Variations of (para-)Hodge Structures and their Period Maps in \(^{tt*}\)-geometry

Lars Schäfer

\(D-20146\) Hamburg, Germany
e-mail: schaefer@math.uni-hamburg.de

Abstract. We introduce the notion of variations of Hodge structures (VHS) in para-complex geometry and define the associated period map. Moreover, we construct VHS from special (para-)complex and (para-)Kähler manifolds and prove that they provide solutions of (metric) \(^{tt*}\)-bundles (cf. [3] for the complex case). In the case of odd weight we relate the period map to the (para-)pluriharmonic maps associated to \(^{tt*}\)-bundles (cf. [18], [19]).

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1. Introduction

In complex geometry it is known that (metric) \(^{tt*}\)-bundles provide a generalization of variations Hodge structures (cf. [3]). Moreover one [18, D] can associate to any metric \(^{tt*}\)-bundle \((E, D, S, g)\) a pluriharmonic map into \(GL(r, \mathbb{R})/O(p, q)\) where \((p, q)\) with \(r = p + q\) is the signature of the metric \(g\). In this paper we relate for a variation of Hodge structures of odd weight this pluriharmonic map to the period map of the variation of Hodge structures.

More recently the author [19] introduced the para-complex notion of \(^{tt*}\)-bundles. Examples of such structures on the tangent bundle of a special para-Kähler manifold were given in the same reference. In the complex setting special Kähler manifolds carry a polarized variation of Hodge structures of weight one.
This is one way to see that they provide $tt^*$-structures. The described information suggest to study the question if one can generalize VHS and their period maps to para-complex geometry, if the tangent bundle of special para-Kähler manifold carries such VHS, if these VHS provide para-$tt^*$-bundles and if one can identify the related para-pluriharmonic maps. This program is carried out in this paper.

2. Para-complex differential geometry

We shortly recall some notions and facts of para-complex differential geometry. For a more complete source we refer to [7].

In para-complex geometry one replaces the complex structure $J$ with $J^2 = -\mathcal{I}$ (on a finite dimensional vector space $V$) by the para-complex structure $\tau \in \operatorname{End}(V)$ satisfying $\tau^2 = \mathcal{I}$ and one requires that the $\pm 1$-eigenspaces have the same dimension. An almost para-complex structure on a smooth manifold $M$ is an endomorphism-field $\tau$, which is a point-wise para-complex structure. If the eigen-distributions $T^\pm M$ are integrable $\tau$ is called para-complex structure on $M$ and $M$ is called a para-complex manifold. As in the complex case, there exists a tensor, also called Nijenhuis tensor, which is the obstruction to the integrability of the para-complex structure.

The real algebra, which is generated by $1$ and by the para-complex unit $e$ with $e^2 = 1$, is called the para-complex numbers and denoted by $C$. For all $z = x + ey \in C$ with $x, y \in \mathbb{R}$ we define the para-complex conjugation as $\bar{z} : C \to C, x + ey \mapsto x - ey$ and the real and imaginary parts of $z$ by $\Re(z) := x, \Im(z) := y$. The free $C$-module $C^n$ is a para-complex vector space where its para-complex structure is just the multiplication with $e$ and the para-complex conjugation of $C$ extends to $\bar{z} : C^n \to C^n, v \mapsto \bar{v}$.

Note, that $z\bar{z} = x^2 - y^2$. Therefore the algebra $C$ is sometimes called the hypercomplex numbers. The circle $S^1 = \{z = x + iy \in \mathbb{C} | x^2 + y^2 = 1\}$ is replaced by the four hyperbola $\{z = x + ey \in C | x^2 - y^2 = \pm 1\}$. We define $S^1$ to be the hyperbola given by the one parameter group $\{z(\theta) = \cosh(\theta) + e\sinh(\theta) | \theta \in \mathbb{R}\}$.

A para-complex vector space $(V, \tau)$ endowed with a pseudo-Euclidean metric $g$ is called para-hermitian vector space, if $g$ is $\tau$-anti-invariant, i.e. $\tau^*g = -g$. The para-unitary group of $V$ is defined as the group of automorphisms

$$U^n(V) := \operatorname{Aut}(V, \tau, g) := \{L \in GL(V) | [L, \tau] = 0 \text{ and } L^*g = g\}$$

and its Lie-algebra is denoted by $u^n(V)$. For $C^n = \mathbb{R}^n \oplus e\mathbb{R}^n$ the standard para-hermitian structure is defined by the above para-complex structure and the metric $g = \text{diag}(\mathcal{I}, -\mathcal{I})$ (cf. Example 7 of [7]). The corresponding para-unitary group is given by (cf. Proposition 4 of [7]):

$$U^n(C^n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} : A, B \in \operatorname{End}(\mathbb{R}^n), A^T A - B^T B = \mathcal{I}_n, A^T B - B^T A = 0 \right\}.$$

(2.1)

There exist two bi-gradings on the exterior algebra: The one is induced by the splitting in $T^{\pm}M$ and denoted by $\Lambda^k T^*M = \bigoplus_{k=p+q} \Lambda^{p+q} T^* M$ and induces an
obvious bi-grading on exterior forms with values in a vector bundle $E$. The second
is induced by the decomposition of the para-complexified tangent bundle $TM^C = TM \otimes_{\mathbb{R}} C$ into the subbundles $T_p^{1,0}M$ and $T_p^{0,1}M$ which are defined as the $\pm \varepsilon$-eigenbundles of the para-complex linear extension of $\tau$. This induces a bi-grading on the $C$-valued exterior forms noted $\Lambda^k T^*M^C = \bigoplus_{k=p+q} \Lambda^{p,q} T^*M$ and finally on the $C$-valued differential forms on $M$ $\Omega^k(M) = \bigoplus_{k=p+q} \Omega^{p,q}(M)$. In the case $(1,1)$ and $(1+,1-)$ the two gradings induced by $\tau$ coincide, in the sense that $\Lambda^{1,1} T^*M = (\Lambda^{1+,1-} T^*M) \otimes C$. The bundles $\Lambda^{p,q} T^*M$ are para-complex vector bundles in the following sense: A para-complex vector bundle of rank $r$ over a para-complex manifold $(M, \tau)$ is a smooth real vector bundle $\pi : E \to M$ of rank $2r$ endowed with a fiber-wise para-complex structure $\tau^E \in \Gamma(\text{End}(E))$. We denote it by $(E, \tau^E)$. In the following text we always identify the fibers of a para-complex vector bundle $E$ of rank $r$ with the free $C$-module $C^r$. One has a notion of para-holomorphic vector bundles [16], too. In Proposition 2 of the same reference we have shown, that a para-complex connection with vanishing $(0,2)$-curvature on a para-complex vector bundle $E$ induces a para-holomorphic structure on $E$. This generalizes a well-known theorem of complex geometry.

