Projectivity and Flatness over the Colour Endomorphism Ring of a Finitely Generated Graded Comodule

T. Guédénon

110 Penworth Drive S.E., Calgary, AB, T2A 5H4, Canada
e-mail: guedenth@yahoo.ca

Abstract. Let $k$ be a field, $G$ an abelian group with a bicharacter, $A$ a colour algebra; i.e., an associative graded $k$-algebra with identity, $C$ a graded $A$-coring that is projective as a right $A$-module, $C^*$ the graded dual ring of $C$ and $\Lambda$ a left graded $C$-comodule that is finitely generated as a graded right $C^*$-module. We give necessary and sufficient conditions for projectivity and flatness of a graded module over the colour endomorphism ring $\mathcal{C}END(\Lambda)$.

0. Introduction

The notion of graded corings (except graded algebras and graded coalgebras) rarely appears in the literature on corings. The only paper we know where this notion appears is [8]. In the present paper we will give some conditions to test projectivity or flatness over the colour endomorphism ring of a finitely generated graded $C$-comodule, where $C$ is a graded coring. Let $k$ be a field, $A$ a $k$-algebra, $C$ an $A$-coring, $^*C$ the left dual ring of $C$ and $\Lambda$ a right $C$-comodule that is finitely generated as a left $^*C$-module. In [11], we gave necessary and sufficient conditions for projectivity and flatness over the endomorphism ring $\text{End}^C(\Lambda)$ of $\Lambda$. In the present paper, we will extend these results to a $G$-graded $A$-coring $C$, where $G$ is an abelian group with a bicharacter and $A$ is a colour algebra; i.e., a graded associative $k$-algebra with identity. More precisely, let us denote by $C^* = \text{HOM}_A(C_A, A_A)$ the largest graded vector space contained in $\text{Hom}_A(C_A, A_A)$. It has a colour algebra structure. Let $\Lambda$ be a graded left $C$-comodule that is finitely generated as a
graded right $\mathbb{C}^*$-module. We give necessary and sufficient conditions for projectivity and flatness over the colour endomorphism ring $\mathcal{C}END(A)$ of $\Lambda$. The presence of the bicharacter makes the difference with the classical gradation. These results are interesting when $\mathcal{C} = A$, or when $\mathcal{C}$ contains a grouplike element, or when $\mathcal{C}$ comes from a graded entwining structure with respect to a bicharacter. If $\mathcal{C} = A$ then $gr^{-A}\mathcal{M}$ is the category of graded left $A$-modules and $A^*$ is isomorphic to the opposite algebra $A^{op}$ of $A$. If $\mathcal{C}$ contains a grouplike element $X$, then $A$ is a graded left $\mathcal{C}$-comodule that is finitely generated as a graded right $\mathbb{C}^*$-module. In this case, $\mathcal{C}END(A)$ is the colour subring of $(\mathcal{C}, X)$-coinvariants of $A$. Our techniques and methods are inspired from [10], [7] and [11].

1. Preliminary results

Throughout the paper, $k$ is a field, $G$ is an abelian group and $(\cdot/\cdot)$ is a bicharacter on $G$; i.e., a map from $G \times G$ into $k^\times$ satisfying:

$$(x/y) = (y/x)^{-1} \quad \text{and} \quad (x/y + z) = (x/y)(x/z).$$

These two relations imply that $(x + y/z) = (x/z)(y/z)$. If $M$ and $N$ are vector spaces $Hom(M, N)$ is the vector space of $k$-linear maps from $M$ to $N$.

A vector space $A$ is $G$-graded or graded if $A = \bigoplus_{x \in G} A_x$, where the $A_x$ are vector subspaces of $A$. An algebra $A$ (not necessarily associative with identity) is said to be graded if $A$ is a graded vector space as above and the $A_x$ satisfy $A_xA_y \subseteq A_{x+y}$. According to [8, Section 1], a colour algebra is an associative graded algebra. In what follows we assume that all colour algebras are unital. We will consider $k$ as a colour algebra with the trivial gradation. Given colour algebras $A$ and $B$, a morphism of colour algebras $A \to B$ is a morphism of algebras which is homogeneous of degree 0. Let $m$ be an element of a graded vector space $M$. If $m$ is homogeneous, we denote by $|m|$ its degree. If $|m|$ occurs in some expression, this means that we regard $m$ as a homogeneous element and that the expression extends to the other elements by linearity. Let $M$ and $N$ be graded vector spaces. An element of $Hom(M, N)$ is homogeneous of degree $x$ if $f(M_y) \subseteq N_{x+y}$ for all $y \in G$. We denote by $HOM(M, N)_x$ the vector subspace of $Hom(M, N)$ whose elements are homogeneous of degree $x$ and we will set $HOM(M, N) = \bigoplus_{x \in G} HOM(M, N)_x$. Clearly, $HOM(M, N)$ is the largest graded vector space contained in $Hom(M, N)$. The space $HOM(M, N)$ is denoted $Hom_k(M, N)_G$ in [9]. By [13, Corollary 1.2.11], $HOM(M, N) = Hom(M, N)$ if $G$ is finite or if $M$ is finite-dimensional. By [9], $HOM(M, M)$ is a colour algebra. If $M, N, M'$ and $N'$ are graded vector spaces and if $f : M \to M'$ and $g : N \to N'$ are homogeneous linear maps then $(f \otimes g)(m \otimes n) = (|g||m|)f(m) \otimes g(n)$. We will denote by $gr^{-k}\mathcal{M}$ the category of graded $k$-vector spaces. The morphisms of $gr^{-k}\mathcal{M}$ are the homogeneous $k$-linear maps of degree 0; we call them the graded $k$-linear maps. Let $N$ be a graded vector space. For every $x$ in $G$, the $x$-suspension of $N$ is the graded vector space $N(x)$ obtained from $N$ by a shift of the gradation by $x$. As vector spaces, $N$ and $N(x)$ coincide but the gradations are related by $N(x)_y = N_{x+y}$ for all $y \in G$. 


Let $A$ be a colour algebra. A left $A$-module $M$ is called a graded left $A$-module if $M$ admits a decomposition as a direct sum of vector spaces $M = \bigoplus_{x \in G} M_x$ such that $A_x M_y \subseteq M_{x+y}$; $\forall x, y \in G$.

**Definition 1.1.** Let $M$, $N$ be graded left $A$-modules. A homogeneous element $f$ of $\text{Hom}(M, N)$ is colour left $A$-linear if $f(am) = (|f|/|a|)af(m)$ for all $a \in A$.

If $M$, $N$ are graded left $A$-modules, we let $\text{A}_{\text{HOM}}(M, N)_x$ denote the vector subspace of $\text{Hom}(M, N)$ whose elements are colour $A$-linear of degree $x$. So the colour left $A$-linear maps of degree 0 are exactly the left $A$-linear maps of degree $0$; i.e., $\text{A}_{\text{HOM}}(M, N)_0 = \text{A}_{\text{Hom}}(M, N)\cap \text{HOM}(M, N)_0$. We define $\text{A}_{\text{HOM}}(M, N)$ to be the sum of these subspaces; the sum is direct: $\text{A}_{\text{HOM}}(M, N) = \bigoplus_{x \in G} \text{A}_{\text{HOM}}(M, N)_x$. We call $\text{A}_{\text{HOM}}(M, N)$ the subspace of colour left $A$-linear maps of $\text{Hom}(M, N)$. Contrary to the classical gradation, if $A \neq k$ and if the bicharacter is not trivial, there is no comparison relation between $\text{A}_{\text{HOM}}(M, N)$ and $\text{A}_{\text{Hom}}(M, N)$ even if $M$ is finitely generated as an $A$-module or if $G$ is finite. If $G = \mathbb{Z}/2\mathbb{Z}$, colour $A$-linear maps are called $A$-superlinear in [16]. We will denote by $\text{gr}_{-A} \mathcal{M}$ the category of graded left $A$-modules. The morphisms of $\text{gr}_{-A} \mathcal{M}$ are the colour left $A$-linear maps of degree 0; we call them the graded left $A$-linear maps. It is well known that $\text{gr}_{-A} \mathcal{M}$ is a Grothendieck category. We can define in a similar way a graded right $A$-module and a graded $A$-bimodule. A colour right $A$-linear map of degree $x$ is just a homogeneous right $A$-linear map of degree $x$. To establish our main results we will need the following well-known results of graded ring theory.

