STRUCTURAL THEOREMS FOR FAMILIES OF FOURIER HYPERFUNCTIONS

B. STANKOVIĆ

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Abstract. A structural characterization of convergent and bounded families of Fourier hyperfunctions is given.

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1. Introduction

Let \( \{f_h; \ h \in H\} \) be a family of Fourier hyperfunctions which is convergent or bounded. It is of interest for the theory or applications to know whether this family can be given by a unique differential operator \( J(D) \) and a family of continuous or smooth functions \( \{p_h; \ h \in H\} \) such that \( f_h = J(D)p_h, \ h \in H, \) where \( \{p_h; \ h \in H\} \) is convergent or bounded but in some space of functions.

This kind of results for distributions one can find already by Schwartz [9] and in [1], [4], [5], [8] for ultradistributions. In [2] some results have been proved which relate to convergent sequences of hyperfunctions with supports belonging to a compact set \( K. \) In [6], [7] convergent sequences of Fourier hyperfunctions have been treated and in [11], Fourier hyperfunctions having
the $S$-asymptotics. In this paper we prove a theorem for any convergent or bounded net without new conditions, which generalizes the results in [6], [7] and [11].

2. Notation and definitions

Let $\mathcal{O}$ be the sheaf of analytic functions defined on $\mathbb{C}^n$.

We denote by $\mathbb{D}^n$ the radial compactification of $\mathbb{R}^n$, and supply it with the usual topology. The sheaf $\mathring{\mathcal{O}}_{\mathbb{D}^n}^{-\delta}$, $\delta \geq 0$, on $\mathbb{D}^n + i\mathbb{R}^n$ is defined as follows: For any open set $U \subset \mathbb{D}^n + i\mathbb{R}^n$, and $\delta \geq 0$, $\mathring{\mathcal{O}}_{\mathbb{D}^n}^{-\delta}(U)$ consists of those elements $F$ of $\mathcal{O}(U \cap \mathbb{C}^n)$ which satisfy $|F(z)| \leq C_{V, \varepsilon} \exp(-\langle \delta - \varepsilon \rangle \text{Re}z)$ uniformly for any open set $V \subset \mathbb{C}^n$, $V \subset U$, and for every $\varepsilon > 0$. By $\mathring{\mathcal{O}}$ we denote the sheaf on $\mathbb{D}^n + i\mathbb{R}^n$, $\mathring{\mathcal{O}}(U) = \mathring{\mathcal{O}}^0(U)$. The derived sheaf $\mathcal{H}_{\mathbb{D}^n}(\mathring{\mathcal{O}})$, denoted by $\mathcal{Q}$, is called the sheaf of Fourier hyperfunctions. It is a flabby sheaf on $\mathbb{D}^n$.

Let $I$ be a convex neighbourhood of $0 \in \mathbb{R}^n$ and $U_j = \{(\mathbb{D}^n + iI) \cap \{\text{Im}z_j \neq 0\}\}$, $j = 1, ..., n$. The family $\{\mathbb{D}^n + iI, U_j; j = 1, ..., n\}$ gives a relative Leray covering for the pair $\{(\mathbb{D}^n + iI), (\mathbb{D}^n + iI) \setminus \mathbb{D}^n\}$ relative to the sheaf $\mathcal{O}$. Thus

$$\mathcal{Q}(\mathbb{D}^n) = \mathring{\mathcal{O}}((\mathbb{D}^n + iI)\#\mathbb{D}^n) / \sum_{j=1}^{n} \mathring{\mathcal{O}}((\mathbb{D}^n + iI)\#_j\mathbb{D}^n),$$

where $(\mathbb{D}^n + iI)\#\mathbb{D}^n = U_1 \cap ... \cap U_n$ and $(\mathbb{D}^n + iI)\#_j\mathbb{D}^n = U_1 \cap ... \cap U_{j-1} \cap U_{j+1} \cap ... \cap U_n$. Similarly, $\mathring{\mathcal{Q}}_{\mathbb{D}^n}^{-\delta}$, $\delta > 0$ is defined using $\mathring{\mathcal{O}}_{\mathbb{D}^n}^{-\delta}$ instead of $\mathring{\mathcal{O}}$ (cf. Definition 8.2.5. in [3]).