Let us transfer some notions of hermitian linear algebra (cf. [21]): A para-hermitian sesquilinear scalar product is a non-degenerate sesquilinear form $h : C^r \times C^r \to C$, i.e. it satisfies (i) $h$ is non-degenerate: Given $w \in C^r$ such that for all $v \in C^r$ $h(v, w) = 0$, then it follows $w = 0$, (ii) $h(v, w) = \overline{h(w, v)}$, $\forall v, w \in C^r$, and (iii) $h(\lambda v, w) = \lambda h(v, w)$, $\forall \lambda \in C$; $v, w \in C^r$. The standard para-hermitian sesquilinear scalar product is given by

$$(z, w)_{C^r} := z \cdot \overline{w} = \sum_{i=1}^r z_i \overline{w}_i, \quad \text{for } z = (z_1, \ldots, z^r), w = (w^1, \ldots, w^r) \in C^r.$$

The para-hermitian conjugation is defined by $C \leftrightarrow C^h = \overline{C^r}$ for $C \in \text{End}(C^r) = \text{End}_C(C^r)$ and $C$ is called para-hermitian if and only if $C^h = C$. We denote by herm$(C^r)$ the set of para-hermitian endomorphisms and by Herm$(C^r) = \text{herm}(C^r) \cap GL(r, C)$. We remark, that there is no notion of para-hermitian signature, since from $h(v, v) = -1$ for an element $v \in C^r$ we obtain $h(ev, ev) = 1$.

**Proposition 1.** Given an element $C \in \text{End}(C^r)$ then it holds $(Cz, w)_{C^r} = (z, C^h w)_{C^r}, \forall z, w \in C^r$. The set herm$(C^r)$ is a real vector space. There is a bijective correspondence between Herm$(C^r)$ and para-hermitian sesquilinear scalar products $h$ on $C^r$ given by $H \mapsto h(\cdot, \cdot) := (H^\cdot, \cdot)_{C^r}.$

A para-hermitian metric $h$ on a para-complex vector-bundle $E$ over a para-complex manifold $(M, \tau)$ is a smooth fiber-wise para-hermitian sesquilinear scalar product.

To unify the complex and the para-complex case we introduce some notations: First we note $J^\varepsilon$ where $J^\varepsilon^2 = \varepsilon^2$ with $\varepsilon \in \{\pm 1\}$. The $\varepsilon$complex unit is denoted by $\hat{\varepsilon}$, i.e. $\hat{\varepsilon} := \varepsilon$, for $\varepsilon = 1$, and $\hat{\varepsilon} = i$, for $\varepsilon = -1$. Further we introduce $\mathbb{C}_\varepsilon$ with $\mathbb{C}_1 = C$ and $\mathbb{C}_{-1} = \mathbb{C}$. In the rest of this work we extend our language by the
following $\epsilon$-notation: If a word has a prefix $\epsilon$ with $\epsilon \in \{\pm1\}$, i.e. is of the form $\epsilon X$, this expression is replaced by

$$\epsilon X := \begin{cases} X, & \text{for } \epsilon = -1, \\ \text{para-}X, & \text{for } \epsilon = 1. \end{cases}$$

The $\epsilon$-unitary group and its Lie-algebra are

$$U^\epsilon(p, q) := \begin{cases} U^\pi(C^r), & \text{for } \epsilon = 1, \\ U(p, q), & \text{for } \epsilon = -1 \end{cases}$$

and $u^\epsilon(p, q) := \begin{cases} u^\pi(C^r), & \text{for } \epsilon = 1, \\ u(p, q), & \text{for } \epsilon = -1. \end{cases}$

where in the complex case $(p, q)$ for $r = p + q$ is the hermitian signature.

Further we use the notation

$$\text{Herm}^\epsilon_{p,q}(\mathbb{C}_\epsilon^r) := \begin{cases} \text{Herm}(C^r), & \text{for } \epsilon = 1, \\ \text{Herm}_{p,q}(C^r), & \text{for } \epsilon = -1, \end{cases}$$

and

$$\text{herm}^\epsilon_{p,q}(\mathbb{C}_\epsilon^r) := \begin{cases} \text{herm}(C^r), & \text{for } \epsilon = 1, \\ \text{herm}_{p,q}(C^r), & \text{for } \epsilon = -1, \end{cases}$$

where, for $p + q = r$, $\text{Herm}_{p,q}(C^r)$ are the hermitian matrices of hermitian signature $(p, q)$ and $\text{herm}_{p,q}(C^r)$ are the hermitian matrices with respect to the standard hermitian product of hermitian signature $(p, q)$ on $C^r$. The standard hermitian sesquilinear scalar product is $(z, w)_{\mathbb{C}_\epsilon^r} := z \cdot \bar{w} = \sum_{i=1}^{r} z^i \bar{w}^i$, for $z = (z^1, \ldots, z^r), w = (w^1, \ldots, w^r) \in \mathbb{C}_\epsilon^r$ and we note

$$\cos_\epsilon(x) := \begin{cases} \cos(x), & \text{for } \epsilon = -1, \\ \cosh(x), & \text{for } \epsilon = 1 \end{cases} \quad \text{and} \quad \sin_\epsilon(x) := \begin{cases} \sin(x), & \text{for } \epsilon = -1, \\ \sinh(x), & \text{for } \epsilon = 1. \end{cases}$$

3. Variations of $\epsilon$Hodge structures

3.1. $\epsilon$Hodge structures and their variations

In this section we introduce the notion of variations of $\epsilon$Hodge structures in para-complex geometry and recall variations of Hodge structures which are classical objects in complex geometry. We follow the notations of [2] which is a reference and contains references for further study of variations of Hodge structures. The para-complex version seems to be new.

Definition 1.

