Aggregation on a Nonlinear Parabolic Functional Differential Equation †

Agregación en una Ecuación Diferencial Funcional No Lineal

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Abstract

In this paper we study the equation

$$u_t = \Delta [\varphi(u(x,[t/\tau])\sigma(x,t))] , \ x \in \Omega , t > 0,$$

with homogeneous Neumann boundary conditions in a bounded domain in $\mathbb{R}^n$. We show existence and uniqueness for the initial value problem, and prove some results that show the aggregating behaviour exhibited by the solutions.

Key words and phrases: parabolic equation, functional differential equation, aggregating populations.

Resumen

En este artículo estudiamos la ecuación

$$u_t = \Delta [\varphi(u(x,[t/\tau])\sigma(x,t))] , \ x \in \Omega , t > 0,$$

con condiciones de frontera homogéneas de tipo Neumann en un dominio acotado en $\mathbb{R}^n$. Probamos la existencia y unicidad del

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Introduction

In this paper we study the equation

\[ u_t = \Delta [\varphi(u(x, [t/\tau]\tau))] u(x, t)], \quad x \in \Omega, \quad t > 0 \tag{1} \]

with boundary conditions

\[ \eta \cdot \nabla [\varphi(u(x, [t/\tau]\tau))] u(x, t)] = 0, \quad x \in \partial \Omega, \quad t > 0 \tag{2} \]

and initial data

\[ u(x, 0) = u_0(x), \quad x \in \Omega. \tag{3} \]

Here \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), \( \tau > 0 \) is a constant, \([\theta]\) denotes the greatest integer less than or equal to \( \theta \) (i.e. \([\theta]\) is an integer such that \( [\theta] \leq \theta < [\theta] + 1 \)), and \( \varphi \) is a non-increasing function. This problem arises on a model for aggregating populations with migration rate \( \varphi \) and constant population. A first attempt to model aggregating behavior using partial differential equations conducts to the following equation (see D. Aronson [1]),

\[ u_t = \Delta f(u), \tag{4} \]

where \( f(u) = u \varphi(u) \), and \( \varphi \) is a non-increasing function of \( u \). Nevertheless, since \( f'(u) \) may be negative for positive values of \( u \), the standard initial-boundary value problems for this equation are ill-posed.

Several models have been proposed to overcome this difficulty. These include models based on systems of difference-differential equations [7], on advection-diffusion equations [3], and on some type of regularization of equation (4) [5, 6, 8].

In this paper we assume that the density dependent dispersal coefficient \( \varphi(u) \) gets actualized at certain predetermined intervals of time, letting us to consider the functional differential equation (1).
In Section 1 we prove existence and uniqueness of the solutions of (1)-(3). We also show some comparison results and study the asymptotic behavior of the solutions of a problem associated to (1)-(3).

In Section 2 we prove some results which show the aggregating behavior that the solutions of (1)-(3) exhibit.

We include an Appendix with the derivation of equation (1).

1 Existence and Uniqueness of Global Solutions

We will assume that the functions $\varphi(u)$ and $f(u) := u\varphi(u)$ satisfy the following hypothesis:

**Hypothesis 1.**

1. $\varphi : [0, \infty) \to (0, \infty)$ is bounded and non-increasing.
2. There exist constants $\alpha_1$ and $\alpha_2$ with $0 < \alpha_1 < \alpha_2 \leq \infty$ such that $f$ is increasing for $u \in (0, \alpha_1)$ and $f$ is decreasing for $u \in (\alpha_1, \alpha_2)$. If $\alpha_2 < \infty$, then $f$ is nondecreasing for $u \in (\alpha_2, \infty)$.

For example, the following functions are admissible: $\varphi(u) = \exp(-u)$; $\varphi(u) = 2^{u+1/2}/(u+1)+1/2$: $\varphi(u) = k_1$ for $0 \leq u \leq \alpha_1$, $\varphi(u) = k_1 + \frac{k_2-k_1}{\alpha_2-\alpha_1}(u-\alpha_1)$ for $\alpha_1 \leq u \leq \alpha_2$, and $\varphi(u) = k_2$ for $\alpha_2 \leq u < \infty$, where $k_1$ and $k_2$ are constants such that $0 < k_2 < k_1$ and $k_1\alpha_1 > k_2\alpha_2$.

In this section we will solve (1)-(3) by the method of steps, i.e., we integrate the equation inductively in $\Omega \times (k\tau, (k+1)\tau]$, for $k = 0, 1, \ldots$. This leads us to solve the parabolic equation:

$$v_t = \Delta[a(x)v(x,t)], \quad x \in \Omega, \quad t \in (0,T],$$

with boundary conditions:

$$\eta \cdot \nabla[a(x)v(x,t)] = 0, \quad x \in \partial \Omega, \quad t > 0$$

and initial data

$$v(x,0) = v_0(x), \quad x \in \Omega.$$  

for any $T > 0$. We will solve (5)-(7) with the following assumptions about the data $a$ and $v_0$:

$A_1$ $a \in L^\infty(\Omega)$ and $0 < \alpha \leq a(x) \leq \beta$ for a.e. $x \in \Omega$.