- If $N$ is a graded left (right) $A$-module, $N(x)$ is a graded left (right) $A$-module which coincides with $N$ as a graded left (right) $A$-module.
- An object of $\text{gr}_{-A} \mathcal{M}$ is projective (resp. flat) in $\text{gr}_{-A} \mathcal{M}$ if and only if it is projective (resp. flat) in $\mathcal{M}$, the category of left $A$-modules.
- An object of $\text{gr}_{-A} \mathcal{M}$ is free in $\text{gr}_{-A} \mathcal{M}$ if it has an $A$-basis consisting of homogeneous elements or equivalently, if it is isomorphic to some $\oplus_{i \in I} A(x_i)$, where $(x_i, i \in I)$ is a family of elements of $G$.
- An object of $\text{gr}_{-A} \mathcal{M}$ is called finitely generated if it is a quotient of a free graded module of finite rank $\oplus_{i \leq m} A(x_i)$, where the $x_i \in G$ and $m$ is a natural integer.
- Any object of $\text{gr}_{-A} \mathcal{M}$ is a quotient of a free object in $\text{gr}_{-A} \mathcal{M}$, and any projective object in $\text{gr}_{-A} \mathcal{M}$ is isomorphic in $\text{gr}_{-A} \mathcal{M}$ to a direct summand of a free object.
- An object of $\text{gr}_{-A} \mathcal{M}$ is flat in $\text{gr}_{-A} \mathcal{M}$ if and only if it is the inductive limit of finitely generated free objects in $\text{gr}_{-A} \mathcal{M}$.
- An object $\Lambda$ of $\text{gr}_{-A} \mathcal{M}$ is called finitely presented if there is an exact sequence $\oplus_{i \leq m} A(x_i) \rightarrow \oplus_{j \leq n} A(y_j) \rightarrow \Lambda \rightarrow 0$ for $x_i, y_j \in G$ and some natural integers $m$ and $n$. A finitely presented graded module is finitely generated.

**Lemma 1.2.** Let $A$ be a colour algebra and $M$ a graded left $A$-module which is generated as $A$-module by a homogeneous element $m$ of degree 0. Then $M$ is finitely generated as a graded left $A$-module.
Proof. We have $M = Am$. The $k$-linear map $f : A \to M$; $a \mapsto am$ is surjective, homogeneous of degree 0 and left $A$-linear. So $f$ is an epimorphism in $\mathcal{M}$. □

An $A$-coring $C$ is an $A$-bimodule together with two $A$-bimodule maps $\Delta_C : C \to C \otimes_A C$ and $\epsilon_C : C \to A$ such that the usual coassociativity and counit properties hold. Let $C$ be an $A$-coring. A left $C$-comodule is a left $A$-module $M$ together with a left $A$-linear map $\rho_{MC} : M \to C \otimes_A M$ such that

$$((\epsilon_C \otimes_A id_M) \circ \rho_{MC}) = id_M, \quad \text{and} \quad ((\Delta_C \otimes_A id_M) \circ \rho_{MC}) = (id_C \otimes_A \rho_{MC}) \circ \rho_{MC}.$$

For more details on corings, we refer to [1], [2], [3], [4] and [5].

An $A$-coring $C$ is called a graded $A$-coring if $C$ admits a decomposition as a direct sum of vector spaces $C = \oplus_C C_\lambda$ such that $C$ is a graded $A$-bimodule, and $\Delta_C$ and $\epsilon_C$ are graded left and right $A$-linear maps. Note that $\epsilon_C(c) = 0$ if $c$ is homogeneous of degree $|c| \neq 0$. We use the notation-type of Sweedler-Heyneman for $\Delta_C$ but we will omit the parentheses on subscripts. So for every homogeneous element $c \in C$ we will write $\Delta_C(c) = \sum c_{(1)} \otimes_A c_{(2)}$, where $\sum |c| = \sum |c_{(1)}| + |c_{(2)}| = |c|$. We have $\sum |c_{(1)}| \sum |c_{(1)}| c_{11} \otimes_A c_{12} \otimes_A c_{22} = \sum |c| \sum_{|c_{(1)}| = |c|} c_{1} \otimes_A c_{21} \otimes_A c_{22}$. Note that $\epsilon_C(c) = 0$ if $|c| \neq 0$.

A left $C$-comodule $M$ is called a graded left $C$-comodule if $M$ admits a decomposition as a direct sum of vector spaces $M = \oplus M_\lambda$ such that $\rho_{MC}$ is homogeneous of degree 0; i.e., $\rho_{MC}$ is a graded left $A$-linear map. We will write $\rho_{MC}(m) = \sum_{|m|} m_{(-1)} \otimes_A m_{(0)}$, where $\sum_{|m|} = \sum_{|m_{(-1)}| + |m_{(0)}| = |m|}$.

Any colour algebra $A$ is a graded $A$-coring called the trivial $A$-coring, and a graded $k$-coalgebra is a graded $k$-coring. A morphism of graded left $C$-comodules $f : M \to N$ is a morphism in $\mathcal{M}$ such that

$$\rho_{NC} \circ f = (id_C \otimes_A f) \circ \rho_{MC},$$

that is

$$\sum_{|m|} f(m)_{(-1)} \otimes_A f(m)_{(0)} = \sum_{|m|} m_{(-1)} \otimes_A f(m)_{(0)} \quad \forall m \in M.$$

A morphism of graded left $C$-comodule will be called a graded left $C$-colinear map. We denote by $\mathcal{C}M$ the category of graded left $C$-comodules. The morphisms of $\mathcal{C}M$ are the graded left $C$-colinear maps. The category $\mathcal{C}M$ has direct sums. If $C$ is projective as a right $A$-module, then $\mathcal{C}M$ is a Grothendieck category ([4] for the ungraded case).

**Definition 1.3.** Let $C$ be a graded $A$-coring and $M$, $N$ be objects of $\mathcal{C}M$. A homogeneous element $f \in Hom(M, N)$ is colour left $C$-colinear if $f$ is colour left $A$-linear and $\rho_{NC} \circ f = (id_C \otimes_A f) \circ \rho_{MC}$.

It follows from Definition 1.3 that a graded left $C$-colinear map is a colour left $C$-colinear map of degree 0. If $M$ and $N$ are objects of $\mathcal{C}M$ and $x \in G$, we will denote by $\mathcal{C}Hom(M, N)_x$ the vector subspace of $Hom(M, N)$ whose elements are colour left $C$-colinear of degree $x$. So we have

$$\mathcal{C}Hom(M, N)_x = \{ f \in \mathcal{A}Hom(M, N) \mid \sum_{|m|} f(m)_{(-1)} \otimes_A f(m)_{(0)} = \}.$$
We will set $\mathcal{C}HOM(M, N) = \bigoplus_{x \in G} \mathcal{C}HOM(M, N)_x$. We call $\mathcal{C}HOM(M, N)$ the subspace of colour left $\mathcal{C}$-comodule maps of $Hom(M, N)$. We can define in a similar way a graded right $\mathcal{C}$-comodule. A homogeneous colour right $\mathcal{C}$-comodule map is just a homogeneous right $\mathcal{C}$-comodule.

If $N$ is a graded left $\mathcal{C}$-comodule, then for every $x \in G$, the $x$-suspending $N(x)$ is a graded left $\mathcal{C}$-comodule which coincides with $N$ as a $\mathcal{C}$-comodule. By [17], the linear map $i_{-x} : N \to N(x)$ defined by $i_{-x}(n) = (-x/|n|)n$ is bijective and homogeneous of degree $-x$. It is obviously colour left $\mathcal{C}$-comodule.

**Lemma 1.4.** Let $\mathcal{C}$ be a graded $A$-coring, and $M$, $N$ be graded left $\mathcal{C}$-comodules. For every $x \in G$, the linear map $\mathcal{C}HOM(M, N)_x \to \mathcal{C}HOM(M, N(x))_0; f \mapsto i_{-x} \circ f$, where $i_{-x}$ is defined above is an isomorphism of vector spaces.

**Lemma 1.5.** Let $P$ be an object of $gr^{-\mathcal{C}}\mathcal{M}$. Then the functor $\mathcal{C}HOM(P, -) : gr^{-\mathcal{C}}\mathcal{M} \to gr_{-k}\mathcal{M}$ is left exact.

**Proof.** Let $0 \to L \to M \to N \to 0$ be an exact sequence in $gr^{-\mathcal{C}}\mathcal{M}$; so $0 \to L(x) \to M(x) \to N(x) \to 0$ is exact in $gr^{-\mathcal{C}}\mathcal{M}$ for every $x \in G$. By [13, Corollary 1.2.2], $P$ is projective in $gr_{-k}\mathcal{M}$. So the sequence

$$0 \to HOM(P, L(x))_0 \to HOM(P, M(x))_0 \to HOM(P, N(x))_0 \to 0$$

is exact for every $x \in G$. It follows from Lemma 1.4 that

$$0 \to HOM(P, L)_x \to HOM(P, M)_x \to HOM(P, N)_x \to 0$$

is an exact sequence for every $x \in G$. We know that $i \circ f \in \mathcal{C}HOM(P, M)_x$ for all $f \in \mathcal{C}HOM(P, L)_x$. This means that the sequence

$$0 \to \mathcal{C}HOM(P, L)_x \to \mathcal{C}HOM(P, M)_x \to \mathcal{C}HOM(P, N)_x$$

is exact for every $x \in G$; i.e.,

$$0 \to \mathcal{C}HOM(P, L) \to \mathcal{C}HOM(P, M) \to \mathcal{C}HOM(P, N)$$

is an exact sequence. So the functor $\mathcal{C}HOM(P, -)$ is left exact. \qed

We say that an object $P$ of $gr^{-\mathcal{C}}\mathcal{M}$ is projective if the functor $\mathcal{C}HOM(P, -)_0$ is exact.

