On the Hadwiger Numbers of Centrally Symmetric Starlike Disks

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Abstract. The Hadwiger number $H(S)$ of a topological disk $S$ in $\mathbb{R}^2$ is the maximal number of pairwise nonoverlapping translates of $S$ that touch $S$. A conjecture of A. Bezdek, K. and W. Kuperberg [2] states that this number is at most eight for any starlike disk. A. Bezdek [1] proved that the Hadwiger number of a starlike disk is at most seventy five. In this note, we prove that the Hadwiger number of any centrally symmetric starlike disk is at most twelve.

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1. Introduction and preliminaries

This paper deals with topological disks in the Euclidean plane $\mathbb{R}^2$. We make use of the linear structure of $\mathbb{R}^2$, and identify a point with its position vector. We denote the origin by $o$.

A topological disk, or shortly disk, is a compact subset of $\mathbb{R}^2$ with a simple, closed, continuous curve as its boundary. Two disks $S_1$ and $S_2$ are nonoverlapping, if their interiors are disjoint. If $S_1$ and $S_2$ are nonoverlapping and $S_1 \cap S_2 \neq \emptyset$, then $S_1$ and $S_2$ touch. A disk $S$ is starlike relative to a point $p$, if, for every $q \in S$, $S$ contains the closed segment with endpoints $p$ and $q$. In particular, a convex disk $C$ is starlike relative to any point $p \in C$. A disk $S$ is centrally symmetric, if $-S$ is a translate of $S$. If $-S = S$, then $S$ is o-symmetric.

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The Hadwiger number, or translative kissing number, of a disk $S$ is the maximal number of pairwise nonoverlapping translates of $S$ that touch $S$. The Hadwiger number of $S$ is denoted by $H(S)$. It is well known (cf. [8]) that the Hadwiger number of a parallelogram is eight, and the Hadwiger number of any other convex disk is six. In [9], the authors showed that the Hadwiger number of a disk is at least six. Recently, Cheong and Lee [4] constructed, for every $n > 0$, a disk with Hadwiger number at least $n$.

A. Bezdek, K. and W. Kuperberg [2] conjectured that the Hadwiger number of any starlike disk is at most eight (see also Conjecture 6, p. 95 in the book [3] of Brass, Moser and Pach). The only result regarding this conjecture is due to A. Bezdek, who proved in [1] that the Hadwiger number of a starlike disk is at most seventy five. Our goal is to prove the following theorem.

**Theorem 1.** Let $S$ be a centrally symmetric starlike disk. Then the Hadwiger number $H(S)$ of $S$ is at most twelve.

In the proof, Greek letters, small Latin letters and capital Latin letters denote real numbers, points and sets of points, respectively. For $u, v \in \mathbb{R}^2$, the symbol $\text{dist}(u, v)$ denotes the Euclidean distance of $u$ and $v$. For simplicity, we introduce a Cartesian coordinate system and, for a point $u \in \mathbb{R}^2$ with $x$-coordinate $\alpha$ and $y$-coordinate $\beta$, we may write $u = (\alpha, \beta)$. The closed segment (respectively, open segment) with endpoints $u$ and $v$ is denoted by $[u, v]$ (respectively, by $(u, v)$). For a subset $A$ of $\mathbb{R}^2$, $\text{int} A$, $\text{bd} A$, $\text{card} A$ and $\text{conv} A$ denotes the interior, the boundary, the cardinality and the convex hull of $A$, respectively.

Consider a convex disk $C$ and two points $p, q \in \mathbb{R}^2$. Let $[t, s]$ be a chord of $C$, parallel to $[p, q]$, such that $\text{dist}(s, t) \geq \text{dist}(s', t')$ for any chord $[s', t']$ of $C$ parallel to $[p, q]$. The $C$-distance $\text{dist}_C(p, q)$ of $p$ and $q$ is defined as

$$\text{dist}_C(p, q) = \frac{2 \text{dist}(p, q)}{\text{dist}(s, t)}.$$ 

For the definition of $C$-distance, see also [10]. It is well known that the $C$-distance of $p$ and $q$ is equal to the distance of $p$ and $q$ in the normed plane with unit disk $\frac{1}{2}(C - C)$. The $o$-symmetric convex disk $\frac{1}{2}(C - C)$ is called the central symmetral of $C$. We note that $C \subset C'$ yields $\text{dist}_C(p, q) \geq \text{dist}_{C'}(p, q)$ for any $p, q \in \mathbb{R}^2$.

