)/2k-2 = (p + 3)/2k-2$. Thus $k \leq 2$ for $p$ even and $k \leq 3$ and $s_k$ is odd for $p \equiv 3 \pmod{4}$. However, by the Hurwitz-Riemann formula for $k = 3$ and $(\Delta, \Gamma)$, we have $2\gamma + r - 3s_3 = 0$, which implies $\gamma = 0$ and $r = 3s_3$ in virtue of obvious $r \geq 3s_3$. Therefore, for $p \equiv 3 \pmod{4}$, an epimorphism $\theta : \Delta \to G_3$ actually can not exist. Consequently $k \leq 2$ if $p \equiv 0 \pmod{2}$ or $p \equiv 3 \pmod{4}$. Furthermore $X$ admits at most 2 $p$-involutions in these cases since, by the proof of the Theorem 3.2, a product of two $p$-involutions cannot be a $p$-involution for $g > 3p$.

Now let $p \equiv 1 \pmod{4}$. First we shall show that $k \leq 5$ and that surfaces whose $p$-involutions generate $G_4$ or $G_5$ exist only for $p \leq 5$. For, let us write $p = 4\alpha + 1$ for some integer $\alpha$. Then $g = 1 + 4(1 + 3\alpha)$ and $s_k = (\alpha + 1)/2k-4$. Let $n$ and $m$ be the greatest integers such that $g \equiv 1 \pmod{2^n}$ and $p \equiv 1 \pmod{2^n}$. Then for even $\alpha$, we have $n = 2$ which by (i) of the Proposition 3.3 implies $k \leq 4$ and for odd $\alpha$, $m = 2$ and consequently $k \leq 5$.

Now let $t = r - ks_k$. Applying the Hurwitz-Riemann formula for $(\Delta, \Gamma)$ and $k = 4$, we obtain $1 = 4\gamma + \alpha + t$. Thus $\gamma = 0$ and either $\alpha = 1, r = 4s_4$, or $\alpha = 0, r = 4s_4 + 1$. Consequently $p = 5$, $s_4 = 2$ and $\sigma(\Delta) = (0; 2, 2, 2, 2, 2, 2, 2)$ or $p = 1, s_4 = 1$ and $\sigma(\Delta) = (0; 2, 2, 2, 2, 2)$. So there exists exactly one possible epimorphism $\theta : \Delta \to G_4$ whose image is generated by $p$-involution and it is given by the assignment

$$\theta(x_i) = \rho_j \text{ for } 1 \leq j \leq k, (j - 1)s_k < i \leq js_k,$$

in the first case and by the assignment

$$\theta(x_i) = \rho_j, \theta(x_{ksk+1}) = \rho_1 \cdot \cdots \cdot \rho_k \text{ for } 1 \leq j \leq k, (j - 1)s_k < i \leq js_k,$$

in the second one, where $k = 4$. Thus the surface whose $p$-involution generate $G_4$ exists only for $p = 1$ or $p = 5$ and the corresponding group $G_4$ admits exactly five or four $p$-involution respectively.

Similarly for $k = 5$ we obtain $4\gamma + \alpha + t = 2$. Since for even $\alpha$ we have $k \leq 4$, it follows that $\alpha = 1, \gamma = 0$ and $r = 5s_5 + 1$. Thus $p = 5, s_5 = 1$ and $\Delta$ has the signature $(0; 2, 2, 2, 2, 2, 2)$. Now the assignment (7) defines the only possible epimorphism onto $G_5$. Thus the surface whose $p$-involution generate $G_5$ exists only for $p = 5$ and the corresponding group $G_5$ admits exactly six $5$-involution.
Thus if central, it follows that actually $\rho \neq \theta$. Thus the assignment (6) for $k = 3$, defines the only possible epimorphism $\Delta \rightarrow G_3$ whose image is generated by $p$-involutions and so the group $G_3$ contains exactly 3 $p$-involutions. \hfill \Box

Let us notice that for arbitrary positive integer $k \geq 5$, we can find integers $p$ and $g$ such that there exists a Riemann surface of genus $g$ admitting $k$ pairwise commuting $p$-involutions. Indeed for $g = 1+(k-4)2^{k-3}$ and $p = 1+(k-5)2^{k-4}$ we can take a Fuchsian group $\Delta$ with the signature $(0;2, \ldots, 2)$ and define an epimorphism $\theta : \Delta \rightarrow Z_2 \oplus \cdots \oplus Z_2 = \langle \rho_1 \rangle \oplus \cdots \oplus \langle \rho_{k-1} \rangle$ by the assignment $\theta(x_i) = \rho_i$ for $i = 1, \ldots, k$ and $\theta(x_k) = \rho_1 \cdots \rho_{k-1}$. Then $\Gamma = \ker \theta$ is a surface Fuchsian group of orbit genus $g$ and $\rho_i$ are $p$-involutions of a Riemann surface $X = \mathcal{H}/\Gamma$.

At the end of the paper we give a bound for the number of all central $p$-involutions of a surface $X$.

**Theorem 3.6.** Let $X$ be a $p$-hyperelliptic Riemann surface of genus $g \geq 2$ and let $G$ be its automorphism group of order $2N$. Assume that the canonical projection $X \rightarrow X/G$ is ramified at $r$ points with multiplicities $m_1, \ldots, m_r$. Then for $g \neq 2p - 1$, the number of central $p$-involutions of $X$ does not exceed

$$(N \sum_{i=1}^{r} 1/m_i) / (g + 1 - 2p).$$

**Proof.** Here $X = \mathcal{H}/\Gamma$ for some Fuchsian surface group $\Gamma$ with the signature $(g; -)$ and $G = \Delta/\Gamma$ for some Fuchsian group $\Delta$ with the signature $(\delta; m_1, \ldots, m_r)$. Let $x_1, \ldots, x_r$ be canonical elliptic generators of $\Delta$ and let $\theta : \Delta \rightarrow G$ be the canonical epimorphism. Assume that $X$ admits a central $p$-involution $\rho$. If $g \neq 2p - 1$ then $\rho$ has fixed points and so it is conjugate to $\theta(x_i)^{m_i/2}$ for some $x_i$ corresponding to an even period $m_i$. However since $\rho$ is central, it follows that actually $\rho = \theta(x_i)^{m_i/2}$. In particular for distinct $p$-involutions $\rho$ and $\rho'$, $\varepsilon_i(\rho) \neq \varepsilon_i(\rho')$. Moreover by Theorem 2.1, $N \sum_{i=1}^{r} \varepsilon_i(\rho)/m_i = g + 1 - 2p = N \sum_{i=1}^{r} \varepsilon_i(\rho')/m_i$. Thus if $n$ is the number of all $p$-involutions of $X$ then $n(g + 1 - 2p) \leq N \sum_{i=1}^{r} 1/m_i$ and so the theorem is proved. \hfill \Box

**Acknowledgement.** The author wishes to thank the referee for his comments and suggestion.

**References**


Received August 2, 2004