NORDHAUS-GADDUM-TYPE RELATIONS FOR THE ENERGY AND LAPLACIAN ENERGY OF GRAPHS

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A b s t r a c t. Let $\overline{G}$ denote the complement of the graph $G$. If $I(G)$ is some invariant of $G$, then relations (identities, bounds, and similar) pertaining to $I(G) + I(\overline{G})$ are said to be of Nordhaus-Gaddum type. A number of lower and upper bounds of Nordhaus-Gaddum type are obtained for the energy and Laplacian energy of graphs. Also some new relations for the Laplacian graph energy are established.

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1. Introduction

In this paper we are concerned with simple graphs. Let $G$ be such a graph, and let $n$ and $m$ denote, respectively, the number of its vertices and edges. Then $G$ is said to be an $(n,m)$-graph.

The (ordinary) spectrum of $G$ is the spectrum of its adjacency matrix [6], and consists of the numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The Laplacian spectrum of $G$ is the spectrum of its Laplacian matrix [10, 11, 21, 22], and consists of the numbers $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. 
The energy of a graph $G$, denoted by $E(G)$, is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$  

This graph-spectrum-based invariant has its origin in theoretical chemistry (for details see [13, 14]), but has recently attracted the interest of mathematicians. The basic mathematical properties of graph energy can be found in the review [12], whereas some most recent mathematical studies in the papers [3, 4, 25–30, 32, 33, 35].

The Laplacian energy of a graph $G$, denoted by $LE(G)$, has been recently defined as [15]

$$LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$$

and was aimed at being the Laplacian-spectral analog of graph energy. Until now, only two papers [15, 37] are devoted to the study of Laplacian graph energy.

As usual, $\overline{G}$ will symbolize the complement of the graph $G$. The number of vertices and edges of the complement of an $(n, m)$-graph will be denoted by $\overline{n}$ and $\overline{m}$, respectively.

Nordhaus and Gaddum [23] reported bounds for the sum of the chromatic numbers of a graph and its complement. Eventually, Norhhaus-Gaddum-type relations were established for many other graph invariants [1, 2, 5, 8, 9, 16, 17, 20, 31, 34, 36]. In this paper we obtain bounds of this kind for the graph energy and Laplacian graph energy.

2. Nordhaus-Gaddum-Type Bounds for Graph Energy

Let $\overline{\lambda_1}$ be the largest eigenvalue of $\overline{G}$. Nosal [24] demonstrated that for a graph $G$ with $n$ vertices,

$$n - 1 \leq \lambda_1 + \overline{\lambda_1} < \sqrt{2n}$$

which itself is a Nordhaus-Gaddum-type relation. In connection with the right-hand side inequality in (1), it was shown in [17] that

$$\lambda_1 + \overline{\lambda_1} \leq \sqrt{\left(2 - \frac{1}{\omega} - \frac{1}{\overline{\omega}}\right) n(n-1)},$$

where $\omega$ and $\overline{\omega}$ denote the clique numbers of $G$ and $\overline{G}$, respectively.
Theorem 2.1. Let $G$ be a graph with $n$ vertices. Then
\[ E(G) + E(G^c) \geq 2(n - 1) \quad (3) \]
with equality if and only if $G$ is the complete graph $K_n$ or its complement, the empty graph (the $n$-vertex graph without edges).

Proof. We first observe that $E(G) \geq 2\lambda_1$ with equality if and only if $G$ has at most one positive eigenvalue, i.e., if $G$ is the empty graph or a complete multipartite graph [6]. Therefore,
\[ E(G) + E(G^c) \geq 2(\lambda_1 + \overline{\lambda_1}) \geq 2(n - 1). \]
If equality holds in (3), then both $G$ and $G^c$ are empty or complete multipartite graphs, and so $G$ must be the complete graph or the empty graph. Conversely, knowing the spectrum of $K_n$ and $K_n^c$, see [6], it is easily shown that (3) is an equality if $G \cong K_n$ or $G \cong K_n^c$. $\square$

In [19] it was shown that for an $(n, m)$-graph $G$,
\[ E(G) \leq \lambda_1 + \sqrt{(n - 1) \left(2m - \lambda_1^2\right)} \quad (4) \]
From this upper bound it could be deduced that [18]
\[ E(G) \leq \frac{n}{2} (\sqrt{n} + 1) \]
which immediately implies
\[ E(G) + E(G^c) \leq n (\sqrt{n} + 1) \quad . \]
In what follows we improve the latter upper bound.

