SETS OF COSPECTRAL GRAPHS WITH LEAST EIGENVALUE
AT LEAST −2 AND SOME RELATED RESULTS

D. CVETKOVIĆ, M. LEPOVIĆ

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Abstract. In this paper we study the phenomenon of cospectrality
in generalized line graphs and in exceptional graphs. The paper contains a
table of sets of cospectral graphs with least eigenvalue at least −2 and at
most 8 vertices together with some comments and theoretical explanations
of the phenomena suggested by the table. In particular, we prove that the
multiplicity of the number 0 in the spectrum of a generalized line graph \( L(G) \)
is at least the number of petals of the corresponding root graph \( G \).

AMS Mathematics Subject Classification (2000): 05C50
Key Words: graphs, eigenvalues, least eigenvalue, cospectral graphs

1. Introduction

The spectrum of a graph is the spectrum of its adjacency matrix. Cospec-
tral graphs are graphs having the same spectrum.

Both subjects contained in the title, cospectral graphs and graphs with
least eigenvalue −2, have been studied since very beginnings of the develop-
ment of the theory of graph spectra.

Both subjects, although present in the investigations all the time, have recently attracted special attention. In the first case it was the power of nowadays computers which enabled some investigations which were not possible in the past [18], while in the second case the reason was the constructive enumeration of maximal exceptional graphs [12].

In this paper we consider the intersection of these two subjects and study the phenomenon of cospectrality in generalized line graphs and in exceptional graphs. The paper contains a table of sets of cospectral graphs with least eigenvalue at least $-2$ and with 6, 7 or 8 vertices together with some comments and theoretical explanations of the phenomena suggested by the table.

2. Basic notions

Let $G = (V, E)$ be a simple graph with $n$ vertices. The characteristic polynomial $\det(xI - A)$ of the adjacency matrix $A$ of $G$ is called the characteristic polynomial of $G$ and denoted by $P_G(x)$. The eigenvalues of $A$ (i.e., the zeros of $\det(xI - A)$) and the spectrum of $A$ (which consists of the $n$ eigenvalues) are also called the eigenvalues and the spectrum of $G$, respectively. The eigenvalues of $G$ are usually denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$; they are real because $A$ is symmetric. Unless we indicate otherwise, we shall assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and use the notation $\lambda_i = \lambda_i(G)$ for $i = 1, 2, \ldots, n$.

Graphs with the same spectrum are called isospectral or cospectral graphs. The term "(unordered) pair of isospectral non-isomorphic graphs" will be denoted by PING. More generally, a "set of isospectral non-isomorphic graphs" is denoted by SING. A two element SING is a PING. A SING may be empty (of course, if it has no elements) or trivial (if it consists of just one graph). A graph $H$, cospectral but non-isomorphic to a graph $G$, is called a cospectral mate of $G$.

As usual, $K_n, C_n$ and $P_n$ denote respectively the complete graph, the cycle and the path on $n$ vertices. Further, $K_{m,n}$ denotes the complete bipartite graph on $m + n$ vertices. The cocktail-party graph $CP(n)$ is the unique regular graph with $2n$ vertices of degree $2n - 2$; it is obtained from $K_{2n}$ by deleting $n$ mutually non-adjacent edges.

The union of disjoint graphs $G$ and $H$ is denoted by $G \cup H$. The joint $G \triangledown H$ of (disjoint) graphs $G$ and $H$ is the graph obtained from $G$ and $H$ by joining each vertex of $G$ with each vertex of $H$. 
3. \(L\)-graphs and graphs with blossoms

Let \(L(\mathcal{L}^+ \cup \mathcal{L}^0)\) be the set of graphs whose least eigenvalue is greater than or equal to \(-2\) (greater than \(-2\), equal to \(-2\)). A graph is called an \(L\)-graph (\(\mathcal{L}^+\)-graph, \(\mathcal{L}^0\)-graph) if its least eigenvalue is greater than or equal to \(-2\) (greater than \(-2\), equal to \(-2\)).

The line graph \(L(H)\) of any graph \(H\) is defined as follows. The vertices of \(L(H)\) are the edges of \(H\) and two vertices of \(L(H)\) are adjacent whenever the corresponding edges of \(H\) have a vertex of \(H\) in common.

Interest in the study of graphs with least eigenvalue \(-2\) began with an elementary observation that line graphs have the least eigenvalue greater than or equal to \(-2\). A natural problem arose to characterize the graphs with such a remarkable property. It appeared that line graphs share this property with generalized line graphs and with some exceptional graphs.

