Generalized Adjoint Semigroups of a Ring

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Abstract. In this paper, we introduce generalized adjoint semigroups (GA-semigroups) of a ring $R$. We construct generalized adjoint semigroups on a ring $R$ by means of bitranslations of $R$. It is shown that GA-semigroups of a $\pi$-regular ring are $\pi$-regular. As an application we deduce that in any ring, idempotents can be lifted modulo $\pi$-regular ideals. GA-semigroups containing idempotents are described in terms of the ring of a Morita context.

1. Introduction

Let $R$ be a ring not necessarily with identity. The composition defined by $a \circ b = a + b + ab$ for any $a, b \in R$ is usually called the circle or adjoint multiplication of $R$, which plays a role in the theory of Jacobson radical. It is well-known that $(R, \circ)$ is a monoid with identity 0, called the circle or adjoint semigroup of $R$. There are many interesting connections between a ring and its adjoint semigroup, which were studied in several papers, for example, [8, 13, 14, 16, 22, 23, 24, 30, 31]. Typical results are to describe the adjoint semigroup of a given ring and the ring with a given semigroup as its adjoint semigroup.

The circle multiplication of a ring satisfies the following generalized distributive laws:

$$a \circ (b + c - d) = a \circ b + a \circ c - a \circ d,$$

$$b + c - d \circ a = b \circ a + c \circ a - d \circ a,$$

References

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or equivalently,

\[ a \circ (b + c) = a \circ b + a \circ c - a \circ 0, \]
\[ (b + c) \circ a = b \circ a + c \circ a - 0 \circ a, \]

which was observed in [1]. Thus as generalizations of the circle multiplication of a ring, a binary operation \( \circ \) (associative or nonassociative) on an Abelian group \( A \) satisfying the generalized distributive laws have been studied by several authors making use of different terminologies, for example, pseudo-ring in [33], weak rings in [10], quasirings in [11], prerings in [3, 4, 29]. In particular, the so-called \((m,n)\)-distributive rings studied in [5, 26, 27, 36] also satisfy the generalized distributive laws (1) and (2). To such a system \((A, +, \circ)\) there corresponds a unique associated ordinary ring. But, in general, even if \( A \) is a ring, there may exist no relation between the ring \( A \) and the associated ring of \((A, +, \circ)\). In this paper, we are interested in a binary operation \( \circ \) on a ring \( R \), satisfying the associative law, the generalized distributive laws as (1) and (2), and the compatibility:

\[ xy = x \circ y - x \circ 0 - 0 \circ y + 0 \circ 0. \]

This is equivalent to say that \((R, +, \circ)\) is a weak ring such that the ring \( R \) is exactly the associated ring of \((R, +, \circ)\). Such a binary operation \( \circ \) is called a generalized adjoint multiplication on \( R \) and the semigroup \((R, \circ)\) is called a generalized adjoint semigroup of \( R \), abbreviated GA-semigroup, which is a generalization of the multiplicative semigroup and the adjoint semigroup of a ring \( R \). Essentially, the multiplicative and adjoint semigroup of \( R \) are exactly generalized adjoint semigroups of \( R \) with zero and identity, respectively (cf. Theorem 2.14). The other generalization of adjoint multiplication was studied in [21].

The aim of this paper is to describe generalized adjoint semigroups of a ring \( R \). In Section 2, we present a way to construct generalized adjoint multiplications on a ring \( R \) by means of bitranslations of \( R \), characterize a GA-semigroup with identity or zero and describe GA-semigroups of a ring with 1.

In Section 3, we prove that GA-semigroups of a \( \pi \)-regular ring are \( \pi \)-regular.

In Section 4, we first prove that a GA-semigroup containing idempotents can be represented as a GA-semigroup of the ring of a Morita context. Then we present a sufficient condition and a necessary condition for a GA-semigroup to contain idempotents, in virtue of which we prove that in any ring, idempotents can be lifted modulo a \( \pi \)-regular ideal. This generalizes a classical result in ring theory which states that idempotents modulo a nil ideal can be lifted ([28]) and the ring-theoretic analogue of a result of Edwards ([19, Corollary 2]) which extends the well-known Lallement’ lemma to eventually regular semigroups (i.e., \( \pi \)-regular semigroups). Finally, we prove that GA-semigroups of rings with DCC on principal right ideals contain idempotents.

In the forthcoming paper [17], we characterize the rings with a GA-semigroup having a property \( P \) and its such GA-semigroups, where \( P \) stands for orthodox, right inverse, inverse, pseudoinverse, \( E \)-unitary, and completely simple, respectively.
Although a ring $R$ in this paper needs not contain identity, it is convenient to use a formal identity $1$, which can be regarded as the identity of a unitary ring containing $R$, since $R$ can be always embedded into a ring with identity $1$; for example, we can write $a \circ b = (1 + a)(1 + b) - 1$ for any $a, b \in R$ and write $x^0 = 1$ for any $x \in R$ by making use of a formal $1$.

For $x \in R$ and a positive integer $n$ we denote by $x^{[n]}$ the $n$-th power of $x$ with respect to a generalized adjoint multiplication $\circ$, and $x^{[0]}$ stands for an empty word.

A radical ring means a Jacobson radical ring.

For the algebraic theory and terminology on semigroups we will refer to [9, 20, 25].

2. A construction of GA-semigroups

**Definition 2.1.** Let $R$ be a ring. A binary operation $\circ$ on $R$ is called a generalized adjoint multiplication on $R$, if it satisfies the following conditions:

(i) the associative law: $x \circ (y \circ z) = (x \circ y) \circ z$;

(ii) the generalized distributive laws:

\[
\begin{align*}
  x \circ (y + z) &= x \circ y + x \circ z - x \circ 0, \\
  (y + z) \circ x &= y \circ x + z \circ x - 0 \circ x;
\end{align*}
\]

(iii) the compatibility: $xy = x \circ y - x \circ 0 - 0 \circ y + 0 \circ 0$.

The semigroup $(R, \circ)$ is called a generalized adjoint semigroup of $R$, abbreviated GA-semigroup and denoted by $R^\circ$.

We now remark that for a binary operation $\circ$ on $R$, the generalized distributive laws are equivalent to

\[
\begin{align*}
  w \circ (x + y - z) &= w \circ x + w \circ y - w \circ z, \\
  (x + y - z) \circ w &= x \circ w + y \circ w - z \circ w.
\end{align*}
\]

**Example 2.2.** The multiplicative semigroup $R^*$ of a ring $R$ is a GA-semigroup of $R$ with zero $0$. The adjoint semigroup $R^\circ$ of $R$ is a GA-semigroup of $R$ with identity $0$.