Theorem 2.2. Let $G$ be a graph with $n$ vertices. Then
\[ E(G) + E(G^c) < \sqrt{2} n + (n - 1)\sqrt{n - 1} \quad . \quad (5) \]

Proof. Let $m$ and $\overline{m}$ denote, respectively, the number of edges of $G$ and $G^c$. By (4) and (1), we have
\[ E(G) + E(G^c) \leq \lambda_1 + \overline{\lambda_1} + \sqrt{(n - 1) \left(2m - \lambda_1^2\right)} + \sqrt{(n - 1) \left(2\overline{m} - \overline{\lambda_1}^2\right)} \]

Nordhaus-Gaddum-type relations for the energy and Laplacian energy of graphs
\[ \lambda_1 + \lambda_1 + \sqrt{2(n-1) \left[ 2m + 2\lambda_1 - (\lambda_1^2 + \lambda_1^2) \right]} \]
\[ \leq \lambda_1 + \lambda_1 + \sqrt{2(n-1) \left[ n(n-1) - \frac{1}{2} (\lambda_1 + \lambda_1)^2 \right]} \]
\[ < \sqrt{2} n + \sqrt{2(n-1) \left[ n(n-1) - \frac{1}{2} (n-1)^2 \right]} \]
\[ = \sqrt{2} n + (n-1) \sqrt{n-1} . \]

This completes the proof. \[\Box\]

**Remark 2.3.** Let \( G \) be an \( n \)-vertex regular graph of degree \( r \). Then (4) becomes \( E(G) \leq r + \sqrt{(n-1)r(n-r)} \) and we have

\[ E(G) + E(G) \leq n - 1 + \sqrt{(n-1) \left[ \sqrt{r(n-r)} + \sqrt{(r+1)(n-r-1)} \right]} \]
\[ \leq (n-1) \left( \sqrt{n+1} + 1 \right) \]

which for \( n \geq 6 \) is better than (5).

**Remark 2.4.** A strongly regular graph \( G \) with parameters \((n, r, \rho, \sigma)\) is an \( r \)-regular graph on \( n \) vertices, in which each pair of adjacent vertices has \( \rho \) common neighbors and each pair of non-adjacent vertices has \( \sigma \) common neighbors. If \( \sigma \geq 1 \) and \( G \) is non-complete, then the eigenvalues of \( G \) are [6] \( r \), \( s \), and \( t \), with multiplicities 1, \( m_s \), and \( m_t \), where \( s \) and \( t \) are the solutions of the equation \( x^2 + (\sigma - \rho)x + (\sigma - r) = 0 \), and \( m_s \) and \( m_t \) are determined by \( m_s + m_t = n - 1 \) and \( r + m_s s + m_t t = 0 \). If \( G \) is a strongly regular graph with parameters \((n, (n + \sqrt{n})/2, (n + 2\sqrt{n})/4, (n + 2\sqrt{n})/4)\) (for some conveniently chosen value of \( n \)), then

\[ E(G) + E(G) = \frac{n}{2} (\sqrt{n} + 1) + \frac{n}{2} (\sqrt{n} + 1) - \sqrt{n} - 2 = (n-1) (\sqrt{n} + 1) - 1 . \]

If we consider a Paley graph \( H \), which is a strongly regular graph with parameters \((n, (n-1)/2, (n-5)/4, (n-1)/4)\), then

\[ E(H) + E(H) = (n-1)(\sqrt{n} + 1) . \]
The results stated in Remark 2.4 show that the bound (5) is asymptotically tight.