**Lemma 1.6.** Let $\mathcal{C}$ be a graded $A$-coring. An object $P$ of $gr^{-\mathcal{C}}\mathcal{M}$ is projective in $gr^{-\mathcal{C}}\mathcal{M}$ if and only if the functor $\mathcal{C}HOM(P, -)$ is exact.

**Proof.** Assume that $\mathcal{C}HOM(P, -)$ is exact in $gr^{-\mathcal{C}}\mathcal{M}$. Let $0 \to L \to M \to N \to 0$ be an exact sequence in $gr^{-\mathcal{C}}\mathcal{M}$. So the sequence $0 \to \mathcal{C}HOM(P, L) \to \mathcal{C}HOM(P, M) \to \mathcal{C}HOM(P, N) \to 0$ is exact. It follows that the sequence $0 \to \mathcal{C}HOM(P, L)_0 \to \mathcal{C}HOM(P, M)_0 \to \mathcal{C}HOM(P, N)_0 \to 0$ is exact. This means
that \( P \) is a projective object in \( \mathcal{M} \). Assume that \( P \) is projective in \( \mathcal{M} \).

Let \( 0 \to L \to M \to N \to 0 \) be an exact sequence in \( \mathcal{M} \). Clearly, \( 0 \to L(x) \to M(x) \to N(x) \to 0 \) is an exact sequence in \( \mathcal{M} \) for every \( x \in G \). By the projectivity of \( P \), the sequence

\[
0 \to \text{HOM}(P, L(x))_0 \to \text{HOM}(P, M(x))_0 \to \text{HOM}(P, N(x))_0 \to 0
\]

is exact for every \( x \in G \). Using Lemma 1.4, we get that the sequence \( 0 \to \text{HOM}(P, L) \to \text{HOM}(P, M) \to \text{HOM}(P, N) \to 0 \) is exact. \( \square \)

Let us consider \( A \) as a graded right \( A \)-module. By [13], \( \text{HOM}_A(C_A, A_A) = \bigoplus_x \text{HOM}_A(C_A, A_A)_x \) is a graded vector space: it is the largest graded vector space contained in \( \text{Hom}_A(C_A, A_A) \). We write \( C^*_x = \text{HOM}_A(C_A, A_A)_x \) and \( C^* = \text{HOM}_A(C_A, A_A) \). Then \( C^* \) is a colour algebra called the graded right dual of \( C \) (see [5, 7.8] for the ungraded case): the multiplication is defined by \( f \# g = ([f], [g])g \circ (f \otimes_A \text{id}_C) \circ \Delta_C \); i.e., \( f \# g(c) = \sum_{\epsilon}[c](f|\epsilon)[g(f(c_1)c_2) \right) \) for all colour right \( A \)-linear maps \( f, g \): \( C \to A \) and homogeneous element \( c \in C \); where \( \Delta_C(c) = \sum_{\epsilon}c_1 \otimes_A c_2 \). The unit of \( C^* \) is \( e_C \) and there is a morphism of colour algebras \( i: A^p \to C^* \) defined by \( i(a)(c) = ([a], [c])e_C(c)a \). We will denote by \( \mathcal{M}_{gr-C^*} \) the category of graded right \( C^* \)-modules. Any graded left \( C \)-comodule \( M \) is a graded right \( C^* \)-module: the action is defined by \( m.f = \sum_{[m]}([m_{(-1)}], [m])f(m_{(-1)})\).

If \( C \) is projective as a right \( A \)-module, then \( \mathcal{M}_{gr-C^*} \) is a full subcategory of \( \mathcal{M}_{gr-C} \); i.e., \( \text{HOM}(M, N) = \text{HOM}_{C^*}(M, N) \) for any \( M, N \in \mathcal{M}_{gr-C^*} \). As a consequence, an object of \( \mathcal{M}_{gr-C} \) that is projective in \( \mathcal{M}_{gr-C^*} \) is projective in \( \mathcal{M}_{gr-C^*} \). Another consequence is that if \( M \) and \( N \) are objects of \( \mathcal{M}_{gr-C^*} \) with \( M \) finitely generated as a right \( C^* \)-module, then \( \text{HOM}(M, N) = \text{HOM}_{C^*}(M, N) = \text{Hom}_{C^*}(M, N) \).

Given two graded left \( C \)-comodules \( \Lambda \) and \( N \), the graded vector space \( \text{HOM}(\Lambda, N) \) is a graded left module over the colour endomorphism ring \( B = \text{END}(\Lambda) \) of \( \Lambda \): the action is given by \( b.f = ([b], [f])((f \circ b); \forall f \in \text{HOM}(\Lambda, N), b \in B \). This defines a functor \( F' = \text{HOM}(\Lambda, -) : \mathcal{M}_{gr-C} \to \mathcal{M}_{gr-B} \). Let us consider \( \Lambda \) as a graded right \( B \)-module by \( \lambda.b = ([\Lambda], [b])b(\lambda) \). So \( \Lambda \) is a graded \( (A, B) \)-bimodule.

For any \( P \in \mathcal{M}_{gr-B} \), \( \Lambda \otimes_B P \) is a graded left \( C \)-comodule with the coaction \( \rho_{P, \Lambda} = \rho_{\Lambda, \Lambda} \otimes_B \text{id}_P \).

**Lemma 1.7.** Let \( \Lambda \) and \( N \) be graded left \( C \)-comodules and \( P \) be a graded left \( B \)-module. For every \( x \in G \), the canonical linear map

\[
\phi : \text{HOM}(\Lambda \otimes_B P, N)_x \to \text{BOM}(P, \text{HOM}(\Lambda, N))_x
\]

defined by \( \phi(f)(p)(\lambda) = ([p], [\lambda])f(\lambda \otimes_B p) \) is an isomorphism.

**Proof.** The inverse of \( \phi \) is defined by \( \psi(g)(\lambda \otimes_B p) = ([\lambda], [p])g(p)(\lambda) \). \( \square \)

We deduce from Lemma 1.7 that \( \text{HOM}(\Lambda \otimes_B P, N)_0 \cong \text{HOM}(P, \text{HOM}(\Lambda, N))_0 \), and this means that the functor \( F' \) has the left adjoint \( F = \Lambda \otimes_B - : \mathcal{M}_{gr-B} \to \mathcal{M}_{gr-C} \). The unit of the adjunction is given by the graded \( k \)-linear map

\[
u_N : N \to \text{HOM}(\Lambda, \Lambda \otimes_B N), n \mapsto [\lambda \mapsto ([n], [\lambda])(\lambda \otimes n)]
\]
for $N \in _{gr-B}M$, while the counit is given by the graded $k$-linear map (the evaluation map)

$$c_M : \Lambda \otimes_B \cdots \text{HOM}(\Lambda, M) \to M; \lambda \otimes f \mapsto (|\lambda|, |f|) f(\lambda)$$

for $M \in _{gr-C}M$. The adjointness property means that we have

$$F'(c_M) \circ u_{F'(M)} = id_{F'(M)}, \; c_{F(N)} \circ F(u_N) = id_{F(N)}; \; M \in _{gr-C}M, \; N \in _{gr-B}M. \; (\ast)$$

2. The main results

Let $A$ be a colour algebra and $C$ a graded $A$-coring. We keep the notations of the preceding sections.

**Lemma 2.1.** Let $\Lambda$ and $N$ be graded left $C$-comodules. Set $B = \cdots \text{END}(\Lambda)$. For every $x \in G$, we have

1. $\cdots \text{HOM}(\Lambda, N(x)) = \cdots \text{HOM}(\Lambda, N)(x)$
2. $\Lambda \otimes_B B(x) = \Lambda(x)$.

An object $\Lambda \in _{gr-C}M$ is called semi-quasiprojective if the functor $\cdots \text{HOM}(\Lambda, -) : _{gr-C}M \to _{gr-A}M$ sends an exact sequence of the form $\oplus_i \Lambda(x_i) \to \oplus_j \Lambda(x_j) \to N \to 0$ to an exact sequence (see [15]). A projective object in $_{gr-C}M$ is semi-quasiprojective in $_{gr-C}M$.

**Lemma 2.2.** Assume that $C$ is projective as a right $A$-module. Let $\Lambda$ be a graded left $C$-comodule and set $B = \cdots \text{END}(\Lambda)$. Then the functor $\cdots \text{HOM}(\Lambda, -)$ commutes with

1. direct sums if $\Lambda$ is finitely generated as a graded right $C^*$-module,
2. direct limits if $\Lambda$ is finitely presented as a graded right $C^*$-module.