**Lemma 2.3.** For any $x_i, y_j \in R$, and $p_i, q_j \in \mathbb{Z}$ with $\sum p_i = \sum q_j = 0$, we have

\[
\left( \sum p_i x_i \right) \left( \sum q_j y_j \right) = \sum p_i q_j (x_i \circ y_j).
\]
Proof. Set \( p = \sum p_i \) and \( q = \sum q_j \). Then we have that
\[
\left( \sum p_i x_i \right) \left( \sum q_j y_j \right) \\
= \sum p_i q_j (x_i y_j) \\
= \sum p_i q_j (x_i \circ y_j) - \sum p_i q_j (x_i \circ 0) \\
- \sum p_i q_j (0 \circ y_j) + \sum p_i q_j (0 \circ 0) \quad \text{(by the compatibility)} \\
= \sum p_i q_j (x_i \circ y_j) - q \sum p_i (x_i \circ 0) - p \sum q_j (0 \circ y_j) + pq(0 \circ 0) \\
= \sum p_i q_j (x_i \circ y_j),
\]
as desired. \( \square \)

**Corollary 2.4.** If \( x \circ y = y \circ x \), then \((x - y)^n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} x[i] \circ y[n-i] \).

*Proof.* As the usual binomial theorem, the corollary can be proved by use of an induction on \( n \) and Lemma 2.3. \( \square \)

Recall that a bitranslation is a pair \((\lambda, \rho) \in \text{End}(R_R) \times \text{End}(R_R)\) such that \( x\lambda(y) = \rho(x)y \) for any \( x, y \in R \). The set \( \Omega(R) \) of all bitranslations of \( R \) is a subring of \( \text{End}(R_R) \times \text{End}(R_R) \) with identity \((1_R, 1_R)\), called the translational hull of \( R \). For \( a \in R \), let \( \lambda_a \) and \( \rho_a \) be the left and right multiplications by \( a \), respectively. Then \((\lambda_a, \rho_a)\) is a bitranslation of \( R \), denoted by \( \pi_a \), and \( \pi : a \mapsto \pi_a \) defines a ring homomorphism form \( R \) into \( \Omega(R) \) such that the image \( \pi(R) \) is an ideal of \( \Omega(R) \) and the kernel is \( \text{Ann}(R) = \{ x \in R | xR = Rx = 0 \} \). Hence we can identify \( a \in R \) with \( \pi_a \) and \( R \) with \( \pi(R) \) whenever \( \text{Ann}(R) = 0 \). A bitranslation \( \theta = (\lambda, \rho) \) will be considered as a double operator on \( R \) defined by \( \theta x = \lambda(x) \) and \( x\theta = \rho(x) \) for any \( x \in R \). Then \( \theta = \theta' \) if and only if \( \theta x = \theta' x \) and \( x\theta = x\theta' \) for any \( x \in R \). A bitranslation \( \theta \) is called self-permutable if \( (\theta x)\theta = \theta(x\theta) \) for any \( x \in R \) ([32, 34, 35]).

For a self-permutable bitranslation \( \theta \), there is no ambiguity if we write \( \theta xy\theta^2 z \), for example.

By an associated pair of \( R \) we mean a pair \((\theta, \vartheta) \in \Omega(R) \times R\) satisfying the following conditions:

\( (i) \) \( \theta \vartheta = \vartheta \theta \);

\( (ii) \) \( \theta \) is self-permutable;

\( (iii) \) \( \theta^2 = \theta + \pi_{\vartheta} \).

**Theorem 2.5.** Let \((\theta, \vartheta)\) be an associated pair of a ring \( R \) and define
\[
x \circ y = xy + x\theta + \theta y + \vartheta
\]
for any \( x, y \in R \). Then \( \circ \) is a generalized adjoint multiplication on \( R \) (called one induced by \((\theta, \vartheta)) \). Conversely, every generalized adjoint multiplication \( \circ \) on \( R \) can be obtained in this fashion by setting \( \vartheta = 0 \circ 0 \), \( \theta x = 0 \circ x - 0 \circ 0 \) and \( x\theta = x \circ 0 - 0 \circ 0 \). Moreover, the correspondence \((\theta, \vartheta) \mapsto \circ \) is a 1-1 correspondence between the associated pairs of \( R \) and generalized adjoint multiplications on \( R \).
Proof. Suppose that \((\theta, \vartheta)\) is an associated pair of \(R\) and the operation \(\diamond\) is given by (3). Then the associative law is verified as follows:

\[
(x \diamond y) \diamond z = (xy + x\theta + \theta y + \vartheta) \diamond z \quad \text{(by (3))}
\]

\[
= xyz + x\theta z + \theta y z + \vartheta z + x\theta y + \theta y \theta + \vartheta \theta + \theta z + \vartheta
\]

\[
= xyz + x\theta y + x\theta z + x\vartheta + x\theta + \theta y z + \theta y \theta + \vartheta z + \vartheta \theta + \vartheta
\]

\[
= x \diamond (yz + y\theta + \theta z + \vartheta) \quad \text{(by (3))}
\]

\[
= x \diamond (y \diamond z).
\]

For the generalized distributive laws, we have that

\[
x \diamond (y + z) = xy + xz + x\theta + \theta y + \theta z + \vartheta \quad \text{(by (3))}
\]

\[
= (xy + x\theta + \theta y + \theta z + \vartheta) + (xz + x\theta + \theta z + \vartheta) - (x\theta + \vartheta)
\]

\[
= x \diamond y + x \diamond z - x \diamond 0, \quad \text{(by (3))}
\]

and similarly \((y + z) \diamond x = y \diamond x + z \diamond x - 0 \diamond x\). The compatibility follows from

\[
x \diamond y - x \diamond 0 - 0 \diamond y + \vartheta
\]

\[
= (xy + x\theta + \theta y + \vartheta) - (x\theta + \vartheta) - (\theta y + \vartheta) + \vartheta \quad \text{(by (3))}
\]

\[
= xy.
\]

Thus \(\diamond\) is a generalized circle multiplication on \(R\).