We prove the theorem in Section 2. During the proof we present two remarks, showing that as we broaden our knowledge of $S$, we are able to prove better and better upper bounds on its Hadwiger number.

**2. Proof of the theorem**

Let $S$ be an $o$-symmetric starlike disk. Let $\mathcal{F} = \{S_i : i = 1, 2, \ldots, n\}$ be a family of translates of $S$ such that $n = H(S)$ and, for $i = 1, 2, \ldots, n$, $S_i = c_i + S$ touches $S$ and does not overlap with any other element of $\mathcal{F}$. Let $K = \text{conv} S$, $X = \{c_i : i = 1, 2, \ldots, n\}$, $C = \text{conv} X$ and $\bar{C} = \text{conv} (X \cup (-X))$. Furthermore, let $R_i$ denote the closed ray $R_i = \{\lambda c_i : \lambda \in \mathbb{R} \text{ and } \lambda \geq 0\}$.

First, we prove a few lemmas.
Lemma 1. The disk $S$ is starlike relative to the origin $o$. Furthermore, $o \in \text{int} S$.

Proof. Let $S$ be starlike relative to $p \in S$, and assume that $p \neq o$. By symmetry, $S$ is starlike relative to $-p$. Consider a point $q \in S$. Since $S$ is starlike relative to $p$ and $-p$, the segments $[p, q]$ and $[-p, q]$ are contained in $S$. Thus, any segment $[p, r]$, where $r \in [-p, q]$, is contained in $S$. In other words, we have $\text{conv}\{p, -p, q\} \subset S$, which yields that $[o, q] \subset S$. The second assertion follows from the first and the symmetry of $S$. □

Lemma 2. If $x + S$ and $y + S$ are nonoverlapping translates of $S$, then we have $\text{dist}_K(x, y) \geq 1$.

Proof. Without loss of generality, we may assume that $x = o$. Suppose that $y \in \text{int} K$. Note that there are points $p, q \in S$ such that $y \in \text{int} \text{conv}\{o, p, q\}$. By the symmetry of $S$, $[y - p, y]$ and $[y - q, y]$ are contained in $y + S$. Since $y \in \text{int} \text{conv}\{o, p, q\}$, the segments $[y - p, y]$ and $[o, q]$ cross, which yields that $S$ and $y + S$ overlap; a contradiction. Hence, $y \notin \text{int} K$. Since $\text{int} K$ is the set of points in the plane whose distance from $o$, in the norm with unit ball $K$, is less than one, we have $\text{dist}_K(o, y) \geq 1$. □

Remark 1. The Hadwiger number $H(S)$ of $S$ is at most twenty four.

Proof. Note that, for every value of $i$, $K$ and $c_i + K$ either overlap or touch. Since $K$ is $o$-symmetric, it follows that $c_i \in 2K$, and $c_i + \frac{1}{2}K$ is contained in $\frac{1}{2}K$. By Lemma 2, $\{c_i + \frac{1}{2}K : i = 1, 2, \ldots, n\} \cup \{\frac{1}{2}K\}$ is a family of pairwise nonoverlapping translates of $\frac{1}{2}K$. Thus, $n \leq 24$ follows from an area estimate. □

Lemma 3. If $j \neq i$, then $R_i \cap \text{int} S_j = \emptyset$. Furthermore, $R_i \cap S_j \subset (o, c_i)$.