(a) A real $\epsilon$Hodge structure of weight $w \in \mathbb{N}$ is a real vector space $H$ on the $\epsilon$complexification of which there is a decomposition into $\epsilon$complex vector spaces

$$H^\epsilon^C = \bigoplus_{w=p+q} H^{p,q} \text{ with } p, q \in \mathbb{N}$$

and where

$$H^{p,q} = H^{q,p} \text{ with } p, q \in \mathbb{N}. \quad (3.2)$$

The $\epsilon$complex conjugation $\bar{\cdot}$ is relative to the real structure on $H^\epsilon^C = H \otimes \mathbb{C}_\epsilon$. \(\square\)
Suppose, that an $\epsilon$Hodge structure of weight $w$ carries a bilinear form $b : H \times H \to \mathbb{R}$ which satisfies the following Riemannian bilinear relations

(i) The $\mathbb{C}_\epsilon$-linear extension of the bilinear form $b$, also denoted by $b$, satisfies $b(x, y) = 0$ if $x \in H^{p,q}$ and $y \in H^{r,s}$ for $(r, s) \neq (w - p, w - q) = (q, p)$.

(ii) The bilinear form $b$ defines an $\epsilon$hermitian sesquilinear scalar product (compare Section 2) on $H^{p,q}$ by $h(x, y) = (-1)^{(w-1)/2}b(x, \bar{y})$.

Then we call this $\epsilon$Hodge structure weakly polarized.

(c) Suppose, that a (complex) Hodge structure of weight $w$ carries a bilinear form $b : H \times H \to \mathbb{R}$ which satisfies the first Riemannian bilinear relation (i) and in addition

(ii) The bilinear form $b$ defines a positive definite hermitian sesquilinear form on $H^{p,q}$ by $h(x, y) = (-1)^{(w-1)/2}i^{q-p}b(x, \bar{y})$.

Then we call this Hodge structure strongly polarized.

(d) An $\epsilon$Hodge structure of weight $w$ is called polarized if it is weakly polarized or strongly polarized.

Closely related to the $\epsilon$Hodge decomposition is the following filtration

$$F^p = \bigoplus_{a \geq p} H^{a,b}, \quad p = 0, \ldots, w, \quad (3.3)$$

which satisfies for an $\epsilon$Hodge structure of weight $w$ the relation

$$H^{C_\epsilon} = F^p \oplus \overline{F^{w-p+1}}, \quad p = 1, \ldots, w. \quad (3.4)$$

Any filtration which obeys equation (3.4) is called an $\epsilon$Hodge filtration. Such as an $\epsilon$Hodge decomposition induces an $\epsilon$Hodge filtration we obtain from an $\epsilon$Hodge filtration an $\epsilon$Hodge decomposition by $H^{p,q} = F^p \cap \overline{F^q}$, with $p + q = w$. This $\epsilon$Hodge decomposition satisfies the relation (3.3).

We remark further, that the first Riemannian bilinear relation (cf. Definition 1) is equivalent to $(F^p)\perp = F^{w-p+1}$, $p = 1, \ldots, w$, where $\perp$ is taken with respect to the bilinear from $b$. Now we are going to consider deformations of these structures:

**Definition 2.** A (real) variation of $\epsilon$Hodge structures ($\epsilon$VHS) is a triple $(E, \nabla, F^p)$, where $E$ is a real vector bundle over an (connected) $\epsilon$complex base manifold $(M, J^\epsilon)$, $\nabla$ is a flat connection and $F^p$ is a filtration of $E^{C_\epsilon}$, which is a point-wise $\epsilon$Hodge structure satisfying the infinitesimal period relation or the Griffiths transversality

$$\nabla_X F^p \subset F^{p-1}, \quad \forall X \in T^{1,0}M. \quad (3.5)$$

A polarization of a variation of $\epsilon$Hodge structures $(E, \nabla, F^p)$ consists of a non-degenerate bilinear form $b \in \Gamma(E^* \otimes E^*)$ having the following properties

(i) $b$ induces a polarization on each fiber obeying the first and the second bilinear relation,
Remark 1. In complex geometry VHS roughly arise on the Hodge-decomposition of the cohomology of smoothly varying families of Kähler manifolds $X_t$ where $t$ is the parameter of the variation (cf. [2] Chapter 4 for details). To ensure that the Hodge-numbers $h^{p,q}(X_t)$ are constant in $t$ one needs a result, which states that the kernel of a family $D_t$ of elliptic differential operators depends upper-semi-continuously on $t$. Unfortunately this does not generalize to para-complex geometry for the following reason: If we consider an (almost) para-complex manifold $M^{2n}$ endowed with a para-hermitian metric $g$ the metric is forced to have split signature, i.e. signature $(n, n)$. As a consequence the naturally associated differential operators are no longer elliptic and we are not able to use the above cited theory.

A class of $\epsilon$VHS which is related to the special geometry of (Euclidean) supersymmetry is discussed in the next subsection.

### 3.2. $\epsilon$VHS and special $\epsilon$Kähler manifolds

Each fiber of the $\epsilon$complex tangent bundle

$$TM^\epsilon = T^{1,0}M \oplus T^{0,1}M$$

carries a natural $\epsilon$Hodge structure of weight 1:

$$0 = F^2_x \subset F^1_x = T^{1,0}_x M \subset F^0_x = T^{\epsilon}_x M.$$  \hspace{1cm} (3.6)

We recall that an affine special $\epsilon$Kähler manifold $(M, J, \nabla, g)$ (cf. [1, 15, 7]) is an $\epsilon$Kähler manifold endowed with a flat torsion-free connection $\nabla$, such that $(\nabla, J^\epsilon)$ is special, i.e. $\nabla J^\epsilon$ is symmetric and $\nabla \omega = 0$, where $\omega$ is the $\epsilon$Kähler form. The complex version of the next lemma and proposition was proved in [3] and we generalize it to the para-complex case.

**Lemma 1.** Let $\nabla$ be a torsion-free flat connection on the $\epsilon$complex manifold $(M, J^\epsilon)$. Then $F^1 = T^{1,0}M$ is an chomomorphic subbundle of $F^0 = T^\epsilon M$ with respect to the chomomorphic structure defined by $\nabla$ (compare Section 2) if and only if $(\nabla, J^\epsilon)$ is special, i.e. $\nabla J^\epsilon$ is symmetric.