Conversely, suppose \(\diamond\) is a generalized adjoint multiplication on \(R\). Set \(\vartheta = 0 \diamond 0\), \(\lambda(x) = 0 \diamond x - 0 \diamond 0\), \(\rho(x) = x \diamond 0 - 0 \diamond 0\) and \(\theta = (\lambda, \rho)\). For any \(a, x, y \in R\), we have that

\[
\lambda(x + y) = 0 \diamond (x + y) - 0 \diamond 0 = 0 \diamond x + 0 \diamond y - 2\vartheta = \lambda(x) + \lambda(y),
\]

\[
\lambda(x)a = (0 \diamond x - 0 \diamond 0)(a - 0)
\]

\[
= 0 \diamond x \diamond a - 0 \diamond x \diamond 0 - 0 \diamond 0 \diamond a + 0 \diamond 0 \diamond 0 \quad \text{(by Lemma 2.3)}
\]

\[
= 0 \diamond (x \diamond a - x \diamond 0 - 0 \diamond a + 0 \diamond 0) - 0 \diamond 0
\]

\[
= 0 \diamond (xa) - 0 \diamond 0
\]

\[
= \lambda(xa),
\]

which imply that \(\lambda \in \text{End}(R_R)\). Symmetrically, \(\rho \in \text{End}(R_R)\). Note that

\[
x \lambda(y) = (x - 0)(0 \diamond y - 0 \diamond 0)
\]

\[
= x \diamond 0 \diamond y - x \diamond 0 \diamond 0 - 0 \diamond 0 \diamond y + 0 \diamond 0 \diamond 0 \quad \text{(by Lemma 2.3)}
\]

\[
= (x \diamond 0 - 0 \diamond 0)(y - 0) \quad \text{(by Lemma 2.3)}
\]

\[
= \rho(x)y.
\]
Hence \( \theta \) is a bitranslation of \( R \) such that \( \theta x = 0 \circ x - 0 \circ 0 \) and \( x \theta = x \circ 0 - 0 \circ 0 \).

Thus \( \theta \theta = 0 \circ \vartheta - 0 \circ 0 = \vartheta \circ 0 - 0 \circ 0 = \vartheta \theta \). Since

\[
(\theta x)\theta = (0 \circ x - 0 \circ 0) \circ 0 - 0 \circ 0 = 0 \circ x - 0 \circ 0 \circ 0,
\theta(x\theta) = 0 \circ (x \circ 0 - 0 \circ 0) \circ 0 = 0 \circ x - 0 \circ 0 \circ 0,
\]

we have that \( (\theta x)\theta = \theta(x\theta) \), that is, \( \theta \) is self-permutable. Observing that

\[
(\theta + \pi_{\varphi})x = \theta x + \vartheta x
= 0 \circ x - 0 \circ 0 + \vartheta \circ x - \vartheta \circ 0 - 0 \circ x + \vartheta
= \vartheta \circ x - 0 \circ 0 - 0 \circ \vartheta + 0 \circ 0
= \theta(0 \circ x) - \theta \vartheta
= \theta(0 \circ x - \vartheta)
= \theta^2 x,
\]

and similarly \( x(\theta + \pi_{\varphi}) = x\theta^2 \), we see that \( \theta^2 = \theta + \pi_{\varphi} \). It follows that \( (\theta, \vartheta) \) is an associated pair of \( R \). Since

\[
x \circ y = xy + x \circ 0 + 0 \circ y - \vartheta = xy + x \theta + \theta y + \vartheta
\]

we see that \( \circ \) is induced by \( (\theta, \vartheta) \).

If two associated pairs \( (\theta, \vartheta) \) and \( (\theta', \vartheta') \) of \( R \) induce the same generalized adjoint multiplication on \( R \), then for any \( x, y \in R \) we have

\[
xy + x \theta + \theta y + \vartheta = xy + x \theta' + \theta' y + \vartheta',
\]

and so we have \( \vartheta = \vartheta' \) by taking \( x = y = 0, x \theta = x \theta' \) by taking \( y = 0 \), and \( \theta y = \theta' y \) by taking \( x = 0 \), whence \( (\theta, \vartheta) = (\theta', \vartheta') \). Thus the correspondence \( (\theta, \vartheta) \rightarrow \circ \) is a 1-1 correspondence.

Theorem 2.5 is an analogue of results in [26, 27].

**Corollary 2.6.** If \( \text{Ann}(R) = 0 \), then any generalized adjoint multiplication on \( R \) is induced by a bitranslation \( \theta \) of \( R \) such that \( \theta^2 - \theta \in R \), and further there exists a 1-1 correspondence between the set of bitranslations being idempotent modulo \( \pi(R) \) and generalized adjoint multiplications on \( R \).

**Proof.** If \( \text{Ann}(R) = 0 \), then \( \Omega(R) \) is an ideal extension of \( R \). Let \( \circ \) be the generalized adjoint multiplication on \( R \) induced by an associated pair \( (\theta, \vartheta) \). Then \( \theta^2 - \theta \in R \), and \( \theta^2 = \theta + \pi_{\varphi} \) implies \( \vartheta = \theta^2 - \theta \) since \( \text{Ann}(R) = 0 \). It is clear that \( x \circ y = (x + \theta)(y + \theta) - \theta \). From Theorem 2.5 the correspondence \( \theta \rightarrow \circ \) is a 1-1 correspondence.

The following corollary will be used freely throughout the rest of this paper.

**Corollary 2.7.** For any \( x_i, y_j \in R \), and \( p_i, q_j \in \mathbb{Z} \) with \( \sum p_i = \sum q_j = 1 \), we have

\[
\left( \sum p_i x_i \right) \circ \left( \sum q_j y_j \right) = \sum p_i q_j (x_i \circ y_j).
\]
Proof. For any \( x_i, y_j \in R \), and \( p_i, q_j \in \mathbb{Z} \) with \( \sum p_i = \sum q_j = 1 \), we have
\[
\sum p_i q_j (x_i \circ y_j) = \sum p_i q_j (x_i y_j) + \sum p_i q_j (x_i \theta) + \sum p_i q_j (\theta y_j) + \sum p_i q_j \theta = (\sum p_i x_i) (\sum q_j y_j) + (\sum p_i x_i) \theta + \theta (\sum q_j y_j) + \theta = (\sum p_i x_i) \circ (\sum q_j y_j),
\]
as desired. \( \square \)

Corollary 2.8. If \( x, y \in R^\circ \) such that \( x \circ y = y \circ x \) and \( p, q \in \mathbb{Z} \) such that \( p + q = 1 \), then
\[
(px + qy)^n = \sum_{i=0}^{n} p^i q^{n-i} \left( \begin{array}{c} n \\ i \end{array} \right) x^i \circ y^{n-i}.
\]
Proof. As the usual binomial theorem, the corollary can be proved by use of an induction on \( n \) and Corollary 2.7. \( \square \)

By an affine subsemigroup of \( R^\circ \) we mean a subsemigroup \( M \) of \( R^\circ \) such that \( x + y - z \in S \) for any \( x, y, z \in M \).