**Remark 2.5.** Using (2), from the proof of Theorem 2.2, we have

\[
E(G) + E(G) \leq \sqrt{\left( 2 - \frac{1}{\omega} - \frac{1}{\omega'} \right) n(n-1) + (n-1) \sqrt{n-1}}.
\]

3. Some Properties of the Laplacian Graph Energy

Details of the theory of Laplacian graph spectra are found in the reviews [10, 11, 21, 22]. For the following consideration we need the properties:

\[ \mu_n = 0 \text{ for all graphs, and } \mu_{n-1} > 0 \text{ if and only if } G \text{ is connected.} \]

Let \( G_1 \ast G_2 \) denote the join of the graphs \( G_1 \) and \( G_2 \), i.e., the graph obtained from the disjoint union of \( G_1 \) and \( G_2 \) by adding all possible edges between vertices of \( G_1 \) and vertices of \( G_2 \).

**Theorem 3.1.** Let \( G_1 \) and \( G_2 \) be \((n,m)\)-graphs. Then

\[
LE(G_1 \ast G_2) = LE(G_1) + LE(G_2) + 2n - \frac{4m}{n}.
\]

**Proof.** Let \( \mu'_1, \mu'_2, \ldots, \mu'_n \) be the Laplacian eigenvalues of \( G_1 \) and \( \mu''_1, \mu''_2, \ldots, \mu''_n \) the Laplacian eigenvalues of \( G_2 \). Then the Laplacian eigenvalues of \( G_1 \ast G_2 \) are [22]

\[
2n, n + \mu'_1, n + \mu'_2, n + \mu'_2, \ldots, n + \mu'_n, n + \mu''_{n-1}, n + \mu''_{n-1}, 0.
\]

Note that \( G_1 \ast G_2 \) is a \((2n, 2m + n^2)\)-graph with average vertex degree \((2m + n^2)/n\). Therefore,

\[
LE(G_1 \ast G_2) = 2n + \sum_{i=1}^{n-1} \left| n + \mu'_i - \frac{2m + n^2}{n} \right| + \sum_{i=1}^{n-1} \left| n + \mu''_i - \frac{2m + n^2}{n} \right|
\]

\[
= 2n + \sum_{i=1}^{n-1} \left| \mu'_i - \frac{2m}{n} \right| + \sum_{i=1}^{n-1} \left| \mu''_i - \frac{2m}{n} \right|
\]

\[
= 2n + LE(G_1) - \frac{2m}{n} + LE(G_2) - \frac{2m}{n}.
\]

The result follows. \( \square \)
Remark 3.2. Let $G_1$ and $G_2$ be regular graphs of degrees $r'$ and $r''$, respectively, with $n'$ and $n''$ vertices, respectively. Then

$$E(G_1 \ast G_2) = E(G_1) + E(G_2) + \sqrt{(r' - r'')^2 + 4n'n'' - r' - r''}.$$ 

Let $G_1 \times G_2$ denote the Cartesian product of graphs $G_1$ and $G_2$. Then $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $(u_1, u_2)$ is adjacent to $(v_1, v_2)$ if and only if $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$, or $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$.

**Theorem 3.3.** Let $G_1$ and $G_2$ be, respectively, $(n, m_1)$– and $(n, m_2)$-graphs. Then

$$\text{LE}(G_1 \times G_2) \leq n \text{LE}(G_1) + n \text{LE}(G_2).$$

**Proof.** Let the notation be the same as in the proof of Theorem 3.1. Then the Laplacian eigenvalues of $G_1 \times G_2$ are $\mu'_i + \mu''_j$, $i, j = 1, 2, \ldots, n$.