A generalized line graph \(L(H; a_1, \ldots, a_n)\) is defined (in [19]) for graphs \(H\) with vertex set \(\{1, \ldots, n\}\) and non-negative integers \(a_1, \ldots, a_n\) by taking the graphs \(L(H)\) and \(CP(a_i)\) \((i = 1, \ldots, n)\) and adding extra edges: a vertex \(e\) in \(L(H)\) is joined to all vertices in \(CP(a_i)\) if \(i\) is an end-vertex of \(e\) as an edge of \(H\). We include as special cases an ordinary line graph \((a_1 = a_2 = \cdots = a_n = 0)\) and the cocktail-party graph \(CP(n)\) \((n = 1\) and \(a_1 = n)\). We introduce the abbreviation GLG for a generalized line graph.

Let \(a = (a_1, a_2, \ldots, a_n)\). Consider a generalized line graph \(L(G; a)\), where \(G\) is connected and \(\sum_{i=1}^{n} a_i > 0\). The root graph of \(L(G; a)\) is defined in [8] as the multigraph \(H\) obtained from \(G\) by adding \(a_i\) pendant double edges (petals) at vertex \(v_i\) for each \(i = 1, \ldots, n\). Then \(L(G; a) = L(H)\) if we understand that in \(L(H)\) two vertices are adjacent if and only if the corresponding edges in \(H\) have exactly one vertex in common.

It is convenient to reformulate slightly the concept of the root graph of a GLG.

A pendant double edge is called a petal. A blossom \(B_n\) consists of \(n\) \((n \geq 0)\) petals attached at a single vertex. An empty blossom \(B_0\) has no petals and is reduced to the trivial graph \(K_1\). A graph in which to each vertex a blossom (possibly empty) is attached is called a graph with blossoms or a \(B\)-graph. The set of \(B\)-graphs includes as a subset the set of (undirected) graphs without loops or multiple edges. A graph \(G\) is a generalized line graph if \(G = L(H)\) is the line graph of a \(B\)-graph \(H\) called the root graph of \(G\). The definition of \(L(H)\) remains as given above. We have \(L(B_n) = CP(n)\). A GLG is called a line graph if there exists a \(B\)-graph \(H\) with no petals such that \(G = L(H)\) while in the opposite case \(G\) is a proper
generalized line graph. Hence, the set of generalized line graphs is the union of two disjoint sets: the set of line graphs and the set of proper generalized line graphs.

An *exceptional* graph is a connected graph with least eigenvalue greater than or equal to $-2$ which is not a generalized line graph. A *generalized exceptional* graph is a graph with least eigenvalue greater than or equal to $-2$ in which at least one component is an exceptional graph.

An important graph invariant is the *star value* $S$ of an $L$-graph $G$. It is defined by

$$S = \frac{(-1)^n}{(n-k)!} P_G^{(n-k)}(-2) = (\lambda_1 + 2)(\lambda_2 + 2) \cdots (\lambda_k + 2),$$

where $f^{(p)}(x)$ denotes the $p$-th derivative of the function $f(x)$.

Since the characteristic polynomial of a disconnected graph $G$ is equal to the product of characteristic polynomials of its components, the star value of $G$ is the product of star values of components of $G$ as well.

In 1976 the key paper [3] by P.J.Cameron, J.M.Goethals, J.J.Seidel and E.E.Shult introduced root systems into the study of graphs with least eigenvalue $-2$. These graphs can be represented by sets of vectors at 60 or 90 degrees via the corresponding Gram matrices. Maximal sets of lines through the origin with such mutual angles are closely related to the root systems known from the theory of Lie algebras. Using such a geometrical characterization one can show that graphs in question are either generalized line graphs (representable in the root system $D_n$ for some $n$) or exceptional graphs (representable in the exceptional root system $E_8$). The main result is that an exceptional graph can be represented in the exceptional root system $E_8$. In particular, it is proved in this way that an exceptional graph has at most 36 vertices and each vertex has degree at most 28.

Much information on these problems can be found in the books [1], [4], [7], [6], [14], in the expository papers [2], [5] and in the new book [15].

### 4. Table of cospectral graphs

Before presenting some details from our table of cospectral $L$-graphs we shall give some definitions.