For example, for an ideal extension \( \tilde{R} \) of \( R \) (i.e., \( \tilde{R} \) is a ring containing \( R \) as an ideal) and \( a \in \tilde{R} \) such that \( a^2 - a \in R \), then \( (R + a, \bullet) \) is an affine subsemigroup of \( \tilde{R}^\bullet \). The semigroup \( (R + a, \bullet) \) was studied in [18] to deal with lifting idempotents.

Definition 2.9. Let \( M \) and \( N \) be affine subsemigroups of \( GA \)-semigroups \( R^\circ \) and \( S^\circ \) of rings \( R \) and \( S \), respectively. If there exists a bijection \( \phi \) from \( M \) onto \( N \) such that
\[
\phi(x + y - z) = \phi(x) + \phi(y) - \phi(z) \quad \text{and} \quad \phi(x \circ y) = \phi(x) \circ \phi(y)
\]
for any \( x, y, z \in M \), then \( M \) and \( N \) are called affinely isomorphic, notationally \( M \cong N \).

Corollary 2.10. Let \( \tilde{R} \) be an ideal extension of \( R \). Then any \( a \in \tilde{R} \) such that \( a^2 - a \in R \) induces a generalized adjoint multiplication on \( R \) given by
\[
x \circ y = (x + a)(y + a) - a
\]
for \( x, y \in R \), and \( R^\circ \) is affinely isomorphic to the affine subsemigroup \( (R + a, \bullet) \) of \( \tilde{R}^\bullet \).

Proof. It is clear that \( a \) induces a bitranslation \( \theta \) of \( R \) by \( \theta x = ax \) and \( x \theta = xa \). If \( a^2 - a \in R \), then \( (\theta, a^2 - a) \) is an associated pair of \( R \) and the induced generalized adjoint multiplication on \( R \) given by \( x \circ y = xy + xa + ay + \theta = (x + a)(y + a) - a \). Let \( \phi \) be a map from \( R \) into \( R + a \) given by \( \phi(x) = x + a \) for any \( x \in R \). Then it is easy to check that \( \phi \) is an affine isomorphism from \( R^\circ \) onto the affine subsemigroup \( (R + a, \bullet) \) of \( \tilde{R}^\bullet \). \( \square \)
Lemma 2.11. Let $M$ be an affine subsemigroup of $R^\circ$. Then

$$M - M = M - a = \left\{ \sum p_i s_i \mid s_i \in M, \text{ and } p_i \in \mathbb{Z} \text{ with } \sum p_i = 0 \right\}$$

for any $a \in M$, and $M - M$ is a subring of $R$.

Proof. The proof is a routine computation. \qed

Theorem 2.12. Let $M$ and $N$ be affine subsemigroups of GA-semigroups $R^\circ$ and $S^\circ$ of rings $R$ and $S$, respectively. If $M \simeq N$, then the rings $M - M$ and $N - N$ are isomorphic to each other. In particular, if $R^\circ \simeq S^\circ$, then $R \simeq S$.

Proof. Suppose $\phi$ is an affine isomorphism from $M$ onto $N$. Take a fixed $a \in M$ and let $\phi^*$ be the mapping from $M$ into $N$ defined by $\phi^*(x - a) = \phi(x) - \phi(a)$ for any $x \in M$. Then we see that $\phi^*$ is a bijection. Since for any $x, y \in M$,

$$\phi^*((x - a) - (y - a))$$

$$= \phi^*((x - y + a) - a)$$

$$= \phi(x - y + a) - \phi(a)$$

$$= \phi(x) - \phi(y) + \phi(a) - \phi(a)$$

$$= \phi^*(x - a) - \phi^*(y - a),$$

we have that $\phi^*$ is a ring isomorphism from the ring $M - M$ onto $N - N$ by Lemma 2.11. \qed

Lemma 2.13. Let $M$ be an affine subsemigroup of $R^\circ$.

(i) If $M$ has identity, then $M \simeq (M - M, \circ)$;

(ii) If $M$ has zero, then $M \simeq (M - M, \bullet)$.

Proof. Given $e \in (M, \circ)$, we define $\phi : M \to M - M$ by $\phi(x) = x - e$. It is clear that $\phi(x + y - z) = \phi(x) + \phi(y) - \phi(z)$. Note that for any $x, y \in M$

$$(x - e)(y - e) = x \circ y - x \circ e - e \circ y + e \circ e. \quad (4)$$

Thus, if $e$ is identity of $M$, then

$$\phi(x \circ y) = x \circ y - e$$

$$= (x - e)(y - e) + x + y - 2e \quad \text{(by (4))}$$

$$= (x - e) \circ (y - e)$$

$$= \phi(x) \circ \phi(y);$$
while if \( e \) is zero of \( M \), then by (4),
\[
\phi(x \circ y) = x \circ y - e = (x - e)(y - e) = \phi(x)\phi(y).
\]
Hence \( \phi \) is an affine isomorphism if \( e \) is identity or zero of \( M \). \( \square \)

**Theorem 2.14.** Let \( R^\circ \) be a GA-semigroup of a ring \( R \). Then

(i) \( R^\circ \) has identity if and only if \( R^\circ \cong R^\bullet \);

(ii) \( R^\circ \) has zero if and only if \( R^\circ \cong R^\bullet \);

(iii) if \( R \) has identity, then \( R^\circ \cong R^\bullet \cong R^\circ \).

**Proof.** (i) and (ii) are immediate results of Lemma 2.13. If \( R \) has 1, then \( R = \Omega(R) \) and so by Corollary 2.10 there is \( a \in R \) such that \( x \circ y = (x + a)(y + a) - a \) for any \( x, y \in R \). Clearly, \( -a \) is zero of \( R^\circ \). Thus \( R^\circ \cong R^\bullet \) by (ii), and \( R^\circ \cong R^\bullet \) under the affine isomorphism \( x \to 1 + x \) from \( R^\circ \) onto \( R^\bullet \), proving (iii). \( \square \)

**3. GA-semigroups of \( \pi \)-regular rings**

Recall that a semigroup \( S \) is (left, right, completely) \( \pi \)-regular if and only if for any \( x \in S \) there exists a positive integer \( n \) such that \( x^n \in Sx^{n+1} \), \( x^n \in x^{n+1}S \), \( x^n \in Sx^{n+1} \cap x^{n+1}S \) \( x^n \in x^nSx^n \).