**Proof.** (2) We know that $_{gr-C}M$ is a Grothendieck category so the functor $\cdots \text{HOM}(\Lambda, -)_0$ preserves direct limits. We also know from Lemma 1.4 that $\text{HOM}_{C^*}(\Lambda, N)_x = \text{HOM}_{C^*}(\Lambda, N(x))_0$ for every $x \in G$. Let $(N_i)_{i \in I}$ be a directed system of right graded $C^*$-modules. It is easy to show that $(\lim_i N_i)(x) = \lim_i (N_i(x))$ for every $x \in G$. Now the result follows from the fact direct limit commutes with direct sum. \qed

**Lemma 2.3.** Assume that $C$ is projective as a right $A$-module. Let $\Lambda$ be a graded left $C$-comodule that is finitely generated as a graded right $C^*$-module, and let $B = \cdots \text{END}(\Lambda)$. For every index set $I$,

1. $c_{\oplus_i \Lambda(x_i)}$ is an isomorphism for every $x_i \in G$;
2. $u_{\oplus_i B(x_i)}$ is an isomorphism for every $x_i \in G$;
3. if $\Lambda$ is semi-quasiprojective in $_{gr-C}M$, then $u$ is a natural isomorphism; in other words, the induction functor $F = \Lambda \otimes_B (-)$ is fully faithful.
Proof. (1) By Lemma 2.1(1), \( i^\text{c} \text{HOM}(\Lambda, \Lambda)(x_i) = i^\text{c} \text{HOM}(\Lambda, \Lambda(x_i)) \) for every \( i \in I \). This implies that \( \bigoplus_I B(x_i) = \bigoplus_I i^\text{c} \text{HOM}(\Lambda, \Lambda(x_i)) \). By Lemma 2.2(1), the natural map \( \kappa : \bigoplus_I B(x_i) \rightarrow \bigoplus_I i^\text{c} \text{HOM}(\Lambda, \bigoplus_I \Lambda(x_i)) \) is an isomorphism. Lemma 2.1(2) implies that \( \Lambda \otimes_B (\bigoplus_I B(x_i)) \cong \bigoplus_I \Lambda(x_i) \). It is easy to see that this isomorphism is just \( c_{\bigoplus_I \Lambda(x_i)} \circ (id_{\Lambda} \otimes \kappa) \). So \( c_{\bigoplus_I \Lambda(x_i)} \) is an isomorphism since \( \kappa \) is an isomorphism.

(2) Putting \( M = \bigoplus_I \Lambda(x_i) \) in (\( \ast \)) and using (1), we find
\[
i^\text{c} \text{HOM}(\Lambda, c_{\bigoplus_I \Lambda(x_i)}) \circ u_{\text{HOM}(\Lambda, \bigoplus_I \Lambda(x_i))} = id_{i^\text{c} \text{HOM}(\Lambda, \bigoplus_I \Lambda(x_i))}; \text{i.e.,}
\]
\[
i^\text{c} \text{HOM}(\Lambda, c_{\bigoplus_I \Lambda(x_i)}) \circ u_{\bigoplus_I B(x_i)} = id_{\bigoplus_I B(x_i)}.
\]
From (1), \( i^\text{c} \text{HOM}(\Lambda, c_{\bigoplus_I \Lambda(x_i)}) \) is an isomorphism, hence \( u_{\bigoplus_I B(x_i)} \) is an isomorphism.

(3) Take a graded free resolution \( \bigoplus_J B(x_j) \rightarrow \bigoplus_I B(x_i) \rightarrow N \rightarrow 0 \) of a graded left \( B \)-module \( N \). Since \( u \) is natural, we have a commutative diagram
\[
\begin{array}{cccc}
\bigoplus_J B(x_j) & \longrightarrow & \bigoplus_I B(x_i) & \longrightarrow & N & \longrightarrow & 0 \\
\downarrow u_{\bigoplus_J B(x_j)} & & \downarrow u_{\bigoplus_I B(x_i)} & & \downarrow u_N & & \\
F'F(\bigoplus_J B(x_j)) & \longrightarrow & F'F(\bigoplus_I B(x_i)) & \longrightarrow & F'F(N) & \longrightarrow & 0.
\end{array}
\]
The top row is exact; the bottom row is exact, since
\[
F'F(\bigoplus_I B(x_i)) = i^\text{c} \text{HOM}(\Lambda, \Lambda \otimes_B \bigoplus_I B(x_i)) = i^\text{c} \text{HOM}(\Lambda, \bigoplus_I \Lambda(x_i))
\]
and \( \Lambda \) is semi-quasiprojective. By (2), \( u_{\bigoplus_J B(x_j)} \) and \( u_{\bigoplus_I B(x_i)} \) are isomorphisms; and it follows from the five lemma that \( u_N \) is an isomorphism. \( \square \)

We can now give equivalent conditions for the projectivity and flatness of \( P \in \text{gr}_{-B}M \).

**Theorem 2.4.** Assume that \( C \) is projective as a right \( A \)-module. Let \( \Lambda \) be a graded left \( C \)-comodule that is finitely generated as a graded right \( C^* \)-module, and let \( B = i^\text{c} \text{END}(\Lambda) \). For \( P \in \text{gr}_{-B}M \), we consider the following statements.

1. \( \Lambda \otimes_B P \) is projective in \( \text{gr}^{-c}M \) and \( u_P \) is injective;
2. \( P \) is projective as a graded left \( B \)-module;
3. \( \Lambda \otimes_B P \) is a direct summand in \( \text{gr}^{-c}M \) of some \( \bigoplus_I \Lambda(x_i) \), and \( u_P \) is bijective;
4. there exists \( Q \in \text{gr}^{-c}M \) such that \( Q \) is a direct summand of some \( \bigoplus_I \Lambda(x_i) \), and \( P \cong i^\text{c} \text{HOM}(\Lambda, Q) \) in \( \text{gr}_{-B}M \);
5. \( \Lambda \otimes_B P \) is a direct summand in \( \text{gr}^{-c}M \) of some \( \bigoplus_I \Lambda(x_i) \).
Then $(1) \Rightarrow (2) \iff (3) \iff (4) \Rightarrow (5)$.

If $\Lambda$ is semi-quasiprojective in $^{gr-C}M$, then $(5) \Rightarrow (3)$; if $\Lambda$ is projective in $^{gr-C}M$, then $(3) \Rightarrow (1)$.

Proof. $(2) \Rightarrow (3)$: If $P$ is projective as a graded left $B$-module, then we can find an index set $I$ and $P' \in ^{gr-B}M$ such that $\bigoplus I B(x_i) \cong P \oplus P'$. Then obviously

$$
\left( \bigoplus I \Lambda(x_i) \right) \cong \Lambda \otimes_B \left( \bigoplus I B(x_i) \right) \cong (\Lambda \otimes_B P) \oplus (\Lambda \otimes_B P').
$$

Since $u$ is a natural transformation, we have a commutative diagram:

\[
\begin{array}{ccc}
\bigoplus I B(x_i) & \xrightarrow{\cong} & P \oplus P' \\
\downarrow u_{\bigoplus I B(x_i)} & & \downarrow u_P \oplus u_{P'} \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{C}HOM(\Lambda, \bigoplus I \Lambda(x_i)) & \xrightarrow{\cong} & \mathcal{C}HOM(\Lambda, \Lambda \otimes_B P) \oplus \mathcal{C}HOM(\Lambda, \Lambda \otimes_B P') \\
\end{array}
\]

From the fact that $u_{\bigoplus I B(x_i)}$ is an isomorphism, it follows that $u_P$ (and $u_{P'}$) are isomorphisms.

$(3) \Rightarrow (4)$: Take $Q = \Lambda \otimes_B P$.

$(4) \Rightarrow (2)$: Let $f: \bigoplus I \Lambda(x_i) \to Q$ be a split epimorphism in $^{gr-C}M$. Then

$$
\mathcal{C}HOM(\Lambda, f): \mathcal{C}HOM(\Lambda, \bigoplus I \Lambda(x_i)) \cong \bigoplus I B(x_i) \to \mathcal{C}HOM(\Lambda, Q) \cong P
$$

is also split surjective, hence $P$ is projective as a graded left $B$-module.

$(4) \Rightarrow (5)$: If $(4)$ is true, we know from the proof of $(4) \Rightarrow (2)$ that $P$ is a direct summand of some $\bigoplus I B(x_i)$ in $^{gr-B}M$. So $\Lambda \otimes_B P$ is a direct summand of $\bigoplus I \Lambda(x_i)$.

$(1) \Rightarrow (2)$: Take an epimorphism $f: \bigoplus I B(x_i) \to P$ in $^{gr-B}M$. Then

$$
F(f) = id_{\Lambda} \otimes_B f: \Lambda \otimes_B \left( \bigoplus I B(x_i) \right) \cong \bigoplus I \Lambda(x_i) \to \Lambda \otimes_B P
$$

is an epimorphism in $^{gr-C}M$, and it splits since $\Lambda \otimes_B P$ is projective in $^{gr-C}M$.