If the set of graphs $\{G_1, G_2, \ldots, G_k\}$ is a SING and if $G$ is any connected graph, then the set $\{G_1 \cup G, G_2 \cup G, \ldots, G_k \cup G\}$ is also a SING. Each graph in the later SING has a component isomorphic to a fixed graph (to the graph $G$).
A SING $\mathcal{S}$ is called \textit{reducible} if each graph in $\mathcal{S}$ contains a component isomorphic to a fixed graph. Otherwise, $\mathcal{S}$ is called \textit{irreducible}.

A SING is called \textit{complete} if no graph outside the SING is cospectral to graphs from the SING; otherwise the SING is called \textit{incomplete}. The SINGS whose members belong to a set $X$ of graphs are called $X$–SINGs.

The table of cospectral graphs from this paper contains irreducible SINGs in which graphs have the least eigenvalue at least $-2$ and the number of vertices $n$ is at most 8.

The next table gives some statistic of SINGs.

\begin{center}
\begin{tabular}{cccc}
\textbf{$n$} & 5 & 6 & 7 & 8 \\
all SINGs & 1 & 5 & 54 & 829 \\
\mathcal{L}$-SINGs & 1 & 5 & 32 & 198 \\
irreducible \mathcal{L}$-SINGs & 1 & 4 & 28 & 168
\end{tabular}
\end{center}

Our table contains $4 + 28 + 168 = 200$ irreducible $\mathcal{L}$-SINGs with at most 8 vertices.

Many of the SINGs from our table can be found in already published tables of graphs (cf. [17], [7], [6], [11], [13], [9]).

In the table which follows the SINGs are classified by the number of vertices and by the number of edges. Within a group with fixed numbers of vertices and edges the SINGs are classified lexicographically by their eigenvalues in non-decreasing order (first by non-increasing least eigenvalues, then by the second smallest one, etc.). For each SING, first row contains an identification number, followed by eigenvalues and the star value. Next, a row is related to each member of the SING with exceptions mentioned below. The row first contains the rows of the lower triangle of an adjacency matrix of the graph. In addition, the number of components is given followed by the numbers $c_i, i = 1, 2, 3$ where $c_i$ is the number of components with $i$ vertices for $i = 1, 2, 3$. Further we find a graph classifier: LG for line graphs, GL for proper generalized line graphs and EX for generalized exceptional graphs. For line graphs we come across a B if the root graph is bipartite and NB in the opposite case. In proper generalized line graphs the number of petals is given.

To save the space graphs from some SINGs are omitted if they appear in earlier publications. This applies to SINGs consisting of connected graphs on 7 vertices and to SINGs consisting of (connected) exceptional $\mathcal{L}^+$–graphs on 8 vertices. Deleted graphs are referred to by their identification numbers in the table of connected graphs on 7 vertices from [6] and in Table A2 of
exceptional \( L^+ \)-graphs on 8 vertices. The later table appears also in [13] as Table 1. Identification numbers appear behind the character \& for 7 vertex graphs and behind the character \# for 8 vertex graphs. A part of information on deleted graphs is given behind the mentioned identification numbers. In deleted 8 vertex graphs the star value is always equal to 1 and therefore omitted.

For a complete version of the table see [10].

**A TABLE OF COSPECTRAL GRAPHS WITH LEAST EIGENVALUE AT LEAST \(-2\)**

<table>
<thead>
<tr>
<th>Edges</th>
<th>Cospectral graphs with 6 vertices</th>
</tr>
</thead>
</table>
| 4      | 1. 1.7321  1.0000  0.0000  0.0000  -1.0000  -1.7321  12 0 01 001 0000 00 2 1 0 0 LG B
|        | 0 01 100 0001 00010 2 0 1 0 GL 1                                                             |
| 5      | 2. 2.0000  1.0000  0.0000  0.0000  -1.0000  -2.0000  48 0 01 001 0001 00 2 1 0 LG B |
|        | 1 01 010 1000 01000 1 0 0 0 GL 2                                                             |
| 7      | 3. 1.7321  1.0000  0.0000  0.0000  -1.0000  -1.7321  12 0 01 001 0000 00 2 1 0 0 LG B |
|        | 0 01 100 0001 00010 2 0 1 0 GL 1                                                             |