For a positive integer \( n \), a semigroup \( S \) is called (left, right, completely) \( \pi_n \)-regular if \( (x^n \in Sx^{n+1} \), \( x^n \in x^{n+1}S \), \( x^n \in Sx^{n+1} \cap x^{n+1}S \) \( x^n \in x^nSx^n \) for any \( x \in S \). By a (left, right, completely) \( \pi_0 \)-regular semigroup we mean a (left, right, completely) \( \pi \)-regular semigroup.

For a non-negative integer \( n \), a ring is called (left, right, completely) \( \pi_n \)-regular if its multiplicative semigroup is (left, right, completely) \( \pi_n \)-regular.

In [15] we proved that the adjoint semigroup of a \( \pi \)-regular ring is \( \pi \)-regular and in [16], we proved further that the adjoint semigroup of a (left, right, completely) \( \pi_n \)-regular ring is (left, right, completely) \( \pi_n \)-regular. In this section, we will prove that this is true for GA-semigroups.

**Lemma 3.1.** For any \( a, b, x, y, z \in R \), we have
\[
(a - a \circ x)z(b - y \circ b) \in a \circ R \circ b - a \circ R \circ b.
\]

**Proof.** Noting that \( a \circ R \circ b \) is an affine subsemigroup of \( R^\circ \), we see that
\[
(a - a \circ x)z(b - y \circ b) = (a - a \circ x)(z - 0)(b - y \circ b)
\]
\[
= a \circ z \circ b - a \circ z \circ y \circ b - a \circ 0 \circ b + a \circ 0 \circ y \circ b - a \circ x \circ z \circ b
\]
\[
+ a \circ x \circ z \circ y \circ b + a \circ x \circ 0 \circ b - a \circ x \circ 0 \circ y \circ b \quad \text{(by Lemma 2.3)}
\]
\[
\in a \circ R \circ b - a \circ R \circ b, \quad \text{(by Lemma 2.11)}
\]
completing the proof. \( \square \)
Lemma 3.2. Let $A = b \diamond R \diamond c - b \diamond R \diamond c$. If $x$ commutes with $c$ in $R^e$, then $a - a \diamond x \in A$ implies $a - a \diamond x^n \in A$ for any positive integer $n$.

Proof. To prove the lemma, we proceed with an induction on $n$. It is trivial for $n = 1$. Assume $n > 1$ and $a - a \diamond x^{n-1} \in A$. Let $a - a \diamond x^{n-1} = b \diamond y \diamond c - b \diamond z \diamond c$. Then multiplication (with respect to $\diamond$) by $x$ on the right shows that

$$a \diamond x - a \diamond x^n = b \diamond y \diamond x \diamond c - b \diamond z \diamond x \diamond c,$$

whence by Lemma 2.11

$$a - a \diamond x^n = a - a \diamond x + a \diamond x - a \diamond x^n$$

$$= a - a \diamond x + b \diamond y \diamond x \diamond c - b \diamond z \diamond x \diamond c$$

$$\in A,$$

as desired. \qed

Lemma 3.3. Let $a$ and $x$ commute with each other in $R^e$. Then for any positive integers $m$ and $n$ we have that

$$(a - a^{[m]} \diamond x)^n = a^{[n]} - a^{[n+m-1]} \diamond y,$$

for some $y$ commuting with $a$ and $x$ in $R^e$.

Proof. By Corollary 2.4,

$$(a - a^{[m]} \diamond x)^n$$

$$= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a^{[i]} \diamond (a^{[m]} \diamond x)^{[n-i]}$$

$$= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a^{[i+m(n-i)]} \diamond x^{[n-i]}$$

$$= a^{[n]} - \sum_{i=0}^{n-1} (-1)^{n-i+1} \binom{n}{i} a^{[n+m-1]} \diamond a^{[(m-1)(n-1-i)]} \diamond x^{[n-i]}$$

$$= a^{[n]} - a^{[n+m-1]} \diamond \sum_{i=0}^{n-1} (-1)^{n-i+1} \binom{n}{i} (a^{[(m-1)(n-1-i)]} \diamond x^{[n-i]}),$$

since $\sum_{i=0}^{n-1} (-1)^{n-i+1} \binom{n}{i} = 1$. Let

$$y = \sum_{i=0}^{n-1} (-1)^{n-i+1} \binom{n}{i} (a^{[(m-1)(n-1-i)]} \diamond x^{[n-i]}).$$

Then $(a - a \diamond x)^n = a^{[n]} - a^{[n+m-1]} \diamond y$ and it is clear that $y$ commutes with both $a$ and $x$. \qed
Lemma 3.4. Let \( a, w, x, y, z \in R \) such that \( x, y \) and \( z \) commute with \( a \) in \( R^\circ \), and let \( n \) be a positive integer, and \( k \) and \( m \) be non-negative integers not all zero. If

\[
(a - a \odot x \odot a)^n = (a - a \odot y)^k (a - z \odot a)^m,
\]
then \( a^n = a^k \odot u \odot a^m \) for some \( u \in R \).

Proof. Let \( A = a^k \odot R \odot a^m - a^k \odot R \odot a^m \). Then by Lemma 3.3 and Lemma 3.1, we have

\[
a^n - a^{n+1} \odot r = (a^k - a^k \odot s)w(a^m - t \odot a^m) \in A
\]
for some \( r, s, t \in R \) commuting with \( a \). By Lemma 3.2, \( a^n - a^{n+k+m} \odot r \in A \).

Let \( a^n - a^{n+k+m} \odot r = a^k \odot b \odot a^m - a^k \odot c \odot a^m \). Then we have that

\[
a^n = a^{n+k+m} \odot r + a^{k} \odot b \odot a^m - a^{k} \odot c \odot a^m = a^k \odot a^n \odot r + b - c \odot a^m,
\]
as desired. \( \square \)

Theorem 3.5. For a non-negative integer \( n \), if a ring \( R \) is (left, right, completely) \( \pi_n \)-regular, then so is its any GA-semigroup.