Consider the commutative diagram:

\[
\begin{array}{ccc}
\bigoplus I B(x_i) & \xrightarrow{f} & P \\
\downarrow u_{\bigoplus I B(x_i)} & & \downarrow u_P \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{C}HOM(\Lambda, \bigoplus I \Lambda(x_i)) & \xrightarrow{F(f)} & \mathcal{C}HOM(\Lambda, \Lambda \otimes_B P) \xrightarrow{\cong} 0 \\
\end{array}
\]
The bottom row is split exact, since any functor, in particular, \(\mathcal{C} \text{HOM}(\Lambda, -)\) preserves split exact sequences. By Lemma 2.3(2), \(u_{\otimes_B M} \) is an isomorphism. A diagram chasing tells us that \(u_P\) is surjective. By assumption, \(u_P\) is injective, so \(u_P\) is bijective. We deduce that the top row is isomorphic to the bottom row, and therefore splits. Thus \(P\) is projective in \(\mathcal{C}r-B\mathcal{M}\).

Under the assumption that \(\Lambda\) is semi-quasiprojective in \(\mathcal{C}r-\mathcal{M}\), (5) \(\Rightarrow\) (3) follows from Lemma 2.3(3).

(3) \(\Rightarrow\) (1): By (3), \(\Lambda \otimes_B P\) is a direct summand of some \(\oplus_i \Lambda(x_i)\). If \(\Lambda\) is projective in \(\mathcal{C}r-\mathcal{M}\), then \(\oplus_i \Lambda(x_i)\) is projective in \(\mathcal{C}r-\mathcal{M}\). So \(\Lambda \otimes_B P\) being a direct summand of a projective object of \(\mathcal{C}r-\mathcal{M}\) is projective in \(\mathcal{C}r-\mathcal{M}\). \(\square\)

**Theorem 2.5.** Assume that \(\mathcal{C}\) is projective as a right \(A\)-module. Let \(\Lambda\) be a graded left \(\mathcal{C}\)-comodule that is finitely presented as a graded right \(\mathcal{C}^*\)-module, and let \(B = \mathcal{C} END(\Lambda)\). For \(P \in \mathcal{C}r-B\mathcal{M}\), the following assertions are equivalent.

1. \(P\) is flat as a graded left \(B\)-module;
2. \(\Lambda \otimes_B P = \lim Q_i\), where \(Q_i \cong \oplus_{j \leq n_i} B(x_{ij})\) in \(\mathcal{C}r-\mathcal{M}\) for some positive integer \(n_i\), and \(u_P\) is bijective;
3. \(\Lambda \otimes_B P = \lim Q_i\), where \(Q_i \in \mathcal{C}r-\mathcal{M}\) is a direct summand of some \(\oplus_{j \in I_i} \Lambda(x_{ij})\) in \(\mathcal{C}r-\mathcal{M}\), and \(u_P\) is bijective;
4. there exists \(Q = \lim Q_i \in \mathcal{C}r-\mathcal{M}\), such that \(Q_i \cong \oplus_{j \leq n_i} \Lambda(x_{ij})\) for some positive integer \(n_i\) and \(\mathcal{C} HOM(\Lambda, Q) \cong P\) in \(\mathcal{C}r-B\mathcal{M}\);
5. there exists \(Q = \lim Q_i \in \mathcal{C}r-\mathcal{M}\), such that \(Q_i\) is a direct summand of some \(\oplus_{j \in I_i} \Lambda(x_{ij})\) in \(\mathcal{C}r-\mathcal{M}\), and \(\mathcal{C} HOM(\Lambda, Q) \cong P\) in \(\mathcal{C}r-B\mathcal{M}\).

If \(\Lambda\) is semi-quasiprojective in \(\mathcal{C}r-\mathcal{M}\), these conditions are also equivalent to conditions (2) and (3), without the assumption that \(u_P\) is bijective.

**Proof.** (1) \(\Rightarrow\) (2): \(P = \lim N_i\), with \(N_i = \oplus_{j \leq n_i} B(x_{ij})\) for some positive integer \(n_i\). Take \(Q_i = \oplus_{j \leq n_i} \Lambda(x_{ij})\), then
\[
\lim Q_i \cong \lim (\Lambda \otimes_B N_i) \cong \Lambda \otimes_B \lim N_i \cong \Lambda \otimes_B P.
\]

Consider the following commutative diagram:
\[
\begin{array}{ccc}
P = \lim N_i & \xrightarrow{\lim(u_{N_i})} & \lim \mathcal{C} HOM(\Lambda, \Lambda \otimes_B N_i) \\
\downarrow u_P & & \downarrow f \\
\mathcal{C} HOM(\Lambda, \Lambda \otimes_B (\lim N_i)) & \xrightarrow{\cong} & \mathcal{C} HOM(\Lambda, \lim (\Lambda \otimes_B N_i)).
\end{array}
\]

By Lemma 2.3(2), the \(u_{N_i}\) are isomorphisms; by Lemma 2.2, the natural homomorphism \(f\) is an isomorphism. Hence \(u_P\) is an isomorphism.
(2) \Rightarrow (3) and (4) \Rightarrow (5) are obvious.

(2) \Rightarrow (4) and (3) \Rightarrow (5): Put \( Q = \Lambda \otimes_B P \). Then \( u_P : P \to {}^c \text{HOM}(\Lambda, \Lambda \otimes_B P) \) is the required isomorphism.

(5) \Rightarrow (1): We have a split exact sequence \( 0 \to N_i \to P_i = \oplus_{j \in I_i} \Lambda(x_{ij}) \to Q_i \to 0 \) in \( gr-\mathcal{M} \). Consider the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & FF'(N_i) & \longrightarrow & FF'(P_i) & \longrightarrow & FF'(Q_i) & \longrightarrow & 0 \\
& & c_{N_i} & & c_{P_i} & & c_{Q_i} & \\
0 & \downarrow & N_i & \downarrow & P_i & \downarrow & Q_i & \downarrow & 0.
\end{array}
\]

We know from Lemma 2.3(1) that \( c_{P_i} \) is an isomorphism. Both rows in the diagram are split exact, so it follows that \( c_{N_i} \) and \( c_{Q_i} \) are also isomorphisms. Next consider the commutative diagram:

\[
\begin{array}{cccc}
\Lambda \otimes_B \lim {}^c \text{HOM}(\Lambda, Q_i) & \overset{id \otimes f}{\longrightarrow} & \Lambda \otimes_B {}^c \text{HOM}(\Lambda, Q) \\
\uparrow h & & \downarrow c_Q \\
\lim(\Lambda \otimes_B {}^c \text{HOM}(\Lambda, Q_i)) & \overset{\lim c_{Q_i}}{\longrightarrow} & Q
\end{array}
\]

where \( h \) and \( f \) are the natural homomorphisms. \( h \) is an isomorphism, because \( \Lambda \otimes_B (\cdot) \) preserves inductive limits; by Lemma 2.2, \( f \) is an isomorphism; and \( \lim c_{Q_i} \) is an isomorphism because every \( c_{Q_i} \) is an isomorphism. It follows that \( c_Q \) is an isomorphism, hence \( {}^c \text{HOM}(\Lambda, c_Q) \) is an isomorphism. From (\( \star \)), we get

\[ {}^c \text{HOM}(\Lambda, c_Q) \circ u_{\text{HOM}(\Lambda, Q)} = id_{\text{HOM}(\Lambda, Q)}. \]

It follows that \( u_{\text{HOM}(\Lambda, Q)} \) is also an isomorphism. Since \( {}^c \text{HOM}(\Lambda, Q) \cong P \), \( u_P \) is an isomorphism. Consider the isomorphisms

\[ P \cong {}^c \text{HOM}(\Lambda, \Lambda \otimes_B P) \cong {}^c \text{HOM}(\Lambda, \Lambda \otimes_B {}^c \text{HOM}(\Lambda, Q)) \cong {}^c \text{HOM}(\Lambda, Q) \cong \lim {}^c \text{HOM}(\Lambda, Q_i); \]

where the first isomorphism is \( u_P \), the third is \( {}^c \text{HOM}(\Lambda, c_Q) \) and the last one is \( f \). It follows from Lemmas 2.1(1) and 2.2 that \( {}^c \text{HOM}(\Lambda, P_i) \cong \oplus_{j \in I_i} B(x_{ij}) \) is projective as a graded left \( B \)-module, hence \( {}^c \text{HOM}(\Lambda, Q_i) \) is also projective as a graded left \( B \)-module, and we conclude that \( P \) is flat in \( \mathcal{M} \). The final statement is an immediate consequence of Lemma 2.3(3). \( \square \)
3. Applications

3.1. \( \mathcal{C} \) contains a grouplike element

A grouplike element of \( \mathcal{C} \) is an element \( X \in \mathcal{C}_0 \) such that \( \Delta_c(X) = X \otimes_A X \) and \( \epsilon_c(X) = 1_A \) (see [14]). If \( \mathcal{C} \) contains a grouplike element \( X \), then \( A \) is an object of \( \mathcal{C} \)-comodule: the \( \mathcal{C} \)-coaction is defined by \( \rho_{A,c}(a) = aX = aX \otimes_A 1_A; \forall a \in A \). Conversely, if \( A \) is an object of \( \mathcal{C} \), then \( \rho_{A,c}(1_A) = X \) is a grouplike element of \( \mathcal{C} \).