<table>
<thead>
<tr>
<th>Edges</th>
<th>Cospectral graphs with 7 vertices</th>
</tr>
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<tr>
<td>5</td>
<td>1. 2.0000  1.0000  0.0000  0.0000  0.0000  0.0000  -1.0000  -2.0000  96 0 01 001 0001 1000 00000 3 1 0 0 LG B</td>
</tr>
<tr>
<td></td>
<td>0 01 100 0001 00010 000100 2 0 1 0 GL 2</td>
</tr>
<tr>
<td></td>
<td>0 01 101 0001 00100 000000 2 1 0 0 GL 2</td>
</tr>
<tr>
<td>6</td>
<td>2. 2.0000  1.0000  1.0000  0.0000  0.0000  0.0000  -1.0000  -2.0000  72 0 01 010 0100 10000 00000 3 1 0 0 LG B</td>
</tr>
<tr>
<td></td>
<td>0 01 100 0001 00010 0000000 0 2 1 0 LG B</td>
</tr>
<tr>
<td></td>
<td>1 01 010 0010 10000 0000000 1 0 0 0 EX</td>
</tr>
<tr>
<td></td>
<td>3. 2.0000  1.4142  0.0000  0.0000  0.0000  0.0000  -1.4142  -2.0000  64 0 01 001 0001 10000 0000000 3 1 0 0 LG B</td>
</tr>
<tr>
<td></td>
<td>0 01 100 0001 00010 0000000 2 0 1 0 GL 2</td>
</tr>
<tr>
<td></td>
<td>1 01 010 0010 10000 0000000 1 0 0 0 GL 2</td>
</tr>
<tr>
<td>7</td>
<td>4. 2.4383  1.1386  0.6180  0.0000  0.8202  -1.6180  -1.7566  8 1 01 010 0010 10000 0000000 3 1 0 0 LG B</td>
</tr>
<tr>
<td></td>
<td>0 01 101 1000 0001 0000000 2 1 0 0 LG BB</td>
</tr>
<tr>
<td>Graph</td>
<td>Eigenvalues</td>
</tr>
<tr>
<td>-------</td>
<td>-------------</td>
</tr>
<tr>
<td>5</td>
<td>2.5616, 1.0000, 0.0000, 0.0000, 0.0000, -1.5616, -2.0000</td>
</tr>
<tr>
<td>6</td>
<td>2.7093, 1.4142, 0.1939, 0.0000, -1.0000, -1.4142, -1.9032</td>
</tr>
<tr>
<td>7</td>
<td>2.1208, 1.4626, 0.0000, 0.0000, -1.0000, -1.4626, -1.9354</td>
</tr>
<tr>
<td>8</td>
<td>2.7649, 1.2395, 0.3257, 0.0000, -1.0000, -1.3746, -1.9555</td>
</tr>
<tr>
<td>9</td>
<td>2.8136, 1.0000, 0.0000, 0.0000, -1.0000, -1.3429, -2.0000</td>
</tr>
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<td>10</td>
<td>2.7321, 1.4142, 0.0000, 0.0000, -0.7321, -1.4142, -2.0000</td>
</tr>
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<td>11</td>
<td>2.9032, 0.8061, 0.0000, 0.0000, 0.0000, -1.7093, -2.0000</td>
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<td>2.7649, 1.2395, 0.3257, 0.0000, -1.0000, -1.3746, -1.9555</td>
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<td>2.8136, 1.0000, 0.0000, 0.0000, -1.0000, -1.3429, -2.0000</td>
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<td>14</td>
<td>2.8162, 1.3666, 0.4363, 0.0000, 0.0000, -1.0000, -1.3746</td>
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<td>15</td>
<td>2.8508, 1.2892, 0.0000, 0.0000, -1.0000, -1.3746, -2.0000</td>
</tr>
</tbody>
</table>

Sets of cospectral graphs with 8 vertices

<table>
<thead>
<tr>
<th>Graph</th>
<th>Eigenvalues</th>
<th>Indices</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>1.8478, 1.4142, 0.7654, 0.0000, 0.0000, -1.8478</td>
<td>48, 10, 10, 10, 10, 10</td>
</tr>
<tr>
<td>7</td>
<td>2.0000, 1.0000, 0.0000, 0.0000, -1.0000, -1.0000</td>
<td>48, 10, 10, 10, 10, 10</td>
</tr>
</tbody>
</table>

Cospectral graphs with 8 vertices
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3. 2.0000 1.4142 0.0000 0.0000 0.0000 0.0000 -1.4142 -2.0000 128
   0 01 001 0100 0000001 00000000000 3 1 0 1 LG B
   0 01 100 0001 01000 001000 0010000 2 0 1 0 GL 2
   0 01 101 0100 000100 00100000 000000002 1 1 0 0 GL B