**Proof.** The condition of $F^1$ to be chomorphic is equivalent to

$$\nabla_{\bar{Y}} X = 0 \text{ for all } X, Y \in \mathcal{O}(T^{1,0}M)$$

and the condition of $(\nabla, J^\epsilon)$ to be special is equivalent to

$$(\nabla_X J^\epsilon)(\bar{Y}) = (\nabla_{\bar{Y}} J^\epsilon)(X) \text{ for all } X, Y \in \mathcal{O}(T^{1,0}M),$$

due to the following short argument:

Let $X, Y \in \Gamma(T^{1,0}M)$ or $X, Y \in \Gamma(T^{0,1}M)$

$$(\nabla_X J^\epsilon)(Y) = \nabla_X J^\epsilon Y - J^\epsilon \nabla_X Y = \pm i \nabla_X Y - J^\epsilon \nabla_X Y,$$
which is symmetric as one sees by choosing vector fields \( X \) and \( Y \) such that 
\[ [X, Y] = 0. \]
Let now \( X, Y \in \Gamma(T^{1,0}M) \) be holomorphic vector fields, i.e. \( \mathcal{L}_X(J^*) = 0 \) where \( \mathcal{L} \) is the Lie-derivative. Then it holds
\[
0 = \mathcal{L}_X(J^*)Y = [X, J^*Y] - J^*[X, Y] = \nabla_X J^*Y - \nabla_{J^*Y}X = (\nabla_X J^*)Y - (\nabla_{\nabla J^*})X = (\nabla_X J^*)Y - (\nabla_{J^*}J^*)X + J^*\nabla_{J^*}X = [(\nabla_X J^*)Y - (\nabla_{J^*}J^*)X] + 2i\nabla_{J^*}X.
\]
This finishes the proof. \( \square \)

From the lemma we obtain:

**Proposition 2.** Let \((M, J^*)\) be an complex manifold, \(\nabla\) be a torsion-free flat connection and \(F^*\) defined as in equation (3.6).

1. Then \((M, J^*, \nabla)\) is an affine special complex manifold if and only if \(\nabla\) and \(F^*\) give a variation of \(\epsilon\)Hodge structures of weight 1 on \(TM^\mathbb{C}\).

2. Then \((M, J^*, \nabla, g)\) is an affine special \(\epsilon\)Kähler manifold if and only if \(\nabla\), \(F^*\) and \(\omega(\cdot, \cdot) = g(J^*\cdot, \cdot)\) give a variation of polarized \(\epsilon\)Hodge structures of weight 1 on \(TM^\mathbb{C}\).

In [8] the following notion of a conical special \(\epsilon\)Kähler manifold \((M, J^*, g, \nabla, \zeta)\) is introduced, i.e. an affine special \(\epsilon\)Kähler manifold \((M, J^*, g, \nabla)\) endowed with a vector field \(\zeta\), such that
\[
\nabla\zeta = D\zeta = Id, \quad (3.7)
\]
where \(D\) is the Levi-Civita connection of \(g\). In the same reference it is shown, that
\[
\mathcal{L}_\zeta J^* = 0. \quad (3.8)
\]
This implies that the distribution \(\mathcal{D} = \text{span}\{\zeta, J^*\zeta\}\) is integrable. The space of leaves, i.e. integral manifolds of \(\mathcal{D}\) is denoted by \(\tilde{M}\). If \((M, J^*, g, \nabla, \zeta)\) is a projective special \(\epsilon\)Kähler manifold (cf. [8]) of (real) dimension \(2n + 2\) then the canonical quotient map \(\pi: M \to \tilde{M}\) is an holomorphic submersion onto an complex manifold of (real) dimension \(2n\). The manifold \(\tilde{M}\) inherits an \(\epsilon\)Kähler metric \(\tilde{g}\) from the metric \(g\) such that \(\pi\) is a pseudo-Riemannian submersion. In this case it holds in particular \(g(\zeta, \zeta) = -\epsilon g(J^*\zeta, J^*\zeta) \neq 0\). The affine geometry of conical special para-Kähler manifolds was studied in [9].

In the remaining part of this section we shortly discuss the polarized variation of Hodge structure of weight 3 on \(V = TM \to \tilde{M}\) related to a projective special \(\epsilon\)Kähler manifold \((M, J^*, g, \nabla, \zeta)\):

Let us consider the real line bundle \(L\), which is generated by \(\zeta\). We use \(g(\zeta, \zeta) = -\epsilon g(J^*\zeta, J^*\zeta) \neq 0\) to obtain \(TM = L \oplus J^*(L) \oplus L'\), such that \(L' \cong TM\) is the orthogonal complement of \(L \oplus J^*(L)\) with respect to the pseudo-metric \(g\). From condition (3.7) we conclude
\[
\nabla L|_{\mathcal{L}'} = L'. \quad (3.9)
\]
Now we can define the Hodge filtration: We set $F^0 = TM^{C_\epsilon}$. The relation (3.8) implies that $\zeta + \epsilon \bar{J}' \zeta$ generates the holomorphic line bundle $F^3 = L^{1,0} \subset TM^{C_\epsilon}$. From the Riemannian bilinear relation it follows $(F^3)^\perp = F^1$, where $\perp$ is taken with respect to the $\epsilon$Kähler form $\omega$ which is extended $\mathbb{C}$-bilinearly. It remains to define $F^2 = T^{1,0}M$. The Griffiths transversality $\nabla F^3 \subset F^2$ is a consequence of equation (3.9) and $L' \cong TM$. The condition $\nabla F^2 \subset F^1$ follows from equation (3.9) by similar arguments as in [5]. This means we have defined a variation of $\epsilon$Hodge structures of weight 3 by

$$F^3 = L^{1,0} \subset F^2 = T^{1,0}M \subset F^1 = (F^3)^\perp \subset F^0 = TM^{C_\epsilon},$$

which is polarized by the $\epsilon$Kähler form $\omega$.

4. Period domains of variations of $\epsilon$Hodge structures

We recall some information about period domains of variations of $\epsilon$Hodge structures and have a closer look at the description of these either as homogeneous spaces or as flag manifolds, since this is crucial to understand our later results. A reference for the complex case is the book [2]. Again the complex case is classical and the para-complex case is new.