$A_2$ $v_0 \in L^\infty(\Omega)$ and $v_0(x) \geq 0$ for a.e. $x \in \Omega$.

$A_3$ $av_0 \in W^1_2(\Omega)$

These will be called “Assumptions A”.
Definition 1. A solution of problem (5)-(7) on \([0, T]\) is a function \(v\) with the following properties:

i) \(v \in L^\infty(Q_T),\)

ii) \(av \in C([0, T]; L_2(\Omega)) \cap W_2^{1,0}(Q_T),\)

iii) \(\int_{Q_T} v(x,t)\psi(x,t)dx - \int\int_{Q_T} [v(x,t)\psi_t(x,t) - \nabla(a(x)v(x,t)) \cdot \nabla \psi]dxdt = \int_{\Omega} v_0(x)\psi(x,0)dx, \) for all \(\psi \in W_2^1(Q_T)\) and for all \(t \in (0, T].\)

A solution on \([0, \infty)\) means a solution on each \([0, T]\), and a sub-solution (super-solution) is defined by (i), (ii) and (iii) with equality replaced by \(\leq\) (\(\geq\)).

Here we are using the standard notation \(Q_T := \Omega \times (0, T].\)

Next, we will obtain some comparison results for the solutions of (5)-(7).

Proposition 1. Let \(\hat{v}\) be a supersolution of problem (5)-(7) in \([0, T]\) with initial data \(\hat{v}_0\) and let \(v\) be a sub-solution in \([0, T]\) with initial data \(v_0\). Then, for all \(\lambda > 0\) and \(0 \leq t \leq T\), we have

\[
e^{\lambda t} \int_{\Omega} (v(x,t) - \hat{v}(x,t))^+ \leq \int_{\Omega} (v_0(x) - \hat{v}_0(x))^+ + \int_{Q_T} [\lambda(v - \hat{v})]^+ e^{\lambda s}. (8)
\]

Proof: For any \(\psi \in C^2(\overline{Q_T})\) such that \(\psi_x = 0\) for \((x,t) \in \partial \Omega \times [0, T]\), we have

\[
\int_{\Omega} \psi v - \int\int_{Q_T} (v\psi_t + av\psi_{xx}) \leq \int_{\Omega} v_0\psi(0)
\]

and

\[
- \int_{\Omega} \hat{v}\psi + \int\int_{Q_T} (\hat{v}\psi_t + a\hat{v}\psi_{xx}) \leq - \int_{\Omega} \hat{v}_0\psi(0).
\]

Adding term by term we obtain

\[
\int_{\Omega} (v - \hat{v})\psi - \int\int_{Q_T} (v - \hat{v})(\psi_t + a\psi_{xx}) \leq \int_{\Omega} (v_0 - \hat{v}_0)\psi(0). \quad (9)
\]

We now construct a special sequence of functions \(\{\psi_n\}\) to use in (9). Fix \(T > 0\) and choose a sequence \(\{a_n\}\) of smooth functions such that

\[
0 < \gamma \leq a_n \leq \|a\|_{L^\infty(\Omega)}
\]

and

\[
(a_n - a)/\sqrt{a_n} \rightarrow 0 \text{ in } L^2(\Omega).
\]
Since $|a_n^{-1/2}|_{L^\infty(\Omega)} < 1/\gamma$, for all $n$, it is enough to choose $\{a_n\}$ such that $(a_n - a) \to 0$ in $L^2(\Omega)$. \hfill \Box$

Next, let $\chi \in C^\infty_0(\Omega)$ be such that $0 \leq \chi \leq 1$. Finally let $\psi_n$ be the solution of the backward problem

$$
\psi_{nt} + a_n \psi_{nxx} = \lambda \psi_n \quad \text{for} \quad (x,t) \in \Omega \times [0,T)
$$

$$
\psi_{nx}(x,t) = 0 \quad \text{for} \quad (x,t) \in \partial \Omega \times [0,T)
$$

$$
\psi_n(x,T) = \chi(x) \quad x \in \Omega.
$$

This is a parabolic problem and has a unique solution $\psi_n \in C^\infty(\overline{Q}_T)$ that satisfies the properties stated in the following Lemma.

**Lemma 1.** The function $\psi_n$ has the following properties:

(i) $0 \leq \psi_n \leq e^{\lambda(t-T)}$ in $\overline{Q}_T$

(ii) $\iint_{Q_T} a_n (\psi_{nxx})^2 < c$

(iii) $\sup_{0 \leq t \leq T} \int_{\Omega} (\psi_{nx})^2(t) < c$, where the constant $c$ depends only on $\chi$.

