THE CIRCLE AND DIVISOR PROBLEM

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Abstract. New proofs for the classical bounds

\[ P(x) \ll x^{1/3}, \quad \Delta(x) \ll x^{1/3} \log x \]

are given. Here \( P(x) \) denotes the error term in the classical circle, and \( \Delta(x) \) in the classical divisor problem.

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Recently S.D. Miller and W. Schmid [7] gave a proof of the bound

\[ P(x) \ll \varepsilon x^{1/3+\varepsilon} \]  \hspace{1cm} (1)

in the classical circle problem. Here as usual \( P(x) = \sum_{n \leq x} r(n) - \pi x, \ r(n) \)
is the number of representations of the natural number \( n \) as a sum of two integer squares, and \( \varepsilon > 0 \) denotes arbitrarily small numbers. The bound (1) was obtained by the use of the Voronoi summation formula

\[
\sum_{a \leq n \leq b} r(n) f(n) = \pi \int_a^b f(x) \, dx + \sum_{n=1}^\infty r(n) \int_a^b f(x) J_0(2\pi \sqrt{xn}) \, dx,
\]

\hspace{1cm} (2)
where \(f(x)\) is a suitable smooth function, \(J\) is the Bessel function, and \(\sum'\) denotes that at \(n = a\) and \(n = b\) the summand is to be halved if \(a\) or \(b\) is an integer. In [7], (2) was proved by a two-dimensional Poisson summation formula, but it may be proved analogously like the classical Voronoi formula (see e.g., [2, Chapter 3]). Then, on p. 20, the authors say: “With more effort, one can remove \(\varepsilon\) from these bounds (i.e. (1), and the analogous bound in the divisor problem), and get Voronoi’s result

\[ P(x) \ll x^{1/3}. \]  

The aim of this note is to give a new, simple proof of (3) by using (2). We start by noting that, in view of the non-negativity of \(r(n)\), we have

\[ \sum_{n=1}^{\infty} f_-(n)r(n) \leq \sum_{X<n \leq 2X} r(n) \leq \sum_{n=1}^{\infty} f_+(n)r(n), \]  

where \(f_-(x)\) is a smooth, non-negative function supported in \([X, 2X]\) such that \(f(x) = 1\) for \(x \in [X + G, 2X - G]\) \((X \leq G \leq \sqrt{X})\), while similarly \(f_+(x)\) is supported in \([X - G, 2X + G]\) and satisfies \(f(x) = 1\) for \(x \in [X, 2X]\).

If henceforth we denote by \(f(x)\) either \(f_-(x)\) or \(f_+(x)\), then \(f^{(r)}(x) \ll_r G^{-r} (r = 0, 1, 2, \ldots)\), and by (2) we have

\[ \sum_{n=1}^{\infty} f(n)r(n) = \pi X + O(G) + \sum_{n=1}^{\infty} r(n) \int_{X-G}^{2X+G} f(x) \frac{J_0(2\pi \sqrt{\pi n})}{\sqrt{x}} \, dx. \]  

From the theory of Bessel functions we need only the relation (see e.g., N.N. Lebedev [6])

\[ \frac{d}{dz} [z^{\nu} J_{\nu}(z)] = z^{\nu} J_{\nu-1}(z) \]  

and the asymptotic expansion \((k \in \mathbb{N} \text{ is arbitrary, but fixed, and } |\arg z| \leq \pi - \varepsilon)\)

\[ J_{\nu}(z) = \left( \frac{2}{\pi z} \right)^{1/2} \cos\left( z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi \right) \left( \sum_{j=0}^{k} c_j(\nu) z^{-2j} + O(|z|^{-2k-2}) \right) \]

\[ + \left( \frac{2}{\pi z} \right)^{1/2} \sin\left( z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi \right) \left( \sum_{j=1}^{k} d_j(\nu) z^{-1-2j} + O(|z|^{-2k-1}) \right), \]