We introduce the period domain parameterizing the set of polarized $\epsilon$Hodge decompositions of weight $w$ with fixed $\epsilon$Hodge numbers $h^{p,q}$. Such an $\epsilon$Hodge structure is determined by specifying a flag $F^w \subset F^{w-1} \subset \ldots \subset F^0$ of fixed type satisfying the two bilinear relations. The set of such flags satisfying the first bilinear relation is usually called $\tilde{D}$ and can be described in a homogeneous model $G_{\mathbb{C}}/B$ where $G_{\mathbb{C}}$ is the group of automorphisms of $H^{C_\epsilon}$ fixing the polarization $b$ and $B$ is the stabilizer of some given reference structure $F_0^\epsilon$.

**Proposition 3.** The set $\tilde{D}$ classifying $\epsilon$Hodge decompositions of weight $w$ with fixed $\epsilon$Hodge numbers $h^{p,q}$ which obey the first bilinear relation is a flag manifold of type $(f_w, \ldots, f_v)$, $f_p = \dim F^p$, $v = \left\lceil \frac{w+1}{2} \right\rceil$, such that

(i) in the case of even weight $w = 2v$ each $F^p$, for $p = w, \ldots, v+1$, is isotropic with respect to the bilinear form $b$,

(ii) in the case of odd weight $w = 2v-1$ each $F^p$, for $p = w, \ldots, v$, is isotropic with respect to the bilinear form $b$.

It can also be identified with the homogeneous manifold $G_{\mathbb{C}}/B$.

**Proof.** (i) In the case of even weight we recover the spaces $F^p$, for $p = 0, \ldots, (w-v+1) = v+1$, from $F^p$, for $p = w, \ldots, v$, by using the decomposition

$$H^{C_\epsilon} = F^p \oplus \perp F^{w-p+1},$$

where $\perp$ is taken with respect to the non-degenerate hermitian sesquilinear form $b(\cdot, \cdot)$. The condition on $F^p$, for $p = w, \ldots, v+1$, to be isotropic is the first Riemannian bilinear relation.
(ii) In fact, for odd weight, one can recover the whole flag from $F^p$ for $p = w, \ldots, v$, by using the decomposition

$$H^C = F^p \oplus F^{w-p+1},$$

where $\perp$ is taken with respect to the non-degenerate hermitian sesquilinear form $b(\cdot, \overline{\cdot})$. The condition on $F^p$, for $p = w, \ldots, v$, to be isotropic is in the case of odd weight $w$ inherited from the first Riemannian bilinear relation. □

In the complex case $B$ is a parabolic subgroup. There seems to be no equivalent para-complex notion in the literature. The subset of $\tilde{D}$ classifying $\epsilon$Hodge structures which also satisfy the second bilinear relation is called $D$. As a non-degeneracy or a positivity condition the second bilinear relation defines an open subset of $\tilde{D}$.

**Proposition 4.** The period domain $D$ classifying $\epsilon$Hodge filtrations $F^\bullet$ of fixed dimension $f^p = \dim F^p$ satisfying both bilinear relations is an open subset of $\tilde{D}$ and it is a homogeneous manifold $D = G/V$, where $G$ is the group of linear automorphisms of $H$ preserving $b$ and $V = G \cap B$.

We consider the case of Hodge structures which are strongly polarized. Given the space $G/V$, we call $G/K$ where $K$ is the maximal compact subgroup of $G$ the ‘associated symmetric space’ and denote the canonical map by $\pi : G/V \to G/K$.

**The case of odd weight**

Now we have a glance at the groups $G, V$ and $K$ and the associated flag manifolds for $\epsilon$Hodge structures of odd weight. Using this we describe for strongly polarized variations of Hodge structures the map $\pi$ at the level of flag manifolds. This description is needed later to relate the (classical) period map to the $\epsilon$pluriharmonic maps appearing in $\epsilon tt^*$-geometry.

In the case of odd weight $w = 2l + 1$ for $l = v - 1$ the form $b$ is anti-symmetric due to the first Riemannian bilinear relation and hence a symplectic form on $H$. In particular the real dimension of $H$ is even. Hence the group $G$ is the symplectic group $Sp(H, b) \cong Sp(\mathbb{R}^r)$ with $r = \dim_{\mathbb{R}} H \in 2\mathbb{N}$. The maximal compact subgroup of $Sp(\mathbb{R}^r)$ is $K = U(r)$.

We define the $b$-isotropic $\epsilon$complex vector space $\mathcal{L} = \bigoplus_{p=0}^l H^{w-p,p} = F^{w-l} = F^w$.

One sees by equation (3.4)

$$H^C = \mathcal{L} \oplus \overline{\mathcal{L}}. \quad (4.1)$$

Since they have the same dimension, $\mathcal{L}$ and $\overline{\mathcal{L}}$ are, by the first bilinear relation, Lagrangian subspaces. We further fix a reference structure $F^\bullet_o$. Taking successively $\epsilon$unitary bases\(^1\)

$$\{f_i\}_{i=1}^{\dim(\mathcal{L})}$$

and

$$\{f_i\}_{i=1}^{\dim(\mathcal{L}_o)} \quad (4.2)$$

\(^1\)This means a basis with $b(f_i, f_j) = \pm \delta_{ij}$.  


with respect to the hermitian sesquilinear scalar product \( h(\cdot, \cdot) = (-1)^{w-1/2} b(\cdot, \cdot) \) of the flags
\[
H^{w,0} \subset H^{w,0} \oplus H^{w-1,1} \subset \cdots \subset L
\]
and
\[
H^{w,0}_o \subset H^{w,0}_o \oplus H^{w-1,1}_o \subset \cdots \subset L_o
\]
and extending these with \( \{ \tilde{f}_i \} \) and \( \{ \tilde{f}_i^o \} \) on \( L \) and \( L_o \) to symplectic bases one sees that \( Sp(\mathbb{R}^r) \) acts transitively by change of the basis from \( \{ f_i \} \) to \( \{ f_i^o \} \).

