Remarks on Numerically Positive Line Bundles on Normal Surfaces

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Abstract. Let $L$ be a numerically positive Cartier divisor on a normal complete algebraic surface $X$. We prove that $L$ is ample if $g(L) \leq 1$.

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1. Introduction

Let $k$ be an algebraically closed field of char$(k) \geq 0$, which we fix as the ground field throughout the present article. Let $X$ be a normal complete algebraic surface and $L$ a Cartier divisor on $X$. Then the sectional genus $g(L)$ of $L$ defined by $g(L) = 1 + L(K_X + L)/2$, where $K_X$ is the canonical divisor of $X$. Since $L$ is Cartier, $g(L)$ is an integer.

A Cartier divisor $L$ on a normal complete algebraic surface $X$ is said to be numerically positive or nup for shortness if $LC > 0$ for any irreducible curve $C$ on $X$. It is clear that an ample Cartier divisor is nup. Examples of nup non-ample divisors on smooth projective surfaces were constructed by Mumford (see [3, p. 56]) and by Lanteri-Rondena (see [7, §3]). Nevertheless, it seems that nup non-ample Cartier divisors are very rare. In fact, Lanteri-Rondena [7] proved that a nup divisor $L$ on a smooth complex projective surface with $g(L) \leq 1$ is ample and

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gave some necessary conditions that surfaces containing nup non-ample Cartier divisors.

In the present article, we study nup Cartier divisors on normal complete algebraic surfaces and attempt to generalize some results of Lanteri-Rondena [7]. In Section 3, we prove the following result which is the main result of the present article.

**Theorem 1.1.** Let $L$ be a nup Cartier divisor on a normal complete algebraic surface. Then we have:

1. $g(L) \geq 0$.
2. If $g(L) \leq 1$, then $L$ is ample.

As easy consequences of Theorem 1.1, we obtain the following corollaries. Corollary 1.2 is a special case of [5, Theorem] (see also [8]).

**Corollary 1.2.** Let $X$ be a normal complete $\mathbb{Q}$-Gorenstein algebraic surface. Then $-K_X$ is ample if and only if it is nup.

**Corollary 1.3.** Let $L$ be a nef Cartier divisor on a normal complete $\mathbb{Q}$-Gorenstein algebraic surface $X$. Then $K_X + L$ is ample if and only if it is nup.

Throughout the present article, we employ the following notations:

- $\kappa(X)$: The Kodaira dimension of a smooth projective variety $X$,
- $\sim$: the linear equivalence of Cartier divisors,
- $\equiv$: the numerically equivalence of $\mathbb{Q}$-divisors.

**2. Preliminaries**

A semipolarized normal surface is, by definition, a pair $(X, L)$ of a normal complete algebraic surface $X$ and a nef Cartier divisor $L$ on $X$. A semipolarized normal surface $(X, L)$ is said to be a *scroll* over a smooth curve $B$ if $X$ is a $\mathbb{P}^1$-bundle over $B$ and $Lt = 1$ for a fiber $\ell$ of the ruling $p : X \to B$.

**Lemma 2.1.** Let $(X, L)$ be a scroll over a smooth curve $B$ of genus $g$. Then $g(L) = g$.

*Proof.* See [9, Lemma 3.2].

Now, let $X$ be a normal complete algebraic surface and $\pi : Y \to X$ the minimal resolution of $X$. Let $L$ be a nup Cartier divisor on $X$ and set $M := \pi^*L$. Then $M$ is nef.

**Lemma 2.2.** Let $C$ be an irreducible curve on $Y$. Then $MC = 0$ if and only if $C$ is $\pi$-exceptional.

*Proof.* Since $L$ is nup, we have $MC = L\pi_*C > 0$ provided $C$ is not $\pi$-exceptional. Hence the assertion follows.

The following lemma is a special case of [9, Theorem 1].
Lemma 2.3. With the same notation as above, the pair \((Y, M)\) satisfies one of the following:

1. \(K_Y + M\) is nef.
2. \((Y, M) \cong (X, L) \cong (\mathbb{P}^2, \mathcal{O}(r)), r = 1\) or \(2\).
3. \((Y, M)\) is a scroll over a smooth curve.