Proof. Since $S$ and $S_i$ touch, there is a (possibly degenerate) parallelogram $P$ such that $\text{bd} P \subset (S \cup S_i)$ and $[o, c_i] \subset P$ (cf. Figure 1). Note that if $\text{int}(x + S)$ intersects neither $S$ nor $S_i$, then $x \notin P$ and $\text{int}(x + S) \cap (o, c_i) = \emptyset$.

Figure 1.
If $S_j \cap R_i = \emptyset$, we have nothing to prove. Let $S_j \cap R_i \neq \emptyset$ and consider a point $c_j + p \in S_j \cap R_i$. Since $o \in \text{int} S$, $c_j + p \neq o$ and $c_j + p \neq c_i$. By the previous paragraph, if $c_2 + p \in (o, c_i)$, then $c_j + p \notin \text{int} S_j$. Thus, we are left with the case that $c_j + p \in R_i \setminus [o, c_i]$. By symmetry, $c_i - p \in S_i$. Note that $(c_i, c_i - p) \cap (o, c_j) \neq \emptyset$, which yields that $\text{int} S_i$ intersects $(o, c_j)$; a contradiction. □

**Lemma 4.** We have $o \in \text{int} C$, and $X \subset \text{bd} C$.

**Proof.** Assume that $o \notin \text{int} C$. Note that there is a closed half plane $H$, containing $o$ in its boundary, such that $C \subset H$. Let $p$ be a boundary point of $S$ satisfying $S \subset p + H$. Then, for $i = 1, 2, \ldots, n$, we have $S_i \subset p + H$. Observe that, for any value of $i$, $2p + S$ touches $S$ and does not overlap $S_i$. Thus, $\mathcal{F} \cup \{2p + S\}$ is a family of pairwise nonoverlapping translates of $S$ in which every element touches $S$, which contradicts our assumption that card $\mathcal{F} = n = H(S)$.

Assume that $c_i \notin \text{bd} C$ for some $i$, and note that there are values $j$ and $k$ such that $c_i \in \text{int} \text{conv}\{o, c_j, c_k\}$. Since $S_j$ and $S_k$ touch $S$, $\frac{1}{2}c_j$ and $\frac{1}{2}c_k$ are contained in $K$. Observe that at least one of $d_j = c_i - \frac{1}{2}c_j$ and $d_k = c_i - \frac{1}{2}c_k$ is in the exterior of the closed, convex angular domain $D$ bounded by $R_j \cup R_k$ (cf. Figure 2). Since $d_j$ and $d_k$ are points of $c_i + K$, we obtain $(c_i + K) \setminus D \neq \emptyset$. On the other hand, Lemma 3 yields that $S_i \subset D$, hence, $c_i + K = \text{conv} S_i \subset D$; a contradiction. □

![Figure 2](image-url)

**Remark 2.** The Hadwiger number $H(S)$ of $S$ is at most sixteen.

**Proof.** Golab [7] proved that the circumference of every centrally symmetric convex disk measured in its norm is at least six and at most eight. Fáry and Makai [6] proved that, in any norm, the circumferences of any convex disk $C$ and its central symmetral $\frac{1}{2}(C - C)$ are equal. Thus, the circumference of $C$ measured in the norm with unit ball $\frac{1}{2}(C - C)$ is at most eight.

Since $C \subset 2K$, we have $\text{dist}_C(p, q) \geq \text{dist}_{2K}(p, q) = \frac{1}{2} \text{dist}_K(p, q)$ for any points $p, q \in \mathbb{R}^2$. By Lemma 2, $\text{dist}_K(c_i, c_j) \geq 1$ for every $i \neq j$. Thus, $X = \{c_i : i = 1, 2, \ldots, n\}$ is a set of $n$ points in the boundary of $C$ at pairwise $C$-distances at least $\frac{1}{2}$. Hence, $n \leq 16$. □
Now we are ready to prove our theorem. By [5], there is a parallelogram \( P \), circumscribed about \( \overline{C} \), such that the midpoints of the edges of \( P \) belong to \( \overline{C} \). Since the Hadwiger number of any affine image of \( S \) is equal to \( H(S) \), we may assume that \( P = \{ (\alpha, \beta) \in \mathbb{R}^2 : |\alpha| \leq 1 \text{ and } |\beta| \leq 1 \} \). Note that the points \( e_x = (1, 0) \) and \( e_y = (0, 1) \) are in the boundary of \( \overline{C} \).