Note that $G_1 \times G_2$ is an $(n^2, n(m_1 + m_2))$-graph with average vertex degree $(2m_1 + 2m_2)/n$. Therefore,

$$\text{LE}(G_1 \times G_2) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \mu'_i + \mu''_j - \frac{2m_1 + 2m_2}{n} \right|$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \left| \mu'_i - \frac{2m_1}{n} \right| + \left| \mu''_j - \frac{2m_2}{n} \right| \right)$$

$$= n \text{LE}(G_1) + n \text{LE}(G_2).$$

The result follows. \qed

Let $G$ be an $(n, m)$-graph. Note that $\mu_1 \geq 2m/n$. Then

$$\text{LE}(G) = \mu_1 + \sum_{i=2}^{n-1} \left| \mu_i - \frac{2m}{n} \right|.$$ 

If $G$ is not a complete graph, then $\mu_{n-1} \leq 2m/n$ \cite{7}, and therefore

$$\text{LE}(G) = \mu_1 - \frac{2m}{n} + \sum_{i=2}^{n-2} \left| \mu_i - \frac{2m}{n} \right|.$$
Theorem 3.4. Let $G$ be an $(n, m)$-graph with $n \geq 2$ and $m \geq 1$. Then $\text{LE}(G) \geq \mu_1$, with equality if and only if $G \cong K_{n/2,n/2}$, in which case, of course, $n$ must be even.

Proof. It is easy to see that $\text{LE}(G) \geq \mu_1$, with equality if and only if $n = 2$ or for $n \geq 3$, if $\mu_2 = \cdots = \mu_{n-1} = \frac{2m}{n}$. Suppose that $n \geq 3$ and $\text{LE}(G) = \mu_1$. Then by a result from [37], $G$ is a regular complete $k$-partite graph with $1 < k \leq n$. Then

$$n - \frac{n}{k} + (k-1) \frac{n}{k} = n,$$

implying $k = 2$. Thus, $G \cong K_{n/2,n/2}$. Conversely, if $G \cong K_{n/2,n/2}$, then it is easy to verify that $\text{LE}(G) = \mu_1$.

In a similar manner we arrive at

Theorem 3.5. Let $G$ be an $(n, m)$-graph, such that $n \geq 3$ and $m \geq 1$. Then

$$\text{LE}(G) \geq \mu_1 - \mu_{n-1} + \frac{2m}{n}$$

with equality if and only if $n = 3$ or for $n \geq 4$, if $\mu_2 = \cdots = \mu_{n-2} = \frac{2m}{n}$.

4. Nordhaus-Gaddum-Type Bounds for Laplacian Graph Energy

Lemma 4.1. If $G$ is not the complete graph, and has at least one edge, then $\mu_1 - \mu_{n-1} > 1$.

Proof. Since $G$ has at least one edge, $\mu_1 \geq \Delta + 1$, where $\Delta$ is the maximum vertex degree of $G$ [10, 21]. If $G$ is connected, then equality holds if and only if $\Delta = n - 1$.

Suppose that $G$ is connected. Then $\mu_1 - \mu_{n-1} \geq \Delta - 2m/n + 1 \geq 1$. If $\mu_1 - \mu_{n-1} = 1$, then $2m/n = \Delta = n - 1$ and then it would be $G \cong K_n$, a contradiction.

If $G$ is not connected, then $\mu_1 - \mu_{n-1} = \mu_1 \geq \Delta + 1 > 1$.

Theorem 4.2. Let $G$ be a graph with $n$ vertices. Then

$$\text{LE}(G) + \text{LE}(\overline{G}) \geq 2n - 2$$

with equality if and only if $G$ is isomorphic to $K_n$ or $\overline{K_n}$.