Assume that \( \mathcal{C} \) contains a grouplike element \( X \). Then \( A \) is an object of \( \mathcal{C}_{gr-\mathcal{C}} \) and \( a \cdot \epsilon_C = a \), that is, \( A \) is generated as a right \( \mathcal{C}^* \)-module by the homogeneous element \( \epsilon_C \) of degree 0. Lemma 1.2 implies that \( A \) is finitely generated in \( \mathcal{C}_{gr-\mathcal{C}} \). For any graded left \( \mathcal{C} \)-comodule \( M \), we call \( \mathit{coC}^*M = \{ m \in M, \rho_{M,c}(m) = X \otimes_A m \} \) the vector space of \( \mathcal{C} \)-coinvariants of \( M \). Clearly, \( \mathit{coC}^*A = \{ a \in A, Xa = aX \} \) is a colour subalgebra of \( A \): the colour subalgebra of \( \mathcal{C} \)-coinvariants. For every \( f \in \mathcal{C}^*HOM(A,M) \), \( f(1) \in \mathit{coC}^*M \). The graded \( k \)-linear map \( f \mapsto f(1) \) establishes an isomorphism \( \mathcal{C}^*HOM(A,M) \leftrightarrow \mathit{coC}^*M \) with inverse the graded \( k \)-linear map \( \psi \) defined by \( \psi(m)(a) = (|m|,|a|)am \). We have \( \mathcal{C}END(A) = \mathit{coC}^*A \). Set \( B = \mathit{coC}^*A \). Then we get from Theorems 2.4 and 2.5 necessary and sufficient conditions for projectivity and flatness over the colour algebra \( B = \mathit{coC}^*A \).

3.2. A colour algebra as a trivial coring

A colour algebra \( A \) is a graded \( A \)-bimodule. Let us define \( \Delta_A(a) = a \otimes_A 1_A \) and \( \epsilon_A(a) = a \). Then \( A \) is an \( A \)-coring. A graded left \( A \)-comodule is just a graded left \( A \)-module. The product on \( A^* \) is defined by \( f \# g(a) = \sum_{|a|}(|f|,|g|)g(f(a)1_A) = \sum_{|a|}(|f|,|g|)g(1_A)(f(a)) \). It is easy to show that the algebra \( A^* \) is isomorphic to \( A^{op} \), the opposite algebra of \( A \): this isomorphism is defined by \( f \leftrightarrow f(1_A) \). For graded left \( A \)-modules \( M \) and \( N \), we have \( A^*HOM(M,N) = A^*HOM(M,N) \). Then Theorems 2.4 and 2.5 give necessary and sufficient conditions for projectivity and flatness over \( B = A END(\Lambda) \), where \( \Lambda \) is a finitely generated graded left \( A \)-module. When the gradation is trivial we recover [11]. In many examples, \( \Lambda \) will be a colour algebra and \( A \) will be a graded \( \Lambda \)-ring with a graded left grouplike character.

**Definition 3.1.** (see [6], Section 2) Let \( A \) and \( \Lambda \) be two colour algebras and \( i : \Lambda \to A \) a graded ring morphism. A graded \( k \)-linear map \( \chi : A \to \Lambda \) is called a graded left grouplike character on \( A \) if \( \chi \) is graded left \( \Lambda \)-linear and

\[
\chi(a\chi(a')) = \chi(aa') \quad \text{and} \quad \chi(1_A) = 1_{\Lambda} \quad \forall \ a, a' \in A.
\]

We then say that \( A \) is a graded \( \Lambda \)-ring with a graded left grouplike character \( \chi \).

Let \( A \) be a graded \( \Lambda \)-ring with a graded left grouplike character \( \chi \). Then \( \Lambda \) is a graded left \( A \)-module: the action is given by \( a \to \lambda = \chi(a\lambda) \). Furthermore, \( \Lambda \) is cyclic as a left \( A \)-module, since \( \lambda = (1_A) \to 1_A \). But \( 1_A \) is homogeneous of degree 0, so \( \Lambda \) is a finitely generated as a graded left \( A \)-module (Lemma 1.2). So
we get necessary and sufficient conditions for projectivity and flatness over the colour endomorphism ring \( \text{END}(\Lambda) \) of \( \Lambda \).

Now we will give two examples of this situation. There are other examples in the literature.

- Let \( H \) be a colour algebra, the colour tensor product \( H \otimes H \) is the \( G \)-graded vector space \( H \otimes H = \bigoplus_{x \in G} \bigoplus_{y+z=x} H_x \otimes H_z \) with multiplication \( (h \otimes l)(h' \otimes l') = (|l|/|h'|)hh' \otimes ll' \) for homogeneous elements \( h, h', l, l' \in H \). By [9, Lemma 3.2], \( H \otimes H \) is a colour algebra. A Hopf colour algebra is a colour algebra and a graded coalgebra such that \( \Delta_H \) and \( \epsilon_H \) are morphisms of colour algebras and there exists a graded \( k \)-linear map \( S_H : H \to H \) (called antipode) such that \( (S_H \otimes id_H) \circ \Delta_H = \epsilon_H = (id_H \otimes S_H) \circ \Delta_H \) or equivalently, \( \sum_{|h|} \epsilon(h_1)h_2 = h = \sum_{|h|} h_1 \epsilon(h_2) \) and \( \sum_{|h|} S(h_1)h_2 = \epsilon(h) = \sum_{|h|} h_1 S(h_2) \).

Let \( H \) be a Hopf colour algebra over \( k \) with comultiplication \( \Delta_H \), counit \( \epsilon_H \) and antipode \( S_H \). A colour algebra \( \Lambda \) which is a graded left \( H \)-module such that \( h.(\lambda \lambda') = \sum_{|h|} (|h_2|/|\lambda|)(h_1, \lambda)(h_2, \lambda') \) for all \( h \in H \) and \( \lambda, \lambda' \in \Lambda \) will be called a graded left \( H \)-module algebra. We denote by \( A = \Lambda \# H \) the associated smash product; i.e., the colour algebra generated by \( \Lambda \) and \( H \) whose multiplication is defined by \( (\lambda h)(\lambda' h') = \sum_{|h|} (|h_2|/|\lambda'|)\lambda(h_1, \lambda)(h_2, h') \) (see [12]). A graded vector space \( M \) is a graded left \( A \)-module if and only if it is a graded left \( \Lambda \)-module and a graded left \( H \)-module such that \( h.(\lambda m) = \sum_{|h|} (|h_2|/|\lambda|)(h_1, \lambda)(h_2, m) \). Defining a \( k \)-linear map \( \chi : A \to \Lambda \) by \( \chi(h) = \epsilon_H(h)\lambda \). Since \( \epsilon_H(h) = 0 \) for \( |h| \neq 0 \), \( \chi \) is homogeneous of degree 0. Clearly, \( \chi \) is left \( \Lambda \)-linear. It follows that \( \Lambda \# H \) is a graded \( \Lambda \)-ring with a graded left grouplike character \( \chi \). Note that \( \Lambda \# H \text{END}(\Lambda) \) is exactly the colour subring of invariants of \( \Lambda \); i.e., \( \Lambda \# H \text{END}(\Lambda) = \{ \lambda \in \Lambda; h.\lambda = \epsilon_H(h)\lambda \} \).

- Assume that \( C \) contains a grouplike element \( X \). The linear map \( i : A \to C^* \) defined by \( i(a)(x) = a(e(x)) \) is a morphism of colour algebras. Define \( \chi : C^* \to A \) by \( \chi(f) = f(X) \). Then \( \chi \) is a graded left grouplike character on \( C^* \). So \( C^* \) is a graded \( A \)-ring.

### 3.3. \( C \) comes from a graded entwining structure

In this section, \( A \) is a colour algebra with multiplication \( \mu \) and unit \( \iota \), and \( C \) is a graded coalgebra with comultiplication \( \Delta_C \) and counit \( \epsilon_C \). We denote by \( \tau : A \otimes C \to C \otimes A \) the twist map; that is, \( \tau(a \otimes c) = (|a|/|c|)c \otimes a \). If \( M \) is a left (resp. right \( C \)-comodule, we write \( \rho_{M,C}(m) = m_{-1} \otimes m_0 \) (resp. \( \rho_{M,C}(m) = m_0 \otimes m_1 \)). We remind that \( C \) is a graded \( k \)-coring. Interesting examples of graded corings come from graded entwining structures. We will often refer to [5] for the ungraded case.

- Graded left-left entwined modules.