First, we show that there are two points \( r_x \) and \( s_x \) in \( S \), with \( x \)-coordinates \( \rho_x \) and \( \sigma_x \), respectively, such that \( e_x \in \text{conv}\{o, 2r_x, 2s_x\} \) and \( \rho_x + \sigma_x \geq 1 \).

Assume that \( e_x = c_i \) for some value of \( i \). Since \( S \) and \( S_i \) touch, there is a (possibly degenerate) parallelogram \( P_i = \text{conv}\{o, r_x, s_x, c_i\} \) such that \( c_i = r_x + s_x \), \([o, r_x] \cup [o, s_x] \subset S \) and \([c_i, r_x] \cup [c_i, s_x] \subset S_i \) (cf. Figure 1). Observe that \( c_i \in \text{conv}\{o, 2r_x, 2s_x\} \) and \( \rho_x + \sigma_x = 1 \). If \( e_x = -c_i \), we may choose \( r_x \) and \( s_x \) similarly.

Assume that \( e_x \in (c_i, c_j) \) for some values of \( i \) and \( j \). Consider a parallelogram \( P_i = \text{conv}\{o, r_i, s_i, c_i\} \) such that \( c_i = r_i + s_i \), \([o, r_i] \cup [o, s_i] \subset S \) and \([c_i, r_i] \cup [c_i, s_i] \subset S_i \). Let \( L \) denote the line with equation \( x = \frac{1}{2} \). We may assume that \( L \) separates \( s_i \) from \( o \). We define \( r_j \) and \( s_j \) similarly. If the \( x \)-axis separates the points \( s_i \) and \( s_j \), we may choose \( s_i \) and \( s_j \) as \( r_x \) and \( s_x \). If both \( s_i \) and \( s_j \) are contained in the open half plane, bounded by the \( x \)-axis and containing \( c_i \) or \( c_j \), say \( c_i \), we may choose \( r_j \) and \( s_j \) as \( r_x \) and \( s_x \) (cf. Figure 3). If \( e_x \) is in \((-c_i, c_j)\) or \((-c_i, -c_j)\), we may apply a similar argument.

![Figure 3](image)

Analogously, we may choose points \( r_y \) and \( s_y \) in \( S \), with \( y \)-coordinates \( \rho_y \) and \( \sigma_y \), respectively, such that \( e_y \in \text{conv}\{o, 2r_y, 2s_y\} \) and \( \rho_y + \sigma_y \geq 1 \). We may assume that \( \rho_x \leq \sigma_x \) and that \( \rho_y \leq \sigma_y \).

Let \( Q_1, Q_2, Q_3 \) and \( Q_4 \) denote the four closed quadrants of the coordinate system in counterclockwise cyclic order. We may assume that \( X \cap Q_1 \neq \emptyset \), and that \( Q_1 \) contains the points with nonnegative \( x \)- and \( y \)-coordinates. We relabel the indices of the elements of \( \mathcal{F} \) in a way that \( R_1, R_2, \ldots, R_n \) are in counterclockwise cyclic order, and the angle between \( R_1 \) and the positive half of the \( x \)-axis, measured in the counterclockwise direction, is the smallest amongst all rays in \( \{R_i : i = 1, 2, \ldots, n\} \).