7 edges

4. 2.3429 1.4142 0.4707 0.0000 0.0000 -1.0000 -1.4142 -1.7757 16
   0 01 101 0011 10000 000100 0001000 2 1 0 0 LG B
   0 01 101 0011 10000 000100 0001100 2 0 1 0 GL 1
   0 01 011 1001 00010 100000 1010000 1 0 0 0 LG B
   0 01 001 10000 000001 0000010 2 0 1 0 LG B
   0 01 001 10000 000001 0000010 2 0 1 0 GL B

5. 2.0000 1.6180 0.6180 0.0000 0.0000 -0.6180 -1.6180 -2.0000 80
   0 01 001 0100 0000001 00000000001 2 0 0 0 LG B
   1 10 010 0010 01000 001000 0100000 1 1 0 0 LG 1

8 edges

6. 2.6855 1.4142 0.3349 0.0000 0.0000 -1.2713 -1.4142 -1.7757 16
   0 01 101 0011 10000 000100 0001000 2 1 0 0 LG B
   0 01 101 0011 10000 000100 0001100 2 0 1 0 GL 1
   0 01 011 1001 00010 100000 1010000 1 0 0 0 LG B
   0 01 001 10000 000001 0000010 2 0 1 0 LG B
   0 01 001 10000 000001 0000010 2 0 1 0 GL B

7 edges

10. 2.5554 1.1946 0.7799 0.0000 0.0000 -0.8911 -1.7177 -1.9210 4
   1 10 010 0010 11000 000001 0100000 1 0 0 0 GL 1
   0 01 101 0010 00010 000010 1110000 1 0 0 0 EX

11. 2.4728 1.4626 0.6180 0.0000 0.0000 -1.0000 -1.6180 -1.9354 4
   1 10 010 0010 00010 000010 1010000 1 0 0 0 GL 1
   0 01 101 0010 00010 000010 1110000 1 0 0 0 EX

12. 2.3920 1.5739 0.6852 0.2715 -0.5010 -1.0000 -1.4339 -1.9877 #6-7

13. 2.7321 1.0000 1.0000 0.0000 -0.7321 -1.0000 -1.0000 -2.0000 108
   0 01 101 0011 10000 000010 0000000 3 2 0 0 LG B
   0 01 101 0011 10000 000010 0000000 3 2 0 0 GL B

14. 2.3429 2.0000 0.4707 0.0000 -1.0000 -1.0000 -1.0000 -1.8136 16
   0 01 011 1001 00010 100000 1010000 1 0 0 0 LG 1

15. 2.5554 1.1946 0.7799 0.0000 0.0000 -0.8911 -1.7177 -1.9210 4
   1 10 010 0010 11000 000001 0100000 1 0 0 0 GL 1
   0 01 101 0010 00010 000010 1110000 1 0 0 0 EX

16. 2.6855 1.4142 0.3349 0.0000 0.0000 -1.2713 -1.4142 -1.7757 #6-7
   0 01 101 0011 10000 000100 0001000 2 1 0 0 EX

17. 2.5466 1.5596 0.6180 0.4582 -0.2004 -1.3867 -1.6180 -1.9772 #12-13

18. 2.7741 1.4323 0.7366 0.1853 -0.6028 -1.0000 -1.5415 -1.9841 #22-23

19. 2.7231 1.0000 1.0000 0.0000 -0.7321 -1.0000 -1.0000 -2.0000 108
   0 01 101 0011 10000 000010 0000000 3 2 0 0 LG B
   0 01 101 0011 10000 000010 0000000 3 2 0 0 GL B
   0 01 101 0011 10000 000010 0000000 3 2 0 0 EX

20. 2.4812 1.4142 0.6899 0.0000 0.0000 -1.1701 -1.4142 -2.0000 80
   0 01 101 1101 00000 000001 0000001 2 0 1 0 GL B
   1 10 010 0010 01000 001000 0100001 2 1 0 0 GL B

21. 2.8608 1.2541 0.6180 0.0000 0.0000 -1.1149 -1.6180 -2.0000 56
   0 01 101 1101 01001 000001 0000001 2 1 0 0 GL B
   0 01 101 1101 01001 010010 0000000 2 1 0 0 GL B
   1 10 010 0010 01000 001000 0101100 1 0 0 0 EX