A graded left-left entwining structure over \( k \) is a triple \( (A,C,\psi) \) with a graded \( k \)-linear map \( \psi : A \otimes C \to C \otimes A \); \( a \otimes c \mapsto (|a|/|c|)c^a \otimes a_\alpha \) satisfying the following conditions [5, 32.1]:

\[
\psi \circ (\mu \otimes id_C) = (id_C \otimes \mu) \circ (\psi \otimes id_A) \circ (id_A \otimes \psi)
\]
These relations are respectively equivalent to

\[
(\Delta_C \otimes id_A) \circ \psi = (id_C \otimes \psi) \circ (\psi \otimes id_C) \circ (id_A \otimes \Delta_C) \\
(\epsilon_C \otimes id_A) \circ \psi = id_A \otimes \epsilon_C \\
\psi \circ (id_C \otimes \iota) = \iota \otimes id_C.
\]

The map \( \psi \) is called a graded entwining map, and \( A \) and \( C \) are said to be graded entwined by \( \psi \). By [5, 32.1], \( C = C \otimes A \) is a graded \( A \)-coring with \( A \)-multiplications \( a'(c \otimes a)a'' = \psi(a' \otimes c)aa'' \), coproduct

\[
\Delta_C : C \otimes A \to C \otimes A \otimes A \quad C \otimes A \cong C \otimes C \otimes A; \quad c \otimes a \mapsto \Delta_C(c) \otimes a
\]

and counit \( \epsilon_C(c \otimes a) = \epsilon_C(c)a \).

Let \( M \) be a graded left \( A \)-module. Then \( C \otimes M \) becomes a graded left \( A \)-module if we set \( a(c \otimes m) = (|c|/|a|)c^a \otimes (a_m) \). We say that a vector space \( M \) is a graded left-left \((A, C, \psi)\)-entwined module if \( M \) is a graded left \( A \)-module and a graded left \( C \)-comodule such that \( \rho_{M,C} \) is a graded left \( A \)-linear map; i.e.,

\[
\rho_{M,C}(am) = (|a|/|(m_{-1})|(m_{-1})^a \otimes (a_m)).
\]

We denote by \( \mathcal{M}_A(\psi) \) the category of graded left-left \((A, C, \psi)\)-entwined modules: its morphisms are the graded left \( A \)-linear maps and the graded left \( C \)-colinear maps. We can show that \( \mathcal{M}_A(\psi) \) is isomorphic to \( \mathcal{M}_C(\psi) \).

- Graded right-right entwined modules.

A graded right-right entwining structure over \( k \) is a triple \((A, C, \psi)\) with a graded \( k \)-linear map \( C \otimes A \to A \otimes C; \ c \otimes a \mapsto (|c|/|a|)a_\alpha \otimes c^\alpha \) satisfying the following conditions [5, 32.1]:

\[
\psi \circ (id_C \otimes \mu) = (\mu \otimes id_C) \circ (id_A \otimes \psi) \circ (\psi \otimes id_A) \\
\psi \circ (id_A \otimes \Delta_C) = (\psi \otimes id_C) \circ (id_C \otimes \psi) \circ (\Delta_C \otimes id_A) \\
\psi \circ (id_C \otimes \iota) = \iota \otimes id_C.
\]

These relations are respectively equivalent to

\[
(|c|/|(aa')_\alpha|)(|aa'|_\alpha \otimes c^\alpha) = (|c|/|a_\alpha|)(|c^\alpha'|/|a_\beta'|)(a_\alpha a_\beta' \otimes c^\alpha \beta) \\
(|c|/|a_\alpha|)(a_\alpha \Delta_C(c^\alpha)) = (|c|/|a_\alpha|)(|c^\alpha_1|/|a_\beta_2|)(a_\beta_1 \otimes c^\alpha_1 \otimes c^\alpha_2) \\
(|c|/|a_\alpha|)(a_\alpha \epsilon_C(c^\alpha)) = \epsilon_C(c)a
\]
and a graded right coring \( a \) with \( \mathcal{M}(\psi) \)-entwined modules: its morphisms are the graded right \( A \)-comodule via such that \( \rho_{M,C} \) is a graded right \( A \)-linear map; i.e.,

\[ \rho_{M,C}(ma) = ((m_1)|/|a_\alpha|)(m_0a_\alpha) \otimes (m_1)^\alpha. \]

We denote by \( \mathcal{M}(\psi)^{gr-A}_{gr-C} \) the category of graded right-right \((A,C,\psi)\)-entwined modules: its morphisms are the graded right \( C \)-comodule via such that \( \rho_{M,C} \) is a graded right \( A \)-linear map; i.e.,

\[ \rho_{M,C}(ma) = ((m_1)|/|a_\alpha|)(m_0a_\alpha) \otimes (m_1)^\alpha. \]

We denote by \( \mathcal{M}(\psi)^{gr-C}_{gr-A} \) the category of graded right-right \((A,C,\psi)\)-entwined modules: its morphisms are the graded right \( A \)-linear maps and the graded right \( C \)-comodule with \( \rho_{M,C} \).

- Graded left-right entwined modules.

A graded left-right entwining structure over \( k \) is a triple \((A,C,\psi)\) with a graded \( k \)-linear map \( A \otimes C \rightarrow A \otimes C; a \otimes c \mapsto a \alpha \otimes c^\alpha \) satisfying the following conditions [5, 32.1]:

\[
\psi \circ (\mu \otimes id_C) = (\mu \otimes id_C) \circ (id_A \otimes \tau^{-1}) \circ (\psi \circ id_A) \circ (id_A \otimes \tau) \circ (id_A \otimes \psi)
\]

\[
(id_A \otimes \Delta_C) \circ \psi = (\tau^{-1} \otimes id_C) \circ (id_C \otimes \psi) \circ (\tau \otimes id_C) \circ (\psi \circ id_C) \circ (id_A \otimes \Delta_C)
\]

\[
(id_A \otimes \epsilon_C) \circ \psi = id_A \otimes \epsilon_C
\]

\[
\psi \circ (\iota \otimes id_C) = \iota \otimes id_C.
\]

These relations are respectively equivalent to

\[
(aa')_\alpha \otimes c^\alpha = ((a'_\alpha)|/|c^\alpha|)((a_\alpha)|/|c'_{a_\alpha}|)(a_{a_\alpha}' \alpha \otimes c^\alpha')
\]

\[
a_\alpha \otimes \Delta_C(c^\alpha) = ((a_\alpha)|/|c_1^\alpha|)((a_\alpha)|/|c_{a_\alpha}|)(a_{a_\alpha} \otimes c_1^\alpha \otimes c_2^\beta)
\]

\[
a_\alpha \epsilon_C(c^\alpha) = a \epsilon_C(c)
\]

\[
1_\alpha \otimes c^\alpha = 1 \otimes c.
\]

Let \( M \) be a graded left \( A \)-module. Then \( M \otimes C \) becomes a graded left \( A \)-module if we set \( a(m \otimes c) = (|m|)|/|c|(|c^\alpha|/|m|)(a_\alpha m \otimes c^\alpha) \). A vector space \( M \) is a graded left-right \((A,C,\psi)\)-entwined module if \( M \) is a graded left \( A \)-module and a graded right \( C \)-comodule such that \( \rho_{M,C} \) is a graded left \( A \)-linear map; i.e.,

\[ \rho_{M,C}(am) = ((m_0)|/|m_1|)(|m_1|^\alpha)|/|m_0|)(a_\alpha m_0 \otimes (m_1)^\alpha). \]

We denote by \( \mathcal{M}(\psi)^{gr-C}_{gr-A} \) the category of graded left-right \((A,C,\psi)\)-entwined modules: its morphisms are the graded left \( A \)-linear maps and the graded right \( C \)-comodule with \( \rho_{M,C} \).
• Graded right-left entwined modules.

A graded right-left entwining structure over $k$ is a triple $(A, C, \psi)$ with a graded $k$-linear map $C \otimes A \to C \otimes A$; $c \otimes a \mapsto c^\alpha \otimes a_\alpha$ satisfying the following conditions [5, 32.1]:

$$
\psi \circ (id_C \otimes \mu) = (id_C \otimes \mu) \circ (\tau \otimes id_A) \circ (id_A \otimes \psi) \circ (\tau^{-1} \otimes id_A) \circ (\psi \otimes id_A)
$$

$$
(\Delta_C \otimes id_A) \circ \psi = (id_C \otimes \tau) \circ (\psi \otimes id_C) \circ (id_C \otimes \tau^{-1}) \circ (id_C \otimes \psi) \circ (\Delta_C \otimes id_A)
$$

$$
(\epsilon_C \otimes id_A) \circ \psi = \epsilon_C \otimes id_A
$$

$$
\psi \circ (id_C \otimes \iota) = id_C \otimes \iota
$$

where $\tau : C \otimes A \to A \otimes C$; $c \otimes a \mapsto (|c|/|a|)a \otimes c$. These relations are respectively equivalent to

$$
c^\alpha \otimes (aa')_\alpha = (|c^\alpha|/|a_\alpha|)(|a_\alpha|/|c'^\alpha|)(c'^\alpha \otimes a'_\alpha)
$$

$$
\Delta_C(c^\alpha) \otimes a_\alpha = (|c_2^\alpha|/|a_\alpha|)(|a_\alpha|/|c_2^\alpha|)(c_1^\alpha \otimes c_2^\alpha \otimes a_\alpha)
$$

$$
\epsilon_C(c^\alpha)a_\alpha = \epsilon_C(c)a
$$

$$
c_\alpha \otimes 1^\alpha = c \otimes 1.
$$

Let $M$ be a graded right $A$-module. Then $C \otimes M$ becomes a graded right $A$-module if we set $(c \otimes m)a = (|c|/|m|)(|m|/|c^\alpha|)(c^\alpha \otimes ma_\alpha)$. A vector space $M$ is a graded right-left $(A, C, \psi)$-entwined module if $M$ is a graded right $A$-module and a graded left $C$-comodule such that $\rho_{M,C}$ is a graded right $A$-linear map; i.e.,

$$
\rho_{M,C}(ma) = (|m_{-1}|/|m_0|)(|m_0|/|m_{-1}|^\alpha)(m_{-1})^\alpha \otimes (m_0a_\alpha).
$$

We denote by $gr^r - C \mathcal{M}(\psi)_{gr - A}$ the category of graded right-left $(A, C, \psi)$-entwined modules: its morphisms are the graded right $A$-linear maps and the graded left $C$-colinear maps.