If \( \text{card}(Q_i \cap X) \leq 3 \) for each value of \( i \), the assertion holds. Thus, we may assume that, say, \( j = \text{card}(Q_1 \cap X) > 3 \). By Lemma 3, \([c_i, c_i - s_j]\) does not cross the rays \( R_i \) and \( R_j \) for \( i = 2, 3, \ldots, j - 1 \). Thus, the \( y \)-coordinate of \( c_i \) is at least
$\sigma_y$ (cf. Figure 4, note that $c_i$ is not contained in the dotted region). Similarly, the $x$-coordinate of $c_i$ is at least $\sigma_x$ for $i = 2, \ldots, j - 1$. Thus, $\sigma_x \leq 1$ and $\sigma_y \leq 1$, which yield that $\rho_x \geq 0$ and $\rho_y \geq 0$. Since $\sigma_x \geq 1 - \rho_x$ and $\sigma_y \geq 1 - \rho_y$, each $c_i$, with $2 \leq i \leq j - 1$, is contained in the rectangle $T = \{(\alpha, \beta) \in \mathbb{R}^2 : 1 - \rho_x \leq \alpha \leq 1$ and $1 - \rho_y \leq \beta \leq 1\}$.

\[\text{Figure 4.}\]

Let $B = \{(\alpha, \beta) \in \mathbb{R}^2 : |\alpha| \leq \rho_x$ and $|\beta| \leq \rho_y\}$. Note that if $S$ and $p + S$ are nonoverlapping and $u, v \in S$, then the parallelogram $\text{conv}\{o, u, v, u + v\}$ does not contain $p$ in its interior. Thus, applying this observation with \{u, v\} $\subseteq \{\pm r_x, \pm s_x, \pm r_y, \pm s_y\}$, we obtain that $p \notin \text{int} B$ (cf. Figure 5, the dotted parallelograms show the region “forbidden” for $p$).

\[\text{Figure 5.}\]

Furthermore, if $r_x$ and $s_x$ do not lie on the $x$-axis, and $r_y$ and $s_y$ do not lie on the $y$-axis, then the interiors of these parallelograms cover $B$, apart from some points of $S$, and thus, we have $p \notin B$. If $p$ is on a vertical side of $B$, then $r_y$ or $s_y$ lies on the $y$-axis (cf. Figure 6). Note that if $r_y$ lies on the $y$-axis, then
$e_y \in \text{conv}\{o,2r_y,2s_y\}$ yields $\rho_y \geq \frac{1}{2}$, or that also $s_y$ lies on the $y$-axis. Thus, it follows in this case that $\frac{1}{2}e_y \in S$. Similarly, if $p$ is on a horizontal side of $B$, then $\frac{1}{2}e_x \in S$. We use this observation several times in the next three paragraphs.

Note that $T = (1 - \frac{\rho_x}{2}, 1 - \frac{\rho_y}{2}) + \frac{1}{2}B$. Since for any $2 \leq i < k \leq j - 1$, $c_i + \frac{1}{2}B$ and $c_k + \frac{1}{2}B$ do not overlap, it follows that $c_i$ and $c_k$ lie on opposite sides of $T$. By Lemma 4, we immediately obtain that $j \leq 5$.

Assume that $j = 5$. Then, we have $\text{card}(X \cap T) = 3$, which implies that two points of $X \cap T$ are consecutive vertices of $T$. Without loss of generality, we may assume that $c_4 = (1 - \rho_x, 1)$, $c_3 = (1,1)$ and $c_2 = (\tau, 1 - \rho_y)$ for some $\tau \in [1 - \rho_y, 1]$.

Since $c_3 - c_4$ lies on a vertical side of $B$, we obtain that $\frac{1}{2}e_y \in S$. From the position of $c_3 - c_2$, we obtain similarly that $\frac{1}{2}e_x \in S$. Thus, if $c_1$ is not on the $x$-axis or $c_5$ is not on the $y$-axis, then $R_1 \cap \text{int} S_2 \neq \emptyset$ or $R_5 \cap \text{int} S_4 \neq \emptyset$, respectively; a contradiction. Hence, from $\frac{1}{2}e_x, \frac{1}{2}e_y \in S$, it follows that $c_1 = e_x$ and $c_5 = e_y$. By Lemma 4, we have that $c_2 = (1,1 - \rho_y)$, which yields that, for example, $S_1$ and $S_2$ overlap; a contradiction.