Proof. Let \( R^\circ \) be a GA-semigroup of \( R \). If \( R \) be a right \( \pi_n \)-regular ring for \( n \geq 1 \),
then for any \( x \in R \), there exist \( y \in R \) such that \( (x - x^0)^n = (x - x^0)^{n+1}y \). From Lemma 3.4, we deduce that \( x^n = x^{n+1} \odot z \) for some \( z \in R \), whence \( (R, \odot) \) is a right \( \pi_n \)-regular semigroup. The remainder can be proved similarly. \( \square \)

4. GA-semigroups with idempotents

Let \( R^\circ \) be a GA-semigroup of \( R \). Then \( R^\circ \) is called (centrally) 0-idempotent if the additive 0 of \( R \) is an (central) idempotent in \( R^\circ \). Let \( R^\circ \) be a 0-idempotent GA-semigroup induced by the associated pair \((\theta, \vartheta)\). Then it is clear that \( \vartheta = 0 \) and so \( \theta \) is idempotent. One should note that (centrally) 0-idempotent is not an affine isomorphism invariant.

Lemma 4.1. Every GA-semigroup containing (central) idempotents is affinely isomorphic to a (centrally) 0-idempotent one.

Proof. Suppose \( R^\circ \) is a GA-semigroup containing an (central) idempotent \( e \). Let \( R_e = (R, \boxplus, \ast) \) with

\[
x \boxplus y = x + y - e,
\]

\[
x \ast y = (x - e)(y - e) + e,
\]
for any \( x, y \in R \). Then \( R_e \) is a ring in which \( e \) acts as additive zero and \( \ast \) is clearly an associative binary operation on \( R_e \). Denote by \( \boxminus \) the minus in \( R_e \). Noting that
Then the generalized adjoint multiplication induced by bimodules in a natural way. Let \( \tilde{\varphi} \) be a \( \varphi \)-self-permutable bitranslation. Given two rings \( R \) the usual matrix operations and \( \Psi : V \otimes S U \rightarrow T \) (write \( uv \) for \( \Phi(u \otimes v) \) and \( vu \) for \( \Psi(v \otimes u) \)) such that \( u(vu') = (uv)u' \) and \( v(wv') = (vw)v' \) for any \( u, u', v, v' \in U \) and \( v, v' \in V \). Let \( R = \begin{pmatrix} S & U \\ V & T \end{pmatrix} \) be the set of formal matrices. Thus \( \diamond \) is a GA-multiplication on the ring \( R_e \) such that \( R_e^\circ \) is (centrally) 0-idempotent. It is easy to see that the identity mapping of \( R \) is an affine isomorphism from \( R^\circ \) onto \( R_e^\circ \).

Given two rings \( S \) and \( T \), two bimodules \( sU_T \) and \( T V_S \), an \( S-S \)-homomorphism \( \phi : U \otimes_T V \rightarrow S \) and a \( T-T \)-homomorphism \( \psi : V \otimes_S U \rightarrow T \) (write \( uv \) for \( \phi(u \otimes v) \) and \( vu \) for \( \psi(v \otimes u) \)) such that \( u(vu') = (uv)u' \) and \( v(wv') = (vw)v' \) for any \( u, u', v, v' \in U \) and \( v, v' \in V \). Let \( R = \begin{pmatrix} S & U \\ V & T \end{pmatrix} \). Then \( R \) is a ring with the usual matrix operations, called the ring of the Morita context, or a Morita ring, and denoted by \( \mathcal{M}(S,T,U,V) \). Denote by \( \tilde{S} \) and \( \tilde{T} \) the Dorroh extension of \( S \) and \( T \), respectively. Then \( sU_T \) and \( T V_S \) are unitary bimodules in a natural way. Let \( \tilde{R} = \begin{pmatrix} \tilde{S} & U \\ V & \tilde{T} \end{pmatrix} \). Then \( \tilde{R} \) is a unitary ring with the usual matrix operations and \( R \) is an ideal of \( \tilde{R} \). Let \( E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \tilde{R} \). Then the generalized adjoint multiplication induced by \( E_{11} \) is given by

\[
A \diamond B = AB + AE_{11} + E_{11}B \\
= (A + E_{11})(B + E_{11}) - E_{11} \\
= \begin{pmatrix} s \circ s' + uv' & (1 + s)u + ut' \\ u(1 + s') + tv' & uu' + tt' \end{pmatrix}
\]

for any \( A = \begin{pmatrix} s & u \\ v & t \end{pmatrix}, B = \begin{pmatrix} s' & u' \\ v' & t' \end{pmatrix} \in R \). The semigroup \( R^\circ \) is called the \( E_{11} \)-GA-semigroup of \( R \), denoted by \( \mathcal{M}_{11}(S,T,U,V) \). It is clear that the \( E_{11} \)-GA-semigroup \( \mathcal{M}_{11}(S,T,U,V) \) is 0-idempotent.

**Lemma 4.2.** Let \( R^\circ \) be a 0-idempotent GA-semigroup induced by an idempotent self-permutable bitranslation \( \theta \), and let \( R_{11} = \theta R \theta, R_{00} = \theta R(1 - \theta), R_{01} = (1 - \theta)R \theta, \) and \( R_{00} = (1 - \theta)R(1 - \theta) \). Then

(i) \( R = R_{11} \oplus R_{10} \oplus R_{01} \oplus R_{00} \) as additive groups;

(ii) \( R_{ij}R_{kl} \subset \delta_{jk}R_{ij}, \) where \( \delta_{jk} \) is the Kronecker delta, \( i, j, k, l = 0, 1; \)

(iii) if we write \( x = \sum x_{ij}, y = \sum y_{ij}, \) where \( x_{ij}, y_{ij} \in R_{ij}, i, j = 0, 1, \) then

\[
x \diamond y = (x_{11} \circ y_{11} + x_{10}y_{01}) + (x_{10} + x_{11}x_{10} + x_{10}y_{00}) \\
+ (x_{01} + x_{01}y_{11} + x_{00}y_{01}) + (x_{01}y_{10} + x_{00}y_{00});
\]
(iv) \( R_{ij} \), \( i, j = 0, 1 \), are subrings of \( R \) such that \( R_{11}^\circ = R_{11}^\circ \), \( R_{00}^\circ = R_{00}^\circ \), \( R_{10}^\circ \) is a right zero semigroup, and \( R_{01}^\circ \) is a left zero semigroup.