Proof. By Lemma 2.2, \((K_Y + M)\ell \geq 0\) for any \((-1)\)-curve \(\ell\) on \(Y\). Hence, by using the same argument as in [9, §2], we know that one of the assertions (2) and (3) holds true if \(K_Y + M\) is not nef. \(\square\)

3. Proofs

In this section, we prove the results stated in the introduction.

Let \(L\) be a nup Cartier divisor on a normal complete algebraic surface \(X\). Let \(\pi : Y \to X\) be the minimal resolution of \(X\) and set \(M := \pi^*L\). Then \(g(L) = g(M)\).

In Lemmas 3.1–3.3 below, we retain this situation.

Lemma 3.1. If \(g(L) \leq 0\), then \(g(L) = 0\) and \(L\) is ample.

Proof. Assume that \(g(L)(= g(M)) \leq 0\). Then \(M(K_Y + M) = 2g(M) - 2 \leq -2\), so that \(K_Y + M\) is not nef because \(M\) is nef. Hence one of the cases (2) and (3) in Lemma 1.4 takes place. In the case (2), we can easily see that \(L = M\) is ample and \(g(L) = 0\).

We consider the case (3). If \((Y, M)\) is a scroll over a smooth curve \(B\) of genus \(g\), then \(0 \leq g = g(M)\) by Lemma 2.1. So \(B \cong \mathbb{P}^1\) and \(g(M) = 0\). In particular, \(Y\) is the Hirzebruch surface \(\mathbb{F}_n\) of degree \(n(\geq 0)\). Let \(\ell\) be a fiber of the fixed ruling on \(\mathbb{F}_n\) and \(M_n\) a minimal section of \(\mathbb{F}_n\). Then \(M \sim M_n + b\ell\) for some integer \(b\).

If \(X\) is smooth, then \(0 < LM_n = b - n\). So \(b > n\) and hence \(L\) is ample by [4, Proposition V.2.20]. We assume that \(X\) is not smooth. Then \(X\) is the rational cone obtained from \(\mathbb{F}_n\) \((n \geq 2)\) by contracting the minimal section \(M_n\). Since \(MM_n = 0\), \(b = n\) and so \(L^2 = M^2 = (M_n + n\ell)^2 = n > 0\). Hence \(L\) is ample by the Nakai-Moishezon criterion. \(\square\)

Lemma 3.2. If \(g(L) = 1\) and \(K_Y + M\) is not nef, then \(L\) is ample.

Proof. By Lemmas 2.1 and 2.3, \((Y, M)\) is a scroll over a smooth elliptic curve \(B\). Then \(Y \cong \mathbb{P}_B(\mathcal{E})\), where \(\mathcal{E}\) is a normalized rank two vector bundle on \(B\). Set \(e := -\deg(\det \mathcal{E})\). Then \(e \geq -1\) by [4, p. 384]. Let \(\ell\) be a fiber of the ruling \(p : Y \to B\) and \(C_0\) a minimal section. Then \(M \equiv C_0 + b\ell\) for some integer \(b\).

Since \(M\) is nef, it follows from [9, Lemmas 1.4 and 1.5] that \(b \geq e\) (resp. \(b \geq 0\)) if \(e \geq 0\) (resp. \(e = -1\)). So, if \(e = -1\) then \(M \equiv C_0 + b\ell\) is ample by [4, Proposition V.2.21].

Assume that \(e \geq 0\). If \(b > e\), then \(M \equiv C_0 + b\ell\) is ample by [4, Proposition V.2.20]. So we may assume that \(b = e\). If \(e = 0\) then \(MC_0 = (C_0)^2 = 0\), which contradicts Lemma 2.2. So, \(e > 0\). Then \(L^2 = M^2 = (C_0 + e\ell)^2 = e > 0\) and hence \(L\) is ample by the Nakai-Moishezon criterion. \(\square\)
Lemma 3.3. If \( g(L) = 1 \) and \( KY + M \) is nef, then \( L \) is ample.