22. 2.4989 1.4959 1.0000 0.4249 -0.7574 -1.0000 -1.6624 -2.0000 48
   1 10 010 0010 00010 0001010 1 0 0 0 EX

23. 2.5806 1.5143 0.7890 0.0000 0.0000 -1.0769 -1.8070 -2.0000 32
   0 01 101 1101 01000 001000 0101100 1 0 0 0 EX

24. 3.0000 1.0000 0.0000 0.0000 0.0000 0.0000 -2.0000 -2.0000 240
   0 01 101 1101 11000 000000 0000000 3 2 0 0 LG B
   1 10 010 0010 11000 001000 1010000 1 0 0 0 GL 3
Sets of cospectral graphs with least eigenvalue

<table>
<thead>
<tr>
<th>#</th>
<th>Values</th>
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<td>3.3234 1.4412 0.3679 0.0000 -1.0000 -1.0000 -1.4142 -1.6813 16</td>
</tr>
<tr>
<td>26.</td>
<td>1.8191 1.4270 1.0000 -0.4450 -0.536 0.0000 -1.0000 -1.8019 9</td>
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</tr>
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</tr>
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<td>2.7245 2.1364 0.4982 0.0000 -1.0000 -1.4310 -1.9280 4</td>
</tr>
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</tr>
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</tr>
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<td>3.0000 2.0000 0.0000 0.0000 -1.0000 -1.0000 -2.0000 80</td>
</tr>
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<td>3.2361 1.4412 0.0000 0.0000 -1.2361 -1.4142 -2.0000 64</td>
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14 edges
### Sets of cospectral graphs with least eigenvalue

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5. Some observations

Given two graphs $G$ and $H$, we shall say that $G$ is smaller than $H$ if $|V(G)| < |V(H)|$ and in the case $|V(G)| = |V(H)|$ if $|E(G)| < |E(H)|$. Any set of graphs has one or several smallest graphs in the above order of graphs. Since graphs in any SING have the same number of vertices and the same number of edges, we can compare SINGs as well in the above sense.

Cospectral $L$–graphs could be line graphs, proper generalized line graphs and (generalized) exceptional graphs in all combinations.

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Cospectral $L$–graphs could be line graphs, proper generalized line graphs and (generalized) exceptional graphs in all combinations.
The smallest PING without the limitations on the least eigenvalue, which consists of graphs $K_{1,4}$ and $C_4 \cup K_1$, is the only one with 5 vertices and belongs to $L^0$. Note that $K_{1,4}$ is a proper GLG while $C_4 \cup K_1$ is a line graph.

The first PING which appears in the table consists of disconnected graphs $K_{1,3} \cup K_2$, $P_5 \cup K_1$ and this is the smallest irreducible PING with such a property.

Although reducible SINGs should not be included in tables like our since they can easily be generated from irreducible ones, reducible SINGs are not quite uninteresting. Namely, although the reducible PINGs, for example, $\{K_{1,4} \cup K_1, C_4 \cup 2K_1\}$, $\{K_{1,3} \cup K_2 \cup K_1, P_5 \cup 2K_1\}$ have been deleted from the table, the reducible SING $\{K_{1,4} \cup K_2, C_4 \cup K_1 \cup K_2\}$ appears to be incomplete and can be extended to the triplet of cospectral graphs $\{K_{1,4} \cup K_2, C_4 \cup K_1 \cup K_2, S_6 \cup K_1\}$ which does appear in the table! (Here $S_6$ is the tree on 6 vertices with largest eigenvalue equal to 2).

PING No. 2 with 7 vertices consists of a line graph and of an exceptional graph. This is the smallest such PING. The line graph is $C_6 \cup K_1$; the other is an exceptional tree with largest eigenvalue 2 (and the least eigenvalue $-2$). In addition, these two graphs are switching equivalent.

Next we note that PING No. 10 with 8 vertices consists of a connected line graph and a generalized exceptional graph (having an isolated vertex) while in the PING No. 22 with the same number of vertices both graphs are connected one being a line graph and the other an exceptional graph. In the latter case the least eigenvalue is equal to $-2$ and one can prove that this is not possible in $L^+$-graphs.

6. The number of petals

Looking at the table of $L$–SINGs we have realized that the spectrum of a generalized line graph contains some information on the number of petals in the corresponding root graph. This observation has led to the formulation and the proof of Proposition 1 and Theorem 1.