with suitable constants \(c_j(\nu), d_j(\nu)\). By using (7) and the first derivative test (see e.g., [2, Lemma 2.1]) it is seen that the integral on the right-hand
side of (5) is \( \ll X^{1/4} n^{-3/4} \), hence for \( Y \geq 2 \) we have
\[
\sum_{n \leq Y} r(n) \int_{X-G}^{2X+G} f(x)J_0(2\pi \sqrt{xn}) \, dx \ll \sum_{n \leq Y} r(n)X^{1/4} n^{-3/4} \ll (XY)^{1/4}
\]
on using partial summation and \( \sum_{n \leq x} r(n) \ll x \). Furthermore, by using (6) (with \( \nu = 1, 2 \)), performing two integrations by parts and noting that the support of \( f'' \) has measure \( \ll G \), we obtain that
\[
\sum_{n > Y} r(n) \int_{X-G}^{2X+G} f(x)J_0(2\pi \sqrt{xn}) \, dx = \sum_{n > Y} \frac{r(n)}{n^2} \int_{X-G}^{2X+G} f''(x)J_2(2\pi \sqrt{xn}) \, dx \ll \sum_{n > Y} r(n)n^{-5/4}G^{-1} X^{3/4} \ll X^{3/4} G^{-1} Y^{-1/4}.
\]
Therefore (5) yields
\[
\sum_{n=1}^{\infty} f(n)r(n) = \pi X + O(G) + O((XY)^{1/4}) + O(X^{3/4} G^{-1} Y^{-1/4}) = \pi X + O(X^{1/3})
\]
on choosing \( G = Y = X^{1/3} \). From (4) and the above estimates we have then
\[
\sum_{X < n \leq 2X} r(n) = \pi X + O(X^{1/3}), \quad (8)
\]
and (3) follows from (8) on replacing \( X \) by \( 2^{-j}X \) and summing over \( j = 1, 2, \ldots \).

An analogous reasoning gives also the classical bound
\[
\Delta(X) \ll X^{1/3} \log X, \quad \Delta(X) = \sum_{n \leq X} d(n) - X (\log X + 2\gamma - 1),
\]
in the Dirichlet divisor problem, where \( d(n) \) is the number of divisors of \( n \) and \( \gamma = -\Gamma'(1) \) is Euler’s constant. This follows with the use of the corresponding Voronoi formula (see [2, Chapter 3]) for the divisor function, instead of (2). Namely we have
\[
\sum_{a \leq n \leq b}^* d(n)f(n) = \int_a^b (\log x + 2\gamma)f(x)\,dx + \sum_{n=1}^\infty d(n) \int_a^b f(x)\alpha(xn)\,dx, \quad (9)
\]

where \(0 < a < b < \infty\), \(f(x) \in C^2[a, b]\), and in standard notation of Bessel functions

\[
\alpha(x) = 4K_0(4\pi\sqrt{x}) - 2\pi Y_0(4\pi\sqrt{x}) = -\sqrt{2}x^{-1/4} \left( \sin(4\pi\sqrt{x} - \pi/4) - (32\pi)^{-1} \cos(4\pi\sqrt{x} - \pi/4) \right) + O(x^{-5/4}).
\]

In case we are dealing with a Voronoi formula for a multiplicative function which is not necessarily non-negative, then often one can use the theorem of P. Shiu \[8\] on sums of multiplicative functions in short intervals. For example, such is the function \(a(n)\), the \(n\)-th Fourier coefficient of a holomorphic cusp form with respect to the full modular group, which is a normalized eigenfunction for the Hecke operators (see e.g., M. Jutila \[4\] for Voronoi-type formulas for these functions). More generally, Voronoi-type formulas for smooth functions \(f\) can be obtained for a wide class of arithmetic functions (e.g., the so-called \(S\)-class of A. Selberg; see Kaczorowski-Perelli \[5\] for an comprehensive account). The author \[3\] obtained such a formula for the divisor function \(d_r(n)\), generated by \(\zeta^r(s)\), \(r \in \mathbb{N}\). The key to obtaining Voronoi-type formulas in such situations is the relation

\[
\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s)G(s)\,ds = \int_0^\infty f(x)g(x)\,dx
\]

which holds (see e.g., E.C. Titchmarsh \[9\]) under suitable conditions if \(F(s)\) and \(G(s)\) are Mellin transforms of \(f(x)\) and \(g(x)\), respectively. The ideas used in this note in proving (1) work well also in the general case, enabling one to get bounds for corresponding error terms without the “\(\varepsilon\)”-factor. The simple (explicit) form of the right-hand side of (2) (and (9)-(10)) is due to the fact that

\[
\frac{2^{s-p-1}\Gamma(\frac{1}{2}s)}{\Gamma(p-\frac{1}{2}s+1)} \quad (0 < \Re s = \sigma < p + \frac{3}{2})
\]

is the Mellin transform of \(x^{-p}J_p(x)\), and a corresponding Mellin pair exists also in the case of the Voronoi formula (9). In the general case the function appearing in the summation formula is a ‘generalized’ Bessel function (see e.g., J.L. Hafner \[1\]), and its asymptotics may be found by the method developed in Hafner’s paper or by the author in \[3\].
REFERENCES


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