Proof. By the Nakai-Moishezon criterion, it suffices to show that \( L^2 > 0 \). Suppose that \( L^2(= M^2) = 0 \). Then \( KY + M = 0 \) because \( g(L) = 1 \). Since \( KY + M \) is nef, we have \( 0 \leq (KY + M)^2 = (KY)^2 \). If \( (KY)^2 > 0 \), then \( M \equiv 0 \) by the Hodge index theorem. This contradicts the assumption that \( L \) is nup. Hence \( (KY)^2 = 0 \).

We consider the following two cases separately.

Case 1: \( KY \) is not nef. Since \( (KY)^2 = 0 \), one of the following holds by [11, Theorem 2.1].

(i) \( Y \) is a \( \mathbb{P}^1 \)-bundle over a smooth elliptic curve \( B \).

(ii) \( Y \) contains \((-1\)-curves).

We shall consider the above two subcases separately.

Subcase 1-(i). There exists a normalized rank two vector bundle \( \mathcal{E} \) on \( B \) such that \( Y \cong \mathbb{P}_B(\mathcal{E}) \). Set \( e := -\deg(\det \mathcal{E}) \). Let \( C_0 \) be a minimal section and \( \ell \) a fiber of the ruling \( p : Y \to B \). Then \( KY \equiv -2C_0 - e\ell \). Now we write \( M \equiv aC_0 + b\ell \). Then \( KY + M \equiv (a - 2)C_0 + (b - e)\ell \). Since \( KY + M \) is nef, we have \( a \geq 2 \) by [9, Lemmas 1.4 and 1.5]. Moreover, \( 0 = M^2 = a(2b - ae) \), and so \( 2b = ae \). If \( e \geq 0 \), then \( b \geq ae \geq 0 \) because \( M \) is nef. Hence \( b = e = 0 \). However, this contradicts Lemma 2.2 because \( MC_0 = (C_0)^2 = 0 \). Assume that \( e = -1 \). By [10, Theorem 4], there exists an elliptic fibration \( f : Y \to \mathbb{P}^1 \) onto \( \mathbb{P}^1 \). By [1, Theorem 2], we have \( KY \equiv \alpha F \), where \( \alpha < 0 \) and \( F \) is a fiber of \( f \). This also contradicts Lemma 2.2 because \( MF = (1/\alpha)MKY = 0 \) and \( F^2 = 0 \). Thus we know that Subcase 1-(i) does not take place.

Subcase 1-(ii). We prove the following claim.

Claim. \( Y \) is a rational surface.

Proof. Since \( (KY)^2 = 0 \) and \( Y \) is not relatively minimal, \( Y \) is either a rational surface or a surface of general type. If \( Y \) is of general type, then \( KY \) is numerically equivalent to an effective \( \mathbb{Q} \)-divisor \( H \). Since \( HM = KYM = 0 \), \( (KY)^2 = H^2 < 0 \) by Lemma 2.2. This is a contradiction. \( \square \)

Since \( MKY = M^2 = (KY)^2 = 0 \) and \( KY \) is not nef, it follows from Lemma 2.2 that \( h^0(Y, mM) = h^2(Y, mM) = 0 \) for any integer \( m > 0 \). So \( \chi(mM) = -h^1(Y, mM) \leq 0 \). On the other hand, by the Riemann-Roch theorem and the claim as above, we have

\[
\chi(mM) = \frac{1}{2} mM(mM - KY) + \chi(O_Y) = \chi(O_Y) = 1.
\]

This is a contradiction. Thus we know that Subcase 1-(ii) does not take place.

Case 2: \( KY \) is nef. Since \( (KY)^2 = 0 \), \( Y \) is a minimal surface of \( \kappa(Y) = 0 \) or 1.

Subcase 2-(i): \( \kappa(Y) = 1 \) (cf. Subcase 1-(i)). By the classification theory of smooth projective surfaces in any characteristic (cf. [12]), there exists an elliptic or quasi-elliptic fibration \( f : Y \to B \) onto a smooth projective curve \( B \). By [1, Theorem 2], we have \( KY \equiv \alpha F \), where \( \alpha > 0 \) and \( F \) is a general fiber of \( f \). Then \( MF =
\( (1/\alpha) MK_Y = 0 \) and \( F^2 = 0 \). This contradicts Lemma 2.2. Thus we know that Subcase 2-(i) does not take place.