### 3.3.1. Graded Doi-Hopf modules

In this section, $H$ is a Hopf colour algebra with a bijective antipode $S_H$, $A$ is a colour algebra and $C$ is a graded coalgebra.

We say that $A$ is a graded left $H$-comodule algebra if it is a graded left $H$-comodule via $\rho_{A,H}(a) = a_{[-1]} \otimes a_{[0]}$ such that $\rho_{A,H}(aa') = ([a_{[0]}/|a_{[-1]}|])([a_{[-1]}a_{[-1]}')(a_{[0]} \otimes a'_{[0]})$ and $\rho_{A,H}(1_A) = 1_H \otimes 1_A$. This is equivalent to say that the multiplication and the unit are graded left $H$-colinear, where the left $H$-coaction on $A \otimes A$ is defined by $(a \otimes a')_{[-1]} \otimes (a \otimes a')_{[0]} = ([a_{[0]}/|a'_{[-1]}|])([a_{[-1]}a'_{[-1]})(a_{[0]} \otimes a'_{[0]}).

We say that $A$ is a graded right $H$-comodule algebra if it is a graded right $H$-comodule via $\rho_{A,H}(a) = a_{[0]} \otimes a_{[1]}$ such that $\rho_{A,H}(aa') = ([a_{[1]}/|a'_{[0]}|])a_{[0]} \otimes a'_{[0]} \otimes (a_{[1]}a'_{[1]})$ and $\rho_{A,H}(1_A) = 1_A \otimes 1_H$. This is equivalent to say that the multiplication and the unit are graded right $H$-colinear, where the right $H$-coaction on $A \otimes A$ is defined by $(a \otimes a')_{[0]} \otimes (a \otimes a')_{[1]} = ([a_{[1]}/|a'_{[0]}|])a_{[0]} \otimes a'_{[0]} \otimes (a_{[1]}a'_{[1]})$. 
We say that $C$ is a graded left $H$-module coalgebra if $C$ is a graded left $H$-module such that $\Delta_C(h \mapsto c) = (|h_2|/|c_1|)(h_1 \mapsto c_1) \otimes (h_2 \mapsto c_2)$ and $\epsilon_C(h \mapsto c) = \epsilon_H(h)\epsilon_C(c)$. This is equivalent to say that $\Delta_C$ and $\epsilon_C$ are graded left $H$-linear, where the left $H$-action on $C \otimes C$ is defined by

$$h \mapsto (c \otimes c') = (|h_2|/|c_1|)(h_1 \mapsto c_1) \otimes (h_2 \mapsto c_2).$$

We say that $C$ is a graded right $H$-module coalgebra if $C$ is a graded right $H$-module such that $\Delta_C(c \leftarrow h) = (|c_2|/|h_1|)(c_1 \leftarrow h_1) \otimes (c_2 \leftarrow h_2)$ and $\epsilon_C(c \leftarrow h) = \epsilon_H(h)\epsilon_C(c)$. This is equivalent to say that $\Delta_C$ and $\epsilon_C$ are graded right $H$-linear, where the right $H$-action on $C \otimes C$ is defined by

$$(c \otimes c') \leftarrow h = (|c_2|/|h_1|)(c_1 \leftarrow h_1) \otimes (c_2 \leftarrow h_2).$$

- Graded left-left Doi-Hopf modules.

Let $A$ be a graded left $H$-comodule algebra and $C$ a graded left $H$-module coalgebra. According to [5], we call the triple $(H, A, C)$ a graded left-left Doi-Hopf datum.

The category $\mathcal{M}(H)_{gr-\mathcal{A}}$ of graded left-left Doi-Hopf modules is the category whose objects are the graded left $A$-modules and the graded left $C$-comodules $M$ such that $\rho_{M,C}(am) = (|a|/|m|)(a_{[1]} \rightarrow m_{[1]}) \otimes (a_{[0]}m_0)$. The morphisms of this category are the graded left $A$-linear maps and the graded left $C$-colinear maps. Any graded left-left Doi-Hopf datum $(H, A, C)$ gives rise to a graded left-left entwining structure $(A, C, \psi)$: the map $\psi$ is defined by $\psi(a \otimes c) = (|a|/|c|)(a_{[1]} \rightarrow c) \otimes a_{[0]}$. The corresponding category of graded left-left entwined modules coincides with the category $\mathcal{M}(H)_{gr-\mathcal{A}}$.

- Graded right-right Doi-Hopf modules.

Let $A$ be a graded right $H$-comodule algebra and $C$ a graded right $H$-module coalgebra. According to [5], we call the triple $(H, A, C)$ a graded right-right Doi-Hopf datum.

The category $\mathcal{M}(H)_{gr-\mathcal{A}}$ of graded right-right Doi-Hopf modules is the category whose objects are the graded right $A$-modules and the graded right $C$-comodules $M$ such that $\rho_{M,C}(am) = (|a|/|m|)(m_0a_{[0]}) \otimes (a_{[1]} \leftarrow a_{[0]})$. The morphisms of this category are the graded right $A$-linear maps and the graded right $C$-colinear maps. Any graded right-right Doi-Hopf datum $(H, A, C)$ gives rise to a graded right-right entwining structure $(A, C, \psi)$: the map $\psi$ is defined by $\psi(c \otimes a) = (|c|/|a|)(a_{[0]}) \otimes (c \leftarrow a_{[1]})$. The corresponding category of graded right-right entwined modules coincides with $\mathcal{M}(H)_{gr-\mathcal{A}}$.

- Graded left-right Doi-Hopf modules.

Let $A$ be a graded right $H$-comodule algebra and $C$ a graded left $H$-module coalgebra. According to [5], we call the triple $(H, A, C)$ a graded left-right Doi-Hopf datum.

The category $\mathcal{M}(H)_{gr-\mathcal{A}}$ of graded left-right Doi-Hopf modules is the category whose objects are the graded left $A$-modules and the graded right $C$-comodules $M$ such that $\rho_{M,C}(am) = (|a|/|m|)(a_{[0]}m_0) \otimes (a_{[1]} \rightarrow m_1)$. The
morphisms of this category are the graded left $A$-linear maps and the graded right $C$-colinear maps. Any graded left-right Doi-Hopf datum $(H, A, C)$ gives rise to a graded left-right entwining structure $(A, C, \psi)$: the map $\psi$ is defined by $\psi(a \otimes c) = a_{[0]} \otimes (a_{[-1]} \hookrightarrow c)$. The corresponding category of graded left-right entwined modules coincides with $gr_{-A} \mathcal{M}(H)_{gr-C}$.

• Graded right-left Doi-Hopf modules.

Let $A$ be a graded left $H$-comodule algebra and $C$ a graded right $H$-module coalgebra. According to [5], we call the triple $(H, A, C)$ a graded right-left Doi-Hopf datum.

The category $gr_{-C} \mathcal{M}(H)_{gr-A}$ of graded right-left Doi-Hopf modules is the category whose objects are the graded right $A$-modules and the graded left $C$-comodules $M$ such that $\rho_{M,C}(ma) = (|m_0|/|a_{[-1]}|)(m_{-1} \hookrightarrow a_{[-1]}) \otimes (m_0a_{[0]}).$ The morphisms of this category are the graded left $A$-linear maps and the graded right $C$-colinear maps. Any graded right-left Doi-Hopf datum $(H, A, C)$ gives rise to a graded right-left entwining structure $(A, C, \psi)$: the map $\psi$ is defined by $\psi(c \otimes a) = (c \leftarrow a_{[-1]}) \otimes a_{[0]}$. The corresponding category of graded right-left entwined modules coincides with $gr_{-C} \mathcal{M}(H)_{gr-A}$.

References


Received January 15, 2009