We shall use the notation $\Lambda$ for the set of $n-$vectors with entry $\{-1, 1\}$; the corresponding open orthants in $\mathbb{R}^n$ will be denoted by $\Gamma_{\sigma}$, $\sigma \in \Lambda$. A global section $f = [F] \in \mathcal{Q}(\mathbb{D}^n)$ is defined by $F \in \mathcal{O}((\mathbb{D}^n + iI)\#\mathbb{D}^n); F = (F_{\sigma}; \sigma \in \Lambda)$, where $F_{\sigma} \in \mathring{\mathcal{O}}(\mathbb{D}^n + iI_{\sigma}), I_{\sigma} = I \cap \Gamma_{\sigma}, \sigma \in \Lambda$. $F$ is the defining function for $f$.

Recall the topological structure of $\mathcal{Q}(\mathbb{D}^n)$. Let $f = [F]$, and $K$ be a compact set in $\mathbb{R}^n$ then by $P_{K, \varepsilon}(F) = \sup_{z \in \mathbb{R}^n+iK} |F(z)| \exp(-\varepsilon |\text{Re}z|), \varepsilon > 0, K \subset I \setminus \{0\}$, is defined as the family of semi-norms in $\mathring{\mathcal{O}}((\mathbb{D}^n + iI)\#\mathbb{D}^n)$; $\mathring{\mathcal{O}}((\mathbb{D}^n + iI)\#\mathbb{D}^n)$ is a Fréchet and Montel space, as well as the quotient space $\mathcal{Q}(\mathbb{D}^n)$ with the family of semi-norms $p_{K, \varepsilon}([F]) = \inf_{G} P_{K, \varepsilon}(F + G)$, where $G$ belongs to the denominator in (1). In $\mathcal{Q}(\mathbb{D}^n)$ a weak bounded set
is bounded. We associate to \( f = [F^*] \)
\[
f(x) \cong \sum_{\sigma \in \Lambda} F_{\sigma}(x + i\Gamma_{\sigma}0), \quad F_{\sigma} \in \tilde{O}(D^n + iI_{\sigma}), \quad F_{\sigma} = \text{sgn}\sigma F^*_\sigma. \tag{2}\]

Let \( P_* = \text{ind lim}_{I \geq 0} \text{ind lim}_{\delta \downarrow 0} \tilde{O}^{-\delta}(D^n + iI). \) \( P_* \) and \( Q(D^n) \) are topologically dual to each other ([3, Theorem 8.6.2]).

The Fourier transform on \( Q(D^n) \) is defined by the use of functions \( \chi_{\sigma} = \chi_{\sigma_1}...\chi_{\sigma_n} \), where \( \sigma_k = \pm 1, \ k = 1,..., n, \) \( \sigma = (\sigma_1,...,\sigma_n) \) and \( \chi_1(t) = e^t/(1 + e^t), \chi_{-1}(t) = 1/(1+e^t), \ t \in \mathbb{R}. \) Let \( f \) be given by (2). The Fourier transform of \( f \) is defined by
\[
\mathcal{F}(f) \cong \sum_{\sigma \in \Lambda} \sum_{\delta \in \Lambda} \mathcal{F}(\chi_{\delta}F_{\sigma})(\xi - i\Gamma_{\delta}0), \tag{3}\]
where \( \mathcal{F}(\chi_{\delta}F_{\sigma}) \in \tilde{O}(D^n - iI_{\delta}) \) and \( \mathcal{F}(\chi_{\delta}F_{\sigma})(z) = O(e^{-w|z|}) \) for a suitable \( w > 0 \) along the real axis outside the closed \( \sigma \)-orthant (cf. Proposition 8.3.2 in [3]).

A function \( v \) defined on \( \mathbb{R}^n \) (on \( C^n \)) is of infra-exponential type if for every \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that \( |v(z)| \leq C_\varepsilon e^{\varepsilon|z|}, \ z \in \mathbb{R}^n \) \( (z \in C^n) \).

A local operator \( J(D) = \sum b_{\alpha}D^\alpha \) with \( \lim_{|\alpha| \rightarrow \infty} \sqrt{|\alpha|}b_{\alpha}/|\alpha| = 0 \) acts on \( Q(D^n) \) as a sheaf homomorphism and continuously on \( Q(D^n) \).

3. Main results

**Theorem 1.** Let \( f_h = [F^*_h] \in Q(D^n), \) \( F^*_h \in \tilde{O}((D^n + iI)^\#D^n), \) \( h \in H. \) If:
   a) The net \( \{f_h\}_{h \in H} \) converges in \( Q(D^n) \) or
   b) \( \{f_h; h \in H\} \) is a bounded set in \( Q(D^n) \).