(i) First we discuss the complex case. If we have a strongly polarized variation of Hodge structures, then the stabilizer of \( F_o \) is the group \( V = \prod_{p=0}^l U(h^{w-p,p}) \). The map \( \pi : G/V \to G/K \) is at this level nothing else than the forgetful map from the flag \( H^{w,0} \subset H^{w,0} \oplus H^{w-1,1} \subset \cdots \subset L \) to the subspace \( L \). We remark, that the stabilizer of \( L_o \) is contained in the group \( U(r) \), if we assume the variation of Hodge structures to be strongly polarized.

If we consider a weakly polarized variation of Hodge structures, then the stabilizer of \( F_o^* \) is the group \( V = \prod_{p=0}^l U(k_p, l_p) \), where \( (k_p, l_p) \) is the hermitian signature of \( h \) restricted to \( H^{w-p,p} \) with \( q = w - p \). The stabilizer of \( L_o \) is in this case an element of the group \( U(r) \), where \( r = 2(k + l) \) and \( (k, l) \) is the hermitian signature of \( h \) on \( L_o \), i.e. \( k = \sum k_p \) and \( l = \sum l_p \).

Given a variation of Hodge structures of odd weight over the complex base manifold \( (M, J) \) we denote by \( L \) the (holomorphic) map
\[
L : M \to Sp(\mathbb{R}^r)/U(k, l), \quad x \mapsto L_x. \tag{4.3}
\]

The Grassmannian of Lagrangian subspaces, on which \( h \) has signature \( (k, l) \) will be denoted by \( Gr^{k,l}(\mathbb{C}^r) \) and on which \( h \) is positive definite will be denoted by \( Gr^{r,0}(\mathbb{C}^r) = Gr^{r,0}(\mathbb{C}^r) \).

(ii) In the para-complex case the stabilizer of \( L_o \) is the group \( U^*(\mathbb{C}^{2n}) \), with \( r = 2n \), compare equation (2.1). As before given a variation of para-Hodge structures of odd weight \( w \) over the para-complex base manifold \( (M, \tau) \) we denote by \( L \) the (para-holomorphic) map
\[
L : M \to Sp(\mathbb{R}^r)/U^{*}(\mathbb{C}^{2n}), \quad x \mapsto L_x. \tag{4.5}
\]

The associated Grassmannian of Lagrangian subspaces will be denoted by \( Gr^{n}(\mathbb{C}^{2n}) \) with \( r = 2n \).

5. \( ett^* \)-bundles and associated pluriharmonic maps

In this section we recall the notion of (metric) \( ett^* \)-bundles and explain the correspondence between metric \( ett^* \)-bundles and pluriharmonic maps, which was given in [18, 19].
Definition 3. An ett* -bundle \((E, D, S)\) over an \(\epsilon\)complex manifold \((M, J^\epsilon)\) is a real vector bundle \(E \rightarrow M\) endowed with a connection \(D\) and a section \(S \in \Gamma(T^*M \otimes \text{End} E)\) which satisfy the ett* -equation

\[
R^\theta = 0 \quad \text{for all} \quad \theta \in \mathbb{R},
\]

where \(R^\theta\) is the curvature tensor of the connection \(D^\theta\) defined by

\[
D^\theta_X := D_X + \cos_s(\theta)S_X + \sin_s(\theta)S_J \cdot X \quad \text{for all} \quad X \in TM.
\]

A metric ett* -bundle \((E, D, S, g)\) is an ett* -bundle \((E, D, S)\) endowed with a possibly indefinite \(D\)-parallel fiber metric \(g\) such that for all \(p \in M\)

\[
g(S_X Y, Z) = g(Y, S_X Z) \quad \text{for all} \quad X, Y, Z \in T_pM.
\]

Remark 2. 1) If \((E, D, S)\) is an ett* -bundle then \((E, D, S^\theta)\) is an ett* -bundle for all \(\theta \in \mathbb{R}\), where \(S^\theta := D^\theta - D = \cos_s(\theta)S + \sin_s(\theta)S_J\). The same remark applies to metric ett* -bundles. 2) The flatness of the connection \(D^\theta\) can be expressed in a set of equations on \(D\) and \(S\) which can be found in [18, 19].

Given a metric ett* -bundle \((E, D, S, g)\), we consider the flat connection \(D^\theta\) for a fixed \(\theta \in \mathbb{R}\). Any \(D^\theta\)-parallel frame \(s = (s_1, \ldots, s_r)\) of \(E\) defines a map

\[
G = G^{(s)} : M \rightarrow \text{Sym}_{p,q}(\mathbb{R}^\epsilon); \quad x \mapsto G(x) := (g_x(s_i(x), s_j(x))),
\]

where \((p, q)\) is the signature of the metric \(g\).

Let \(G/K\) be a pseudo-Riemannian symmetric space with associated symmetric decomposition \(g = p \oplus \mathfrak{f}\). We recall that a map \(f : (M, J^\epsilon) \rightarrow G/K\) is said to be admissible, if the \(\epsilon\)complex linear extension of its differential maps \(T_x^{1,0}M\) (respectively \(T_x^{0,1}M\)) to an Abelian subspace of \(p^C = p \otimes \mathbb{C}\) for all \(x \in M\).

If \(M\) is simply-connected then it was shown in [18, 19], that \(G : M \rightarrow \text{Sym}_{p,q}(\mathbb{R}^\epsilon)\) is \(\epsilon\)pluriharmonic and that it induces an admissible \(\epsilon\)pluriharmonic map \(\tilde{G} : M \rightarrow \text{Sym}_{p,q}(\mathbb{R}^\epsilon) \rightarrow S(p, q)\).

Conversely, we constructed in [18, 19] a metric ett* -bundle \((E = M \times \mathbb{R}^{2r}, D = \partial - \epsilon S, S = \epsilon d\tilde{G}, g = < G_{\cdot, \cdot}, >_{\mathbb{R}^{2r}})\) over a simply-connected manifold from an admissible \(\epsilon\)pluriharmonic map \(\tilde{G} : M \rightarrow S(p, q)\). If \(M\) is not simply-connected, then we have to replace the maps \(G\) and \(\tilde{G}\) by twisted \(\epsilon\)pluriharmonic maps (cf. [19] Theorems 5 and 6).

6. Variations of \(\epsilon\)Hodge structures as ett* -bundles

In this section we recall the result of Hertling [3] that variations of Hodge structures give solutions of metric ett* -bundles and generalize it to para-complex geometry and symplectic ett* -bundles. Our presentation differs from that of [3],...
since we give this result in the language of real differential geometry. Again, the para-complex version seems to be new.