Proposition 1. Let $G_i$, $G'_i$ be the $B$–graphs obtained from the $B$–graph $G$ by adding at vertex $i$ a pendant edge and a petal, respectively. Then we have

$$P_{L(G'_i)}(\lambda) = -2\lambda P_{L(G_i)}(\lambda) - 2\lambda^2 P_{L(G)}(\lambda).$$
Proof. Let \( x, y \) be vertices of \( L(G'_i) \) corresponding to the petal at vertex \( i \) of \( G \). In the determinant defining \( P_{L(G'_i)}(\lambda) \) subtract the entries of the \( y \)--row from the \( x \)--row and perform the same with the corresponding columns. Then develop the determinant by the \( x \)--row.

**Theorem 1.** The multiplicity of the number 0 in the spectrum of a generalized line graph \( L(G) \) is at least the number of petals of \( G \).

Proof. To each petal of \( G \) Proposition 1 can be applied, yielding a factor \( \lambda \) in the characteristic polynomial of \( L(G) \).

However, the number of petals in the root graph is not determined by the spectrum of the corresponding GLG. There are cospectral GLGs with root graphs having different number of petals. The smallest example is just the PING on 5 vertices if we consider a line graph to be a GLG with 0 petals. If we look for such an example with proper GLGs then the PING No. 11 with 7 vertices provides it. Finally, if we want both proper GLGs to be connected, then the PING No. 73 with 8 vertices does the job (2 and 3 petals).

Sometimes it is of interest to recognize a line graph among generalized line graphs.

The following forbidden subgraph characterization is an immediate consequence of the existing results.

**Proposition 2.** A generalized line graph is a line graph if and only if it does not contain, as an induced subgraph, any of graphs \( K_{1,3} \) and \( K_3 \triangleleft 2K_1 \).

Proof. Line graphs are characterized by a set \( B \) of 9 forbidden subgraphs while generalized line graphs are characterized by a set \( C \) of 31 forbidden subgraphs (cf., e.g., [15], Theorems 2.1.3 and 2.3.18). Since \( C \setminus B \) contains just \( K_{1,3} \) and \( K_3 \triangleleft 2K_1 \) we are done.

Note that \( K_{1,3} \) and \( K_3 \triangleleft 2K_1 \) are proper generalized line graphs whose root graphs contain exactly one petal. This observation together with Proposition 2 could be used to define an algorithm for determining the number of petals in the root graph of a generalized line graph.

7. Additional observations

As known (cf., e.g., [15]), if \( G = L(H) \) the \( B \)--graph \( H \) is not unique. It can happen that a line graph can be presented as a generalized line graph of a graph with petals. We call such graphs *polymorphic* generalized line graph.
graphs. There are exactly 5 connected polymorphic generalized line graphs (see [15], Theorem 2.3.4). Disconnected polymorphic GLGs either have as a component one of the 5 connected polymorphic GLGs or contain two isolated vertices since $2K_1 = L(2K_2) = L(B_1)$.

**Proposition 3.** The only regular connected proper generalized line graphs are the cocktail party graphs $CP(k)$, $k = 4, 5, \ldots$.

**Proof.** It is well-known that regular connected generalized line graphs are either line graphs or cocktail party graphs (see, for example, [15], Proposition 1.1.9). The cocktail party graphs $CP(k)$, $k = 1, 2, 3$ are polymorphic, hence line graphs. For $k = 4, 5, \ldots$ they are not line graphs and the assertion of the lemma follows.

A $B$–graph is called **bipartite** if it contains neither odd cycles nor petals.

**Theorem 2.** Let $H$ be a $B$–graph with $n$ vertices and $m$ edges. Then the multiplicity of the eigenvalue $-2$ in $L(H)$ is $m - n$ if $H$ is not bipartite and $m - n + 1$ if $H$ is bipartite.

This theorem has been proved in [16] for line graphs and in [8] for proper generalized line graphs (see Theorems 2.2.4 and 2.2.8 of [15]. The original results were formulated as two apparently non–related results. Our terminology and notation makes it possible to formulate the theorem as a unique result.

**REFERENCES**


Faculty of Electrical Engineering, University of Belgrade, P.O.Box 35–54, 11120 Belgrade, Serbia and Montenegro ecvetkod@etf.bg.ac.yu

Faculty of Sciences, University of Kragujevac, R. Domanovića 12, 34000 Kragujevac, Serbia and Montenegro lepovic@knez.uis.kg.ac.yu