**Proof.** Since \( \theta \) is idempotent, the proof of (i) and (ii) is essentially similar to that of Pierce decomposition of a ring. For \( x = \sum x_{ij}, y = \sum y_{ij} \), where \( x_{ij}, y_{ij} \in R_{ij}, i,j = 0, 1 \), we have by (ii) that

\[
x \circ y = \left( \sum x_{ij} \right) \left( \sum y_{ij} \right) + \theta \left( \sum x_{ij} \right) + \left( \sum y_{ij} \right) \theta
\]

\[
= \left( \sum x_{ij} y_{kl} \right) + x_{11} + x_{10} + y_{11} + y_{01}
\]

\[
= (x_{11} \circ y_{11} + x_{10} y_{01}) + (y_{10} + x_{11} y_{10} + x_{10} y_{00})
\]

\[
+ (x_{01} + x_{01} y_{11} + x_{00} y_{01}) + (x_{01} y_{10} + x_{00} y_{00})
\]

proving (iii). If \( x, y \in R_{11} \), then

\[
x \circ y = xy + x\theta + \theta y = xy + x + y = x \circ y,
\]

whence \( R_{11}^\circ = R_{11}^\circ \), and similarly, \( R_{00}^\circ = R_{00}^\circ \). For any \( x, y \in R_{10} \), we have by (ii) that

\[
x \circ y = xy + x\theta + \theta y = y,
\]

which implies that \( R_{10}^\circ \) is a right zero semigroup, and similarly \( R_{01}^\circ \) is a left zero semigroup, proving (iv). \( \square \)

**Theorem 4.3.** Let \( R^\circ \) be a GA-semigroup of \( R \). If \( R^\circ \) contains idempotents, then there exists a Morita ring \( \mathcal{M}(S,T,U,V) \) such that \( R \simeq \mathcal{M}(S,T,U,V) \) and \( R^\circ \simeq \mathcal{M}^\circ_{11}(S,T,U,V) \).

**Proof.** Let \( R^\circ \) be a GA-semigroup induced by the associated pair \((\theta, \vartheta)\). If \( R^\circ \) contains idempotents, then by Lemma 4.1, without loss of generality, we may assume that \( R^\circ \) is 0-idempotent. By Lemma 4.2, it is a routine matter to verify that \( \mathcal{M}(R_{11}, R_{00}, R_{10}, R_{01}) \) is a Morita ring in a natural way. By Lemma 4.2 straightforward computation shows that the mapping \( \phi : R \rightarrow \mathcal{M}(R_{11}, R_{00}, R_{10}, R_{01}) \) defined by

\[
\phi(x) = \begin{pmatrix}
\theta x \theta \\
(1-\theta) x \theta \\
(1-\theta) x (1-\theta)
\end{pmatrix}
\]

is a ring isomorphism. Noting that

\[
\phi(x \circ y) = \phi(xy + x\theta + \theta y)
\]

\[
= \phi(x) \phi(y) + \phi(x\theta) + \phi(\theta y)
\]

\[
= \phi(x) \phi(y) + \begin{pmatrix}
\theta x \theta \\
(1-\theta) x \theta \\
0
\end{pmatrix} + \begin{pmatrix}
\theta y \theta \\
(1-\theta) y \theta \\
0
\end{pmatrix}
\]

\[
= \phi(x) \circ \phi(y),
\]

we see that \( \phi \) is an affine isomorphism from \( R^\circ \) onto the \( E_{11} \)-GA-semigroup of \( \mathcal{M}(R_{11}, R_{00}, R_{10}, R_{01}) \). \( \square \)
Corollary 4.4. A GA-semigroup $R^e$ is (centrally) 0-idempotent if and only if there exists an ideal extension $R$ with $I$ of $R$ and an idempotent $\varepsilon \in R$ (commuting with elements of $R$) such that $x \circ y = (x + \varepsilon)(y + \varepsilon) - \varepsilon$ for any $x, y \in R$.

Proof. It follows from Theorem 4.3, the definition of the $E_{11}$-GA-semigroup and taking $\varepsilon = E_{11}$. \hfill $\square$

Lemma 4.5. If $(a - a^{[2]})^2 = 0$, then there exists an idempotent $e = \sum p_i a^{[i]}$ with $\sum p_i = 1$ such that $a^{[2]} = e \circ a^{[2]}$.

Proof. By Corollary 2.4, $(a - a^{[2]})^2 = a^{[2]} - 2a^{[3]} + a^{[4]}$, and so

$$a^{[2]} = 2a^{[3]} - a^{[4]} = a^{[2]} \circ (2a - a^{[2]}) = a^{[2]} \circ (2a - a^{[2]})^{[2]} = a^{[2]} \circ (2a - a^{[2]})^{[3]}.$$ 

Note that by Corollary 2.8,

$$(2a - a^{[2]})^{[3]} = 8a^{[3]} - 12a^{[4]} + 6a^{[5]} - a^{[6]} = a^{[2]} \circ (8a - 12a^{[2]} + 6a^{[3]} - a^{[4]}).$$

Let $b = 8a - 12a^{[2]} + 6a^{[3]} - a^{[4]}$. Then $b$ commutes with $a$ and $a^{[2]} = a^{[2]} \circ b \circ a^{[2]}$. Let $e = a^{[2]} \circ b$. Then it is clear that $e$ is an idempotent of $R^e$ such that $a^{[2]} = e \circ a^{[2]}$.

Let $\Gamma(R) = \{ \theta \in \Omega(R) | \theta x = x\theta \text{ for any } x \in R \}$.

Lemma 4.6. A GA-semigroup of $R$ induced by $(\theta, \vartheta)$ has (central) idempotents if and only if $\theta$ can be lifted to an idempotent of $\Omega(R)$ (contained in $\Gamma(R)$).

Proof. Assume semigroup $R^e$ has an idempotent $e$. Then

$$e = e \circ e = e^2 + e\theta + \theta e + \vartheta,$$

whence $\pi_e = \pi_e^2 + \pi_e \theta + \theta \pi_e + \pi_\vartheta = \pi_e^2 + \pi_e \theta + \theta e + \pi_e \theta - \theta = (\pi_e + \theta)^2 - \theta$. Thus $\pi_e + \theta$ is idempotent. Moreover, if $e$ is central in $R^e$, then $e \circ x = x \circ e$ for any $x \in R$, that is, $ex + e\theta + \theta x + \vartheta = xe + x\theta + \theta e + \vartheta$, and particularly, $e\theta = \theta e$ by taking $x = 0$. Thus $(\pi_e + \theta)x = ex + \theta x = xe + x\theta = x(\pi_e + \theta)$, yielding $\pi_e + \theta \in \Gamma(R)$.