We are left with the case $j = 4$. We may assume that $c_2$ and $c_3$ lie, say, on the vertical sides of $T$. Then we immediately have $\frac{1}{2}e_y \in S$. If $c_4$ is not on the $y$-axis, then $R_4 \cap \text{int} S_3 \neq \emptyset$, and thus, it follows that $c_4 = e_y$. We show, by contradiction, that $\text{card}((Q_1 \cup Q_2) \cap X) \leq 6$.

Assume that $\text{card}((Q_1 \cup Q_2) \cap X) > 6$. Note that in this case $\text{card}(Q_2 \cap X) = 4$, and both $c_5$ and $c_6$ are either on the horizontal sides, or on the vertical sides of $T' = (-2 + \rho_x, 0) + T$. If they are on the horizontal sides, then $\frac{1}{2}e_x \in S$, $c_5 = (-1,1)$, $c_7 = -e_x$, and, by Lemma 4, $c_6 = (-1,1 - \rho_y)$. Thus, $S_6$ overlaps both $S_5$ and $S_7$; a contradiction, and we may assume that $c_5$ and $c_6$ are on the vertical sides of $T'$.

Since the $y$-coordinate of $c_2$ is at least $\frac{1}{2}$, and since $(c_3, c_5 - \frac{1}{2}e_y)$ does not intersect the ray $R_2$, we obtain that the $y$-coordinate of $c_3$ is at least $\frac{3}{4}$. Similarly, the $y$-coordinates of $c_4$ and $c_5$ are at least $\frac{1}{4}$, both $c_4$ and $c_5$ are on the vertical side of $T'$, and $c_6$ is not on the horizontal side of $T'$. By the horizontal sides of $T'$, we may assume that $c_5$ and $c_6$ are on the vertical sides of $T'$, and $c_6$ is not on the horizontal side of $T'$. By Lemma 4, $c_6 = (-1,1 - \rho_y)$. Thus, $S_6$ overlaps both $S_5$ and $S_7$; a contradiction, and we may assume that $c_5$ and $c_6$ are on the vertical sides of $T'$.

Figure 6.
coordinate of \(c_5\) is at least \(\frac{3}{4}\). Note that \(c_3 - s_x\) and \(c_5 + s_x\) are on the positive half of the \(y\)-axis. Then it follows from Lemma 3 that \(c_3 - s_x\) and \(c_5 + s_x\) lie on the open segment \((o, c_4)\). If \(c_3 - s_x \notin \left(\frac{1}{2} c_4, c_4\right)\) or \(c_5 + s_x \notin \left(\frac{1}{2} c_4, c_4\right)\), then we have \(c_5 + s_x \notin (o, c_4)\) or \(c_3 - s_x \notin (o, c_4)\), respectively. Thus, both \(c_5 + s_x\) and \(c_3 - s_x\) belong to \(\left(\frac{1}{2} c_4, c_4\right)\), and a neighborhood of \(\frac{1}{2} c_4\) intersects \(S_4\) in a segment, which yields that \(S_4\) is not a disk; a contradiction.

Assume that \(\text{card}(Q_4 \cap X) > 3\). Then \(\text{card}((Q_1 \cup Q_4) \cap X) > 6\) yields that \(\text{card}((Q_3 \cup Q_4) \cap X) \leq 6\), and the assertion follows. Thus, we may assume that \(\text{card}(Q_4 \cap X) \leq 3\).

Finally, assume that \(\text{card}(Q_3 \cap X) > 3\). Then we have \(\text{card}((Q_3 \cup Q_4) \cap X) \leq 6\) or \(\text{card}((Q_2 \cup Q_3) \cap X) \leq 6\). In the first case we clearly have \(\text{card} X \leq 12\). In the second case, by the argument used for \(Q_1 \cap X\), we obtain that \(-e_x \in X\) and \(\text{card}(Q_2 \cap X) \leq 3\), from which it follows that \(\text{card}((Q_1 \cup Q_2 \cup Q_3) \cap X) \leq 9\). Since \(\text{card}(Q_4 \cap X) \leq 3\), the assertion holds.

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References


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