On Groups Satisfying
\[|G'| > [G : Z(G)]^{1/2}\]

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Abstract. We study conditions under which the commutator subgroup
\(G'\) of a finite group \(G\) satisfies the inequality \(|G'| > [G : Z(G)]^{1/2}\).
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1. Introduction

All groups in this paper are finite. We use the standard notation \(Z(G)\), \(\Phi(G)\)
for the centre and the Frattini subgroup of \(G\). We study conditions under which
the commutator subgroup \(G'\) of a group \(G\) satisfies \(|G'| > [G : Z(G)]^{1/2}\). As is
shown in Example A3 of [2] (see also [1]), the inequality \(|G'| > [G : Z(G)]^{1/2}\)
does not hold for all non-abelian groups, and not even for all solvable non-abelian
groups. The groups in that example have order \(2p^\alpha\), where \(p\) is an odd prime, and
their Sylow \(p\)-subgroup is metabelian. Hence, a natural question is whether the
inequality above holds when all the Sylow subgroups of \(G\) are abelian. We prove
the following theorem, which gives an affirmative answer to that question.

Theorem A. Let \(G\) be a non-abelian group with all Sylow subgroups abelian. Then
\(|G'| > [G : Z(G)]^{1/2}\).

Our next result, which extends Corollary B1 in [2], provides another type of
condition ensuring \(|G'| > [G : Z(G)]^{1/2}\).

Theorem B. Let \(G\) be a non-abelian group such that \(G/G'\) is cyclic. Then \(|G'| >
[G : Z(G)]^{1/2}\).

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As has been already mentioned, Theorem A cannot be extended to groups having a non-abelian Sylow subgroup. However, assuming solvability and \( \Phi(G) = 1 \), the following was obtained in [2], Corollary A1:

**Theorem 1.** Let \( G \neq 1 \) be a solvable group such that \( Z(G) = \Phi(G) = 1 \). Then \(|G'| > |G|^{1/3}\).

It was conjectured\(^1\) in [2] that the inequality of Theorem 1 may be strengthened to \(|G'| > |G|^{1/2}\). Our Theorems A and B provide this stronger inequality for certain families of groups, without the assumptions that \( G \) is solvable and \( \Phi(G) = 1 \) (for another type of results on “large” commutator subgroups, see [5]).

We note that the family of Frobenius groups of order \( p(p-1) \), where \( p \) is a prime, shows that the inequality in Theorems A and B cannot be strengthened.

2. Proofs

**Lemma 1.**

(i) Let \( G \) be a group and let \( N \trianglelefteq G \) satisfy \( N \cap G' = 1 \) (notice that \( N \) must be central in \( G \)). Then \( Z(G/N) = Z(G)/N \).

(ii) Let \( G \) be a group with all Sylow subgroups abelian, and let \( N \leq Z(G) \). Then \( Z(G/N) = Z(G)/N \). In particular, \( Z(G/Z(G)) = 1 \).

**Proof.** (i) Let \( gN \in Z(G/N) \). Then \([g, x] \in G' \cap N = 1\) for all \( x \in G \). Hence \( g \in Z(G) \) and the result follows.

(ii) This follows by (i), since a group with all Sylow subgroups abelian satisfies \( Z(G) \cap G' = 1 \) ([7], Ex. 549(i)). \(\square\)

We are ready now for the proofs of Theorems A and B.

**Proof of Theorem A.** We prove by induction on \(|G|\). Suppose first that \( G > Z(G) > 1 \), and notice that \( G' \cap Z(G) = 1 \) ([7], Ex. 549(i)) and \( Z(G/Z(G)) = 1 \) by Lemma 1. Thus, by induction applied on the group \( G/Z(G) \), we have \(|G'| = [G'Z(G) : Z(G)] = [(G/Z(G))'] > |G/Z(G)|^{1/2} \) as required. Hence, we suppose from now on that \( Z(G) = 1 \). We must show that \(|G'| > |G|^{1/2}\).

Assume first that \( F := \text{Fit}(G) > 1 \) (where \( \text{Fit}(G) \) denotes the Fitting subgroup of \( G \)). For any prime divisor \( p \) of \(|G|\) denote \( O_p = O_p(G) \) and \( A_p = G/C_{G}(O_p) \). Then \( A_p \) is a \( p' \)-group since the Sylow \( p \)-subgroups of \( G \) are abelian. Considering the action of \( A_p \) on the abelian group \( O_p \) we have, by a well-known result on coprime actions ([6], 8.4.2), the decomposition \( O_p = [G, O_p] \times (O_p \cap Z(G)) \). Since \( Z(G) = 1 \) we obtain by these decompositions that

\[ F \leq G'. \]  

(1)

Put \( K/F = Z(G/F) \), then the group \( K \) is nilpotent by abelian, and it follows by [2], Theorem D, that \(|F| = |\text{Fit}(K)| > |K|^{1/2} \). Thus, the proof is completed.