Proof. If $G$ is isomorphic to $K_n$ or $\overline{K_n}$, then it is easy to show that $\text{LE}(G) + \text{LE}(\overline{G}) = 2n - 2$. Suppose that $n \geq 3$ and that $G$ is different from
Then
\[
LE(G) + LE(\overline{G}) = \mu_1 - \mu_{n-1} + \frac{2m}{n} + \sum_{i=2}^{n-2} \left| \mu_i - \frac{2m}{n} \right|
\]
\[
+ \mu_1 - \mu_{n-1} + \frac{2m}{n} + \sum_{i=2}^{n-2} \left| n - \mu_i - \frac{2m}{n} \right|
\]
\[
\geq 2(\mu_1 - \mu_{n-1}) + n - 1 + \sum_{i=2}^{n-2} 1 = 2(\mu_1 - \mu_{n-1}) + 2n - 4.
\]

By Lemma 4.1, \( LE(G) + LE(\overline{G}) > 2n - 2 \).

**Theorem 4.3.** Let \( G \) be a graph with \( n \) vertices. Then
\[
LE(G) + LE(\overline{G}) < n \sqrt{n^2 - 1}.
\]

**Proof.** Denote by \( d_1, d_2, \ldots, d_n \) the vertex degrees of \( G \). Assume that \( n \geq 2 \). Let the auxiliary quantity \( M \) be defined as [15]
\[
M = M(G) = m + \frac{1}{2} \sum_{i=1}^{n} \left( d_i - \frac{2m}{n} \right)^2.
\]
Then
\[
M(\overline{G}) = m + \frac{1}{2} \sum_{i=1}^{n} \left( d_i - \frac{2m}{n} \right)^2.
\]
Using the fact
\[
\sum_{i=1}^{n} (d_i)^2 \leq 2(n-1)m
\]
with equality if and only if \( G \) is the empty graph or the complete graph, we have
\[
M(G) + M(\overline{G}) = \frac{1}{2} n(n-1) + \sum_{i=1}^{n} \left( d_i - \frac{2m}{n} \right)^2
\]
\[
= \frac{1}{2} n(n-1) + \sum_{i=1}^{n} (d_i)^2 - \frac{4m^2}{n}
\]
\[
\leq \frac{1}{2} n(n-1) + 2(n-1)m - \frac{4m^2}{n}
\]
\[
\leq \frac{1}{2} n(n-1) + \frac{1}{4} n(n-1)^2 = \frac{1}{4} (n-1)n(n+1).
\]
Now, because for \( n \geq 2 \) the number of edges of \( K_n \) and \( \overline{K_n} \) differs from \( n(n - 1)/4 \), we have

\[
M(G) + M(\overline{G}) < \frac{1}{4} (n - 1)n(n + 1) .
\]

(6)

In [15] it has been shown that \( LE(G) \leq \sqrt{2nM} \), which combined with (6) implies

\[
LE(G) + LE(\overline{G}) \leq \sqrt{4n [M(G) + M(\overline{G})]} < n \sqrt{n^2 - 1} .
\]

Example 4.4. Let \( G \cong K_{n/2} \cup \overline{K_{n/2}} \). Then the Laplacian eigenvalues of \( G \) are

\[
\frac{n}{2} \left( \frac{n}{2} - 1 \right) \text{ times } \quad \text{and} \quad 0 \left( \frac{n}{2} + 1 \right) \text{ times}
\]

and therefore

\[
LE(G) = \left( \frac{n}{2} - 1 \right) \frac{n + 2}{4} + \left( \frac{n}{2} + 1 \right) \frac{n - 2}{4} = \frac{1}{4} (n^2 - 4) .
\]

The Laplacian eigenvalues of \( \overline{G} \) are

\[
\frac{n}{2} \left( \frac{n}{2} \right) \text{ times } \quad \frac{n}{2} \left( \frac{n}{2} - 1 \right) \text{ times } \quad \text{and} \quad 0 \text{ (1 time )}
\]

and therefore

\[
LE(\overline{G}) = \frac{n}{2} \left( \frac{n}{2} + 2 \right) + \left( \frac{n}{2} - 1 \right) \frac{n - 2}{4} + \frac{3n - 2}{4} = \frac{1}{4} (n^2 + 2n) .
\]

This implies

\[
LE(G) + LE(\overline{G}) = \frac{1}{2} (n^2 + n - 2) .
\]

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REFERENCES

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