Let \((E, \nabla, F^p)\) be a (real) variation of \(\epsilon\)Hodge structures of weight \(w\). The \(\epsilon\)complexified connection of \(\nabla\) on \(E^\epsilon = E \otimes \mathbb{C}_\epsilon\) will be denoted by \(\nabla^c\). Griffiths transversality and the \(\epsilon\)holomorphicity of the subbundles \(F^p\) gives

\[
\nabla^c : \Gamma(F^p) \to \Lambda^{1,0}(F^{p-1}) + \Lambda^{0,1}(F^p) \quad (6.1)
\]

and \(\epsilon\)complex conjugation yields

\[
\nabla^c : \Gamma(F^p) \to \Lambda^{0,1}(F^{p-1}) + \Lambda^{1,0}(F^p). \quad (6.2)
\]

Summarizing one obtains with \(H^{p,w-p} = F^p \cap \mathcal{F}^{w-p}\)

\[
\nabla^c : \Gamma(H^{p,w-p}) \to \Lambda^{1,0}(H^{p,w-p}) + \Lambda^{0,1}(H^{p,w-p}) + \left(\begin{array}{l}
\Lambda^{1,0}(H^{p+1,w-1-p}) + \\
\Lambda^{0,1}(H^{p-1,w+1-p})
\end{array}\right). \quad (6.3)
\]

Using the decomposition induced by the \(\epsilon\)Hodge structure and by the bi-degree of differential forms, one can find, that the curvature of \(\nabla^c\) vanishes if and only if \((E^\epsilon, D, S, g = \text{Re} h)\) defines an \(\epsilon tt^*\)-bundle. In addition the \(\epsilon\)complex conjugation \(\kappa = \overline{\cdot}\) respects the \(\epsilon\)Hodge decomposition and it is \(\nabla^c \kappa = 0\). Again the decomposition induced by the \(\epsilon\)Hodge structure and the bi-degree of differential forms imply that \(D \kappa = 0\), i.e. \(D\) leaves \(E\) invariant and that \(S \kappa = \kappa S\), i.e. \(S\) leaves \(E\) invariant, too.

If \(b\) is a polarization of the above variation of \(\epsilon\)Hodge structures \((E, \nabla, F^p)\), then \(\nabla b = 0\) and \(\nabla^c \kappa = 0\) yield after decomposing with respect to the \(\epsilon\)Hodge structure the equations \(Dg = 0\) and \(g(\cdot, \cdot) = g(\cdot, S \cdot)\) with \(g = \text{Re} h\). Concluding we obtain the proposition:

**Proposition 5.** Let \((E, \nabla, F^p)\) be a (real) variation of \(\epsilon\)Hodge structures of weight \(w\) with a polarization \(b\), then \((E^\epsilon, D, S, g = \text{Re} h)\) and \((E, D, S, g = \text{Re} h)\) with \(D\) and \(S\) as defined in equation \((6.3)\) are metric \(\epsilon tt^*\)-bundles.

The above consideration holds for \(\Omega = \text{Im} h\), too. This implies \(D \Omega = 0\) and \(\Omega(\cdot, \cdot) = \Omega(\cdot, S \cdot)\). Hence we have proven

**Proposition 6.** Let \((E, \nabla, F^p)\) be a (real) variation of \(\epsilon\)Hodge structures of weight \(w\) with a polarization \(b\), then \((E^\epsilon, D, S, \Omega = \text{Im} h)\) and \((E, D, S, \Omega = \text{Im} h)\) with \(D\) and \(S\) as defined in equation \((6.3)\) are symplectic \(\epsilon tt^*\)-bundles.

7. The period map of a variation of \(\epsilon\)Hodge structures

Like period domains describe \(\epsilon\)Hodge structures, \(\epsilon\)holomorphic maps into period domains describe variations of \(\epsilon\)Hodge structures, in the sense of the following proposition which is in the complex case due to Griffiths (cf. [2] Chapter 4.5). We only consider the simply connected case:
Proposition 7. Let \((M, J^r)\) be a simply connected \(\epsilon\)complex manifold and \(G/V\) the period domain classifying polarized \(\epsilon\)Hodge structures of given weight and \(\epsilon\)Hodge numbers, then giving a variation of \(\epsilon\)Hodge structures is equivalent to giving an \(\epsilon\)holomorphic map from \(M\) to \(G/V\) which satisfies the Griffiths transversality condition. Such maps are called period maps.

Let \((E, \nabla, F^p)\) be a variation of \(\epsilon\)Hodge structures of odd weight \(w\) over the \(\epsilon\)complex base manifold \((M, J^r)\) endowed with a polarization \(b\) where \(E\) has rank \(r\) and where \(f_p = \dim F_p\). Denote by \((E, D, S, g)\) the corresponding \(\epsilon tt^*\)-bundle constructed in proposition 5. We suppose, that \(M\) is simply connected.

Like in Section 5 we examine the metric \(g\) in a \(D^0 = \nabla\)-parallel frame \(s\) of \(E\). The flat frame is chosen as constructed in Section 4. The metric \(g\) defines a smooth map

\[
G : M \to \text{Sym}_{p,q}(\mathbb{R}^r) = \{ A \in \text{Mat}(\mathbb{R}^r) \mid A = A^t \text{ and } A \text{ has signature } (p,q)\}. \tag{7.1}
\]

In the complex case \((p, q) = (2k, 2l)\) is the symmetric signature of \(g\). We remark that for a variation of para-Hodge structures the metric \(g\) is forced to have split signature \((p, q) = (n, n)\) with \(n = \frac{1}{2} \dim \mathbb{H} \).

The map \(G\) will be called the fundamental matrix of the variation of \(\epsilon\)Hodge structures \((E, \nabla, F^p)\) and as above \(\text{Sym}_{p,q}(\mathbb{R}^r)\) is identified with the pseudo-Riemannian symmetric space \(GL(r, \mathbb{R})/O(p, q)\).

We recall that for odd weight each fiber of \(E\) has the structure of a symplectic vector space and consequently it holds \(\text{rk}_\mathbb{R} E = r = 2n \in 2\mathbb{N}\).