Then there exist an elliptic local operator \( J(D) \) and nets of functions \( \{q_{h,s}\}_{h \in H}, \ s \in \Lambda, \) such that:

1. \( q_{h,s}(x), h \in H, \ s \in \Lambda, \) are smooth functions and of exponential type on \( \mathbb{R}^n. \)

2. \( q_{h,s}(z) \in \tilde{O}(D^n + iI_s), \ s \in \Lambda, \ h \in H, \) where \( I_s, \ s \in \Lambda, \) does not depend on \( h \in H. \)

3. \( f_h = J(D) \sum_{s \in \Lambda} q_{h,s}(x + i\varepsilon s), \) \( h \in H, \) \( 0 < \varepsilon \leq \varepsilon_0. \)

4. There exists \( \varepsilon_0 > 0 \) such that for any compact sets \( K_1 \subset \subset \mathbb{R}^n \) and \( K_2 \subset \subset (0,\varepsilon_0) : \)
In case a) nets \( \{q_{h,s}(x + i\epsilon s)\}_{h \in H}, s \in \Lambda \) converge uniformly in \( x \in K_1 \) and \( \epsilon \in K_2 \).

In case b) sets \( \{q_{h,s}(x + i\epsilon s)\}_{h \in H}, s \in \Lambda \), are uniformly bounded for \( x \in K_1 \) and \( \epsilon \in K_2 \).

Proof. The idea of the proof is the same as in [11]. Let \( f_h = [F_h^\ast] \) be given by (2) and their Fourier transform by (3). Let \( \varphi \) be a monotone increasing continuous, positive valued function \( \varphi(r), r \geq 0 \), which satisfies \( \varphi(0) = 1, \varphi(r) \to \infty, r \to \infty \).

By Lemma 1.2 in [2] there exists an elliptic local operator \( J(D) \) whose Fourier transform \( J(\zeta) \) satisfies the estimate:

\[
|J(\zeta)| \geq C \exp(|\zeta|/\varphi(|\zeta|)), \quad |\text{Im}\zeta| \leq 1.
\]  

(4)

By (4), \( J^{-2}(\zeta) \in \tilde{O}(\mathbb{D}^n + i\{||\mu|| < 1\}) \). Denote by \( g = F^{-1}(1/J^2) \). By Theorem 8.2.6 in [3], \( g \in \mathcal{Q}^{-1}(\mathbb{D}^n) \). Consequently \( \delta = J_0(D)g, J_0 = J^2, \) and

\[
f_h = J_0(D)(g \ast f_h), \quad h \in H.
\]  

(5)

By the properties of the Fourier transform, cited properties of \( \chi_{\tilde{\sigma}}, \tilde{\sigma} \in \Lambda \), and supposition on \( F_h^\ast, h \in H \), we have for every \( h \in H : \)

(a) \( \mathcal{F}(F_{h,\sigma}\chi_{\tilde{\sigma}})J^{-2}(\zeta) \in \tilde{O}(x - iI_{\tilde{\sigma}}) \) and decreases exponentially outside any cone containing \( \Gamma_{\sigma} \) as a proper subcone.

(b) \( \mathcal{F}(F_{h,\sigma}\chi_{\tilde{\sigma}})J^{-2}\chi_{\tilde{s}} \in \tilde{O}(x - iI_{\tilde{s}}) \) and decreases exponentially outside any cone containing \( \Gamma_{\sigma} \) and \( \Gamma_{s} \) as proper subcones.

(c) \( \mathcal{F}^{-1}(\mathcal{F}(F_{h,\sigma}\chi_{\tilde{\sigma}})J^{-2}\chi_{\tilde{s}}) \in \tilde{O}(x + i(I_{\sigma} \cup I_s)) \) and decreases exponentially outside any cone containing \( \Gamma_{\tilde{\sigma}} \) as a proper subcone. We shall use these properties considering Fourier hyperfunctions \( f_h \ast g, h \in H \), given in (5).