Subcase 2-(ii): \( \kappa(Y) = 0 \). By using the same argument as in Subcase 1-(ii) and by [1, p. 25], we may assume that \( \chi(O_Y) = 0 \). Namely, \( Y \) is an abelian, hyperelliptic, or quasi-hyperelliptic surface (cf. [12], [1] and [2]). Then \( Y \) contains no irreducible curves with negative self-intersection number. So \( X \sim Y \). However, this contradicts Lemma 3.4 below. Thus we know that Subcase 2-(ii) does not take place.

\[ \square \]

The following lemma is proved in [7, Lemma (2.2)] in the case \( \text{char}(k) = 0 \). Almost all the part of the proof of [7, Lemma (2.2)] works in arbitrary characteristic.

**Lemma 3.4.** Let \( S \) be a minimal smooth projective surface of \( \kappa(S) = 0 \). Then every nup divisor on \( S \) is ample.

**Proof.** If \( \chi(O_S) > 0 \), then the assertion can be verified by using the same argument as Subcase 1-(ii) in the proof of Lemma 3.3. So we may assume further that \( \chi(O_S) = 0 \), i.e., \( S \) is an abelian, hyperelliptic, or quasi-hyperelliptic surface.

If \( S \) is an abelian surface, then the assertion can be verified by using the same argument as in the proof of [14, Proposition 1.4]. For the reader’s convenience, we reproduce the proof. Let \( L \) be a nup divisor on \( S \). Given a point \( x \in S \), \( T_x : S \to S \) denotes translation by \( x \) according to the group law. Define

\[
\phi_L : S \to \text{Pic}(S)
\]

as \( \phi_L(x) := T_x^*(L) \otimes L^{-1} \). The connected component \( Z \) of \( K(L) \) passing through the origin is a subgroup scheme. Since \( L \) is nup, we know that \( \dim Z = 0 \), \( K(L) \) is finite, and \( \deg \phi_L > 0 \). Since \( \chi(L) = (1/2)L^2 \) by the Riemann-Roch theorem, and also \( \chi(L)^2 = \deg \phi_L \) (cf. [13, p. 150]), it follows that \( L^2 \neq 0 \). Hence \( L \) is ample.

We treat the case where \( S \) is a hyperelliptic or quasi-hyperelliptic surface. Then the Albanese variety \( \text{Alb}(S) \) of \( S \) is an elliptic curve and the Albanese map

\[
f : S \to \text{Alb}(S)
\]

is an elliptic or quasi-elliptic fibration (see [1, Proposition]). Moreover, there exists a second structure \( g : S \to \mathbb{P}^1 \) of \( S \) as an elliptic surface over \( \mathbb{P}^1 \) by [1, Theorem 3]. Let \( F \) (resp. \( G \)) be a fiber of \( f \) (resp. \( g \)). Then \( FG > 0 \). Since \( b_2(S) = 2 \), \( \{F, G\} \) is a basis of \( \text{NS}(S) \otimes \mathbb{Q} \). Let \( L \) be a nup divisor on \( S \). Then \( L \equiv aF + bG \) for some \( a, b \in \mathbb{Q} \). Since \( L \) is nup, we have \( a = LG/FG > 0 \) and \( b = LF/FG > 0 \). So, \( L^2 = 2abFG > 0 \) and hence \( L \) is ample.

\[ \square \]

Theorem 1.1 is thus verified.

**Proof of Corollary 1.2.** Let \( n \) be a positive integer such that \( nK_X \) is Cartier. Assume that \( -K_X \) is nup non-ample. Then \( g(-nK_X) = 1 \) since \( (-K_X)^2 = 0 \). This contradicts Theorem 1.1.

\[ \square \]
Proof of Corollary 1.3. Assume that $K_X + L$ is nup non-ample. Then $(K_X + L)^2 = 0$. Since $(K_X + L)L \geq 0$, we have $(K_X + L)K_X \leq 0$. Then,

$$g(K_X + L) = \frac{1}{2}(K_X + L)K_X + 1 \leq 1.$$  

This contradicts Theorem 1.1. □

References


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