Assume $\theta$ can be lifted to an idempotent of $\Omega(R)$. Then $\pi_a + \theta$ is idempotent for some $a \in R$, whence $\pi_a = \pi_a^2 + \pi_a \theta + \theta \pi_a + \theta^2 - \theta = \pi_a^2 + \pi_a \theta + \theta \pi_a + \pi_\vartheta$. Thus we have $ax = a^2x + (a\theta)x + (\theta a)x + \vartheta x = a^{[2]}x$, forcing $(a - a^{[2]})R = 0$. In particular, $(a - a^{[2]})^2 = 0$, whence $R^e$ contains an idempotent $e = \sum p_i a^{[i]}$ with $\sum p_i = 1$ by Lemma 4.5. Further, if $\pi_a + \theta$ is an idempotent contained in $\Gamma(R)$. Then for any $x \in R$, $(\pi_a + \theta)x = x(\pi_a + \theta)$, that is, $ax + \theta x = xa + x\theta$, and particularly $\theta a = a\theta$ by taking $x = a$, whence

$$a \circ x = ax + \theta x + a\theta + \vartheta = xa + x\theta + \theta a + \vartheta = x \circ a.$$ 

Hence $e \circ x = x \circ e$, that is, $e$ is a central idempotent of $R^e$. \hfill $\square$

Theorem 4.7. Consider the following conditions:

(i) every GA-semigroup of $R$ contains (central) idempotents;
(ii) in any ideal extension \( \hat{R} \) of \( R \), idempotents of \( \hat{R}/R \) can be lifted to idempotents of \( \hat{R} \) (contained in the centralizer of \( R \) in \( \hat{R} \));

(iii) idempotents of \( \Omega(R)/\pi(R) \) can be lifted to idempotents of \( \Omega(R) \) (contained in \( \Gamma(R) \)). Then (iii) \( \Rightarrow \) (i) \( \Rightarrow \) (ii). Moreover, if \( \text{Ann}(R) = 0 \), then (i), (ii) and (iii) are equivalent.

Proof. (iii) \( \Rightarrow \) (i) follows from Lemma 4.6.

(i) \( \Rightarrow \) (ii): If \( a \in \hat{R} \) and \( a^2 - a \in R \), then the pair \((\theta, \vartheta)\) defined by
\[
    \theta x = ax, \quad x\theta = xa, \quad \text{and} \quad \vartheta = a^2 - a
\]
is an associated pair and so \( x \circ y = xy + xa + ay + a^2 - a \) defines a GA-multiplication on \( R \). If \( e \) is an idempotent of \( R^e \), then \( e = e^2 + ea + ae + a^2 - a = (e + a)^2 - a \), and so \( e + a \) is an idempotent of \( \hat{R} \). Further if \( e \) is a central idempotent of \( R^e \), then \( e \circ x = x \circ e \) for any \( x \in \hat{R} \), that is
\[
e x + ea + ax + \vartheta = xe + xa + ae + \vartheta,
\]
and particularly, \( ea = ae \) by taking \( x = 0 \). Thus \( (e + a)x = ex + ax = xe + xa = x(e + a) \), which implies that \( e + a \) is contained in the centralizer of \( R \) in \( \hat{R} \).

The remainder is clear. \( \square \)

The following corollary is independently interesting, which is a generalization of a classical result in ring theory which states that idempotents modulo a nil ideal can be lifted ([28]) and is a generalization of ring-theoretic analogue of a result of Edwards ([19, Corollary 2]) which extends the well-known Lallement’s lemma to eventually regular semigroups (i.e., \( \pi \)-regular semigroups).

**Theorem 4.8.** In any ring, idempotents modulo a \( \pi \)-regular ideal can be lifted.

**Proof.** By Theorem 3.5, any GA-semigroup of a \( \pi \)-regular ring contains idempotent, and so by Theorem 4.7 idempotents modulo a \( \pi \)-regular ideal can be lifted. \( \square \)

If \( R \) is a ring with \( ECI \), then idempotents can be lifted from \( \Omega(R)/R \) to \( \Omega(R) \) ([7, Corollary 3.6]), and so any GA-semigroup of \( R \) contains idempotents by Theorem 4.7. Particularly, every GA-semigroup of a biregular ring contains idempotents. On the other hand, there is a ring such that idempotents modulo the radical cannot be lifted. Hence a GA-semigroup of a radical ring need not contain idempotents.

A semigroup \( S \) is called completely primitive if the left ideal \( Se \) and the right ideal \( eS \) are minimal for every idempotent \( e \) of \( S \) ([6]). A completely primitive semigroup \( S \) has kernel which is completely simple and contains all of idempotents of \( S \) ([9]).

**Lemma 4.9.** Let \( R^e \) be a GA-semigroup of a radical ring \( R \). If \( R^e \) contains idempotents, then \( R^e \) is completely primitive.
Proof. Let $e$ be an idempotent of $R^e$. Then it is sufficient to prove that $e \circ R \circ e$ is a group. Since $e \circ R \circ e \simeq (e \circ R \circ e - e \circ R \circ e, o)$ by Lemma 2.11 and Lemma 2.13, we have to prove that $e \circ R \circ e - e \circ R \circ e$ is a radical ring. By Corollary 4.4, there are an ideal extension $\tilde{R}$ of $R$ and an idempotent $\varepsilon \in \tilde{R}$ such that $x \circ y = (x + \varepsilon)(y + \varepsilon) - \varepsilon$ for any $x, y \in R$. Thus $e \circ R \circ e - e \circ R \circ e = (e + \varepsilon)(R + \varepsilon)(e + \varepsilon) - (e + \varepsilon)(R + \varepsilon)(e + \varepsilon) = (e + \varepsilon)R(e + \varepsilon)$. Since $e \circ e = e$, we have that $e + \varepsilon$ is an idempotent of $\tilde{R}$ and so it is easy to see that $(e + \varepsilon)R(e + \varepsilon)$ is a radical ring since $R$ is a radical ring. □

Lemma 4.9 is a GA-semigroup version of [18, Theorem 1 (b)–(c)]. Actually, many results in [18] can be reexplained in terms of GA-semigroup.

**Theorem 4.10.** Any GA-semigroup of a nil ring is a completely primitive $\pi$-regular semigroup.

**Proof.** It follows from Theorem 3.5 and Lemma 4.9. □

**Theorem 4.11.** Let $R$ be a ring with descending chain condition for principal right ideals. Then any GA-semigroup of $R$ is completely $\pi$-regular. Particularly, any GA-semigroup of a right Artinian ring is completely $\pi$-regular.

**Proof.** If $R$ is a ring with descending chain condition for principal right ideals, then $R$ is completely $\pi$-regular by Dischinger [12, Theorem 1] and Azumaya [2, Lemma 1]. □

**References**


Received November 5, 2005