The analysis of \( f_h \ast g \) is very similar to the analysis of \( f \ast g \) in [11]. However we give it because of the integrity of the proof.

\[
f_h \ast g = \mathcal{F}^{-1}(\mathcal{F}(f_h)\mathcal{F}(g))
\]

\[
\cong \frac{1}{(2\pi)^n} \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda_{\mathbb{R}^n}} \int e^{iz_{\sigma} \zeta_\sigma} \mathcal{F}(\chi_{\sigma} F_{h,\sigma})(\zeta_\sigma)/J^2(\zeta_\sigma) d\xi, h \in H,
\]

where \( \zeta_\sigma = \xi + i\eta_\sigma, \eta_\sigma \in -I_{\tilde{\sigma}} \) and \( z_\sigma \in \mathbb{R}^n + iI_{\sigma} \).

For fixed \( \sigma \), for all \( \tilde{\sigma} \in \Lambda \) and \( z_\sigma \in \mathbb{R}^n + iI_{\tilde{\sigma}} \)

\[
S_{h,\sigma,\tilde{\sigma}}(z_\sigma) = \frac{1}{(2\pi)^n} \int \mathbb{R}^n e^{iz_{\sigma} \zeta_\sigma} \mathcal{F}(\chi_{\tilde{\sigma}} F_{h,\sigma})(\zeta_\sigma)/J^2(\zeta_\sigma) d\xi;
\]
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\[ |S_{h,\sigma,\tilde{\sigma}}(z_\sigma)| \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-x_\sigma \eta - y_\sigma \xi} |\mathcal{F}(\chi_{\tilde{\sigma}} F_{h,\sigma})(\zeta_{\tilde{\sigma}})| J^2(\zeta_{\tilde{\sigma}}) d\xi, h \in H. \]

One can see that \( S_{h,\sigma,\tilde{\sigma}}(z_\sigma), h \in H, \) are continuables to the real axis. The obtained functions \( S_{h,\sigma,\tilde{\sigma}}(x) \) are continuous and of infra exponential type on \( \mathbb{R}^n. \) By Lemma 8.4.7 in [3], \( S_{h,\sigma,\tilde{\sigma}}(x) \cong S_{h,\sigma,\tilde{\sigma}}(x + i\Gamma_\sigma 0), \tilde{\sigma} \in \Lambda, h \in H \) and

\[ (f_h * g)(x) = \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} S_{h,\sigma,\tilde{\sigma}}(x), h \in H. \]  \( (6) \)

The functions \( S_{h,\sigma,\tilde{\sigma}}(z_\sigma) \) can be written in the following form

\[ S_{h,\sigma,\tilde{\sigma}}(z_\sigma) = \frac{1}{(2\pi)^n} \sum_{s \in \Lambda} \int_{\mathbb{R}^n} e^{iz_\sigma \zeta_{\tilde{\sigma}}} \mathcal{F}(\chi_{\tilde{\sigma}} F_{h,\sigma})(\zeta_{\tilde{\sigma}}) \chi_s(\zeta_{\tilde{\sigma}}) J^2(\zeta_{\tilde{\sigma}}) d\xi, h \in H. \]

Denote by

\[ S_{h,\sigma,\tilde{\sigma},s}(z_\sigma) = \frac{1}{(2\pi)^n} \sum_{s \in \Lambda} \int_{\mathbb{R}^n} e^{iz_\sigma \zeta_{\tilde{\sigma}}} \mathcal{F}(\chi_{\tilde{\sigma}} F_{h,\sigma})(\zeta_{\tilde{\sigma}}) \chi_s(\zeta_{\tilde{\sigma}}) J^2(\zeta_{\tilde{\sigma}}) d\xi, h \in H. \]

Functions \( S_{h,\sigma,\tilde{\sigma},s}(z_\sigma), \sigma, \tilde{\sigma}, s \in \Lambda, h \in H, \) are also continuables to the real axis and the obtained functions \( S_{h,\sigma,\tilde{\sigma},s}(x) \) are continuous and of infra exponential type on \( \mathbb{R}^n. \) Moreover, for every \( h \in H \)

\[ S_{h,\sigma,\tilde{\sigma},s}(x) \cong S_{h,\sigma,\tilde{\sigma},s}(x + i\Gamma_\sigma 0) \quad \text{and} \quad S_{h,\sigma,\tilde{\sigma}}(x) = \sum_{s \in \Lambda} S_{h,\sigma,\tilde{\sigma},s}(x). \]  \( (7) \)