\(^1\)In the meantime, this conjecture was proved in [3]
in this case by (1) if \( K = G \). We may assume then that \( G > K \). Note that 
\( Z(G/K) = 1 \) by Lemma 1, since \( G/K \) is isomorphic to \( (G/F)/(Z(G/F)) \). Hence 
by induction \(|G'K/K| = |(G/K)'| > |G/K|^{1/2}|, thus 
\[
[G': G' \cap K] > |G/K|^{1/2}.
\] (2)
By (1) we have \(|G' \cap K| \geq |F| > |K|^{1/2}|. Taking this in account together with (2) 
we obtain \(|G'| > |G|^{1/2}| as required.

It remains to consider the case \( F = \text{Fit}(G) = 1 \). Suppose, by contradiction, 
that the claim does not hold. Then \(|G'| \leq |G|^{1/2}|. By [7], Ex. 622, there exists 
a nilpotent subgroup \( B < G \) such that \( G = G'B \). As \( B \) is a nilpotent group 
with all Sylow subgroups abelian, \( B \) is abelian. Hence, by a result of Zenkov ([8], 
Theorem 1), there exists \( g \in G \) such that \( B \cap B^g \leq \text{Fit}(G) \). Now \( BB^g \neq G \) 
since a group is not a product of two conjugates of a proper subgroup. Since 
\(|B| \geq |G|^{1/2}|, it follows that \( B \cap B^g > 1 \), implying \( \text{Fit}(G) > 1 \), which is the 
desired contradiction. \( \square \)

**Proof of Theorem B.** The case \( Z(G) = 1 \) was proved in [2], Corollary B1. We 
assume then that \( G > Z(G) > 1 \) and apply induction on \(|G|\).

Case (i). \( G' \cap Z(G) = 1 \). Then \( Z(G/Z(G)) = 1 \) by Lemma 1, and \((G/Z(G))/\text{Fit}(G) \) is cyclic. Thus by induction \(|G'| = |G'Z(G/Z(G))| = |(G/Z(G))'| > |G/Z(G)|^{1/2}|, and the proof is completed.

Case (ii). \( N := G' \cap Z(G) > 1 \). Then \((G/N)/(G/N)'\) is cyclic. Hence, if \( G/N \) is abelian then it is also cyclic, and so \( G/Z(G) \) is cyclic. This implies ([7], Ex. 125) 
that \( G \) is abelian, a contradiction. Thus \( G/N \) is non-abelian and by induction 
\[
|G'/N| = |(G/N)'| > |(G/N) : Z(G/N)|^{1/2}.
\] (3)

Put \( L/N = Z(G/N) \), then \( L \geq Z(G) \), and by (3) we have \(|G' : N| > |G : L|^{1/2}|, 
implying \(|G'| > |G|^{1/2}|/[L : N]^{1/2}|. The proof will be completed by showing \([L : N] \leq |Z(G)|\). We should only consider the case \( L > Z(G) \).

Choose \( g \in L \). Since \([G, L] \leq N \leq Z(G) \), the function \( x \mapsto [g, x] \) is a 
homomorphism from \( G \) to \( N \). We denote this homomorphism by \( \theta_g \). Given 
g, h \in L, we note that \( \theta_g \neq \theta_h \) if and only if \( gZ(G) \neq hZ(G) \). Thus \(|\{g \in L \mid [g, x] = [L : Z(G)]\} = [L : Z(G)]\). For any \( g \in L \), the centralizer \( C_G(g) \) is a normal subgroup of 
\( G \), since it is equal to \( C_G((g)Z(G)) \) and \( (g)Z(G) \leq G \). By [4], Corollary 2.1, since 
\( G/G' \) is cyclic, the group \( G \) is generated by one of its conjugacy classes. Hence 
\( G \) is not a union of proper normal subgroups. Consequently, there exists \( u \in G \) 
which does not centralize any element of \( L - Z(G) \). Thus for any \( g, h \in L \) with 
gZ(G) \neq hZ(G) we have \([g, u] \neq [h, u] \), i.e. \( \theta_g(u) \neq \theta_h(u) \). Since \( \theta_k(u) \in N \) for 
all \( k \in L \), we obtain \([L : Z(G)] \leq |N|\), implying \([L : N] \leq |Z(G)|\) as required. \( \square \)

**References**

G. Kaplan; A. Lev: On Groups Satisfying $|G| > |G : Z(G)|^{1/2}$


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