Theorem 1. Let \((E, \nabla, F^p)\) be a polarized variation of \(\epsilon\)Hodge structures of odd weight \(w\) with polarization \(b\) over the \(\epsilon\)complex base manifold \((M, J^r)\). Let \(r = 2n\) be the real rank of \(E\).

Then the fundamental matrix \(G\) takes values in the totally geodesic submanifold

\[
i : G_{\epsilon}^{k,l}(\mathbb{C}^{2n}) = \text{Sp}(\mathbb{R}^{2n})/U(k, l) \to GL(r, \mathbb{R})/O(2k, 2l), \text{ for } \epsilon = -1, \tag{7.2}
i : G_{\epsilon}^{n}(\mathbb{C}^{2n}) = \text{Sp}(\mathbb{R}^{2n})/\text{U}^+ (\mathbb{C}^n) \to GL(r, \mathbb{R})/O(n, n), \text{ for } \epsilon = 1 \tag{7.3}
\]

and coincides with the map \(L\), i.e. \(G = i \circ L : M \to GL(r, \mathbb{R})/O(p, q)\).

Proof. Given a point \(x \in M\) we put \(V = E^\epsilon_x\) and \(V^\mathbb{R} = E_x \cong \mathbb{R}^r\). To any polarized \(\epsilon\)Hodge structure \(F^p\) of odd weight \(w\) with polarization \(b\) the map \(L\) associated a Lagrangian subspace \(L \in \text{Gr}^\epsilon_0(V)\) in the complex and a Lagrangian subspace \(L \in \text{Gr}^\epsilon_0(V)\) in the para-complex case (see Section 4). We define a scalar product \(g^L = \text{Re } h|_L\) on \(L \subset V\). The projection onto the real points

\[
\text{Re} : V \to V^\mathbb{R} \tag{7.4}
\]

induces an isomorphism \(L \cong V^\mathbb{R}\). Its inverse we denote by \(\Phi = \Phi_L : V^\mathbb{R} \to L\).

Claim:

\[
i(L) = \Phi_L^* g^L =: G^L. \tag{7.5}
\]
We first show the $Sp(\mathbb{R}^r)$-equivariance of the map
\[ \mathcal{L} \mapsto G^\mathcal{L}. \]  
(7.6)

From the definition of $\Phi_\mathcal{L}$ we obtain with $\Lambda \in Sp(\mathbb{R}^r)$:
\[ \Phi_{\mathcal{L}} = \Lambda \circ \Phi_\mathcal{L} \circ \Lambda^{-1} |_{\mathbb{R}^r} \]  
(7.7)

and from this the transformation law of $G^\mathcal{L}$
\[ G^{\mathcal{L}^\epsilon} = \Phi^*_\mathcal{L} g^{\mathcal{L}^\epsilon} = (\Lambda^{-1})^* \Phi^*_\mathcal{L} \Lambda^* g^{\mathcal{L}^\epsilon} = (\Lambda^{-1})^* \Phi^*_\mathcal{L} g^\mathcal{L} = (\Lambda^{-1})^* G^\mathcal{L} = \Lambda \circ G^\mathcal{L}. \]  
(7.8)

Let $F^p_\mu$ be the reference flag of $V_{\sigma}^{\mathcal{L}^\epsilon}$ with dim $F^p_\mu = f_p$. We calculate $G^\mathcal{L}_o$ in the basis $\{f_{\sigma}\}_{i=1}^{\text{dim}(\mathcal{L}_o)}$ constructed in equation (4.2)
\[ (G^\mathcal{L}_o(\text{Re} f_{\sigma}, \text{Re} f_{\sigma}')) = \gamma_{p,q}, \text{ after permutation.} \]  
(7.9)

This yields
\[ \Phi^*_\mathcal{L}_o g^\mathcal{L}_o = \gamma_{p,q}. \]  
(7.10)

The proof is finished, since $G(x) = G^L(x) = i(L(x))$. \hfill \Box

**Corollary 1.** Let $(E, \nabla, F^p)$ be a polarized variation of Hodge structures of odd weight $w$ with polarization $b$ over the complex base manifold $(M, J^r)$. Then the map $L : M \to G_{0, r}(\mathbb{C}^*) = Sp(\mathbb{R}^r)/U(r, \mathbb{R})$ is $\epsilon$pluriharmonic.

**Proof.** This follows from the pluriharmonicity of the fundamental matrix $G : M \to GL(r, \mathbb{R})/O(p, q)$, since $G = i \circ L$, where $i$ is a totally geodesic immersion and consequently, by a well-known result about pluriharmonic maps (cf. [18, 19]), the pluriharmonicity of $L$ is equivalent to that of $G$. \hfill \Box

The last theorem and the last corollary can be specialized for variations of Hodge structures (this means $\epsilon = -1$), which are strongly polarized:

**Theorem 2.** Let $(E, \nabla, F^p)$ be a strongly polarized variation of Hodge structures of odd weight $w$ with polarization $b$ over the complex base manifold $(M, J)$. Then the fundamental matrix $G$ takes values in the totally geodesic submanifold
\[ i : G_{0, r}(\mathbb{C}^*) = G_{0, r, 0}(\mathbb{C}^*) = Sp(\mathbb{R}^r)/U(r, \mathbb{R}) \to GL(r, \mathbb{R})/O(r) \]  
(7.11)

and coincides with the map $L = \pi \circ \mathcal{P} : M \to G/K$, i.e. $G = i \circ L : M \to GL(r, \mathbb{R})/O(r)$.

With the same argument as before, we obtain the

**Corollary 2.** Let $(E, \nabla, F^p)$ be a strongly polarized variation of Hodge structures of odd weight $w$ with polarization $b$ over the complex base manifold $(M, J)$. Then the map $L : M \to G_{0, r}(\mathbb{C}^*) = G_{0, r, 0}(\mathbb{C}^*) = Sp(\mathbb{R}^r)/U(r)$ is pluriharmonic.

This means our results generalize the following result for strongly polarized complex variations of Hodge structures of odd weight:

**Theorem 3.** (cf. [2] Theorem 14.4.1) Let $f : M \to G/V$ be a period mapping and $\pi : G/V \to G/K$, as defined in Section 4 the canonical map to the associated locally symmetric space. Then $\pi \circ f$ is pluriharmonic.
References


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