Let us analyse the functions

\[ I_{s,\epsilon}(\zeta) = J^{-2}(\zeta) e^{-\epsilon s \zeta} \chi_s(\zeta), \zeta \in \mathbb{R}^n + i\{|\eta| < 1\}, \]

where \( 0 < \epsilon < 1. \) These functions are elements of \( P_* \) because of

\[ |I_{s,\epsilon}(\zeta)| = |J^{-2}(\zeta)| \exp(-\epsilon \sum_{i=1}^n s_i \xi_i) \prod_{i=1}^n |\chi_{s_i}(\xi_i)| \]

\[ \leq |J^{-2}(\zeta)| \prod_{i=1}^n |\chi_{s_i}(\xi_i)| \exp(-\epsilon s_i \xi_i) \]

\[ \leq C \exp(-\epsilon \sum_{i=1}^n |\xi_i|), \quad |\eta| < 1, \zeta = \xi + i\eta, s \in \Lambda. \]
Therefore, \( I_{s,\epsilon} \in \tilde{O}^{-\epsilon}(D^n + i\{|\eta| < 1\}) \), \( s \in \Lambda \). Since the Fourier transform maps \( P_* \) onto \( P_* \), there exists \( \psi_{s,\epsilon} \in P_* \) such that \( \mathcal{F}(\psi_{s,\epsilon}) = I_{s,\epsilon} \), \( s \in \Lambda \). By Proposition 8.2.2 in [3],

\[
\psi_{s,\epsilon} \in \tilde{O}^{-1}(D^n + i\{|\eta| < \epsilon\}), \ s \in \Lambda. \tag{8}
\]

Denote by

\[
q_{h,s}(x) = \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} S_{h,\sigma,\tilde{\sigma},s}(x)
\]

\[
\cong \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} \mathcal{F}^{-1}(\mathcal{F}(F_{h,\sigma} \chi_{\tilde{\sigma}})J^{-2}\chi_s)(x + i(\Gamma_\sigma \cup \Gamma_s)0), s \in \Lambda, h \in H. \tag{9}
\]

Let us prove that the functions \( q_{h,s}, s \in \Lambda, h \in H \) have properties 1. - 4. cited in Theorem.

Property 1 follows from (9) and (c). Property 2 is satisfied because of (6) and (7). Property 3 follows by (5), (6) and (9). It remains only the property 4. Let us prove it.

If \( f_h \in Q(D^n), h \in H \), and \( \varphi \in P_* \), then, because of the supposition on \( F_{h}^\prime, h \in H, f_h \ast \varphi \in \tilde{O}(D^n + iI') \) (cf. [10]), where \( I' \) is an interval containing zero. We shall use this fact and the properties of the functions \( I_{s,\epsilon} \), we analysed.

For a fixed \( s \in \Lambda \) and \( h \in H \) there exists \( \epsilon_0 > 0 \), such that \( \epsilon s \) belongs to all infinitesimal wedges of the form \( R^n + i(\Gamma_\sigma \cup \Gamma_s)0 \) which appear in (9). For \( \epsilon, \ 0 < \epsilon \leq \epsilon_0 \) we have

\[
q_{h,s}(x + i\epsilon s) = \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} \frac{1}{(2\pi)^n} \int_{R^n} e^{ix + i\epsilon s} \mathcal{F}(F_{h,\sigma} \chi_{\tilde{\sigma}})(\zeta_{\tilde{\sigma}})J^{-2}(\zeta_{\tilde{\sigma}})\chi_s(\zeta_{\tilde{\sigma}})d\xi
\]

\[
= \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} \int_{R^n} e^{ix} \mathcal{F}(F_{h,\sigma} \chi_{\tilde{\sigma}})(\zeta_{\tilde{\sigma}})\mathcal{F}(\psi_{s,\epsilon})(\zeta_{\tilde{\sigma}})d\xi \tag{10}
\]

\[
= \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} (F_{h,\sigma} \chi_{\tilde{\sigma}}) * \psi_{s,\epsilon}(x) = ((F_{h,\sigma}) * \psi_{s,\epsilon})(x)
\]

\[
= (f_h * \psi_{s,\epsilon})(x) = \langle f_h(t), \psi_{s,\epsilon}(x - t) \rangle, \ s \in \Lambda, h \in H
\]

Now, 4. a) and 4. b) follows from (10).
REFERENCES


Institute of Mathematics
University of Novi Sad
Trg Dositeja Obradovića 4
21000 Novi Sad
Yugoslavia