The proof of this Lemma is similar to the proof of Lemma 10 in D. Aronson, M. G. Crandall and L. A. Peletier [2] and it is omitted. If we set $t = T$ and $\psi = \psi_n$ in (9) we obtain:

$$
\int_{\Omega} (v - \hat{v}) \chi - \iint_{Q_T} (v - \hat{v})(a - a_n) \psi_{nxx}
$$

$$
\leq \int_{\Omega} (v_0 - \hat{v}_0) \psi_n(0) + \iint_{Q_T} \lambda(v - \hat{v}) \psi_n
$$

$$
\leq \int_{\Omega} (v_0 - \hat{v}_0)^+ e^{-\lambda T} + \iint_{Q_T} [\lambda(v - \hat{v})]^+ e^{\lambda(s-T)}.
$$

Since

$$
\iint_{Q_T} |a - a_n| |\psi_{nxx}| = \iint_{Q_T} \frac{|a - a_n|}{\sqrt{a_n}} |\psi_{nxx}|,
$$

we have, by Lemma 1 (ii),

$$
\| (a - a_n) \psi_{nxx} \|_{L^1(Q_T)} \leq \frac{\| a - a_n \|_{L^\infty(Q_T)}}{\sqrt{a_n}} \| \psi_{nxx} \|_{L^2(Q_T)}
$$

$$
= T^{1/2} \frac{\| a - a_n \|_{L^\infty(\Omega)}}{\sqrt{a_n}} \| \psi_{nxx} \|_{L^2(Q_T)}
$$

$$
\leq (cT)^{1/2} \frac{\| a - a_n \|_{L^\infty(\Omega)}}{\sqrt{a_n}} \| \psi_{nxx} \|_{L^2(Q_T)}.
$$
which tends to zero as \( n \to \infty \) by the choice of \( a_n \). Thus, letting \( n \to \infty \) in (10) we obtain

\[
\int_\Omega (v(T) - \hat{v}(T)) \chi \leq \int_\Omega (v_0 - \hat{v}_0)^+ e^{-\lambda T} + \int_0^T \int_{Q_T} [\lambda(v - \hat{v})]^+ e^{\lambda(S-T)} d\tau.
\]

This inequality holds for any \( \chi \in C_0^\infty(\Omega) \) with \( 0 \leq \chi \leq 1 \). Hence, it continues to hold for \( \chi(x) = 1 \) on \( \{ x : v(T) > \hat{v}(T) \} \) and \( \chi = 0 \) otherwise (i.e., \( \chi = \text{sign}(v(T) - \hat{v}(T))^+ \)). Here we have used the fact that \( C_0^\infty(\Omega) \) is dense in \( L^1(\Omega) \). Replacing \( T \) by any \( t \leq T \) and applying the same argument we complete the proof of the Proposition.

\[ \square \]

**Theorem 1.** (i) Let \( v, \hat{v} \) be solutions problem (5)-(7) on \([0,T]\) with initial data \( v_0 \) and \( \hat{v}_0 \) respectively. Then

\[
\|v(t) - \hat{v}(t)\|_{L^1(\Omega)} \leq \|v_0 - \hat{v}_0\|_{L^1(\Omega)}
\]

Thus, in particular, the solution of problem (5)-(7) is unique.

(ii) Let \( v \) be a sub-solution and \( \hat{v} \) a super-solution of problem (5)-(7) with initial data \( v_0 \), and \( \hat{v}_0 \) respectively. Then if \( v_0 \leq \hat{v}_0 \) it follows that

\[ v \leq \hat{v} \]

**Proof:** With the assumptions of (ii), Proposition 1 yields

\[
e^{\lambda t} \int_\Omega (v(t) - \hat{v}(t))^+ \leq \int_\Omega (v_0 - \hat{v}_0)^+ + \int_0^t \int_\Omega e^{\lambda \tau}[\lambda(v - \hat{v})]^+. \quad (12)
\]

Thus if we write

\[
h(t) = e^{\lambda t} \int_\Omega (v(t) - \hat{v}(t))^+.
\]

(12) implies, by Gronwall’s Lemma, that \( h(t) \leq h(0) e^{\lambda t} \) or

\[
\int_\Omega (v(t) - \hat{v}(t))^+ \leq \int_\Omega (v_0 - \hat{v}_0)^+.
\]

This proves (ii). The assertion (i) follows by adding the corresponding inequality for \( (\hat{v} - v)^+ \).

\[ \square \]
Remark 1. Since $v_0 \geq 0$ and zero is a solution of (5)-(7), we obtain that the solutions of (5)-(7) are non negative.

Remark 2. Since $v_0 \in L^\infty(\Omega)$, let $K$ be a constant such that $v_0 \leq K$. Let $\hat{v}_0 = K$ then $\hat{v}(x,t) = e^{Mt}$ is a super-solution (in fact, a solution) of the problem (5)-(7). Then, by the theorem, $\hat{v}(x,t) \leq e^{Mt}$. In particular, $v \in L^\infty(\Omega)$.

Now we proceed to the proof of the following theorem:

Theorem 2. If the Assumptions A are fulfilled, then the problem (5)-(7) has a unique solution $v \in [0,T]$ for any $T > 0$. Moreover, $v$ satisfies the following energy relation

$$\frac{1}{2} \int_\Omega av^2 + \int_{Q_T} (av)^2 = \frac{1}{2} \int_\Omega (av_0)^2,$$

and the estimate

$$\operatorname{ess} \sup_{0 \leq t \leq T} ||a(\cdot)v(\cdot, t)||_{L^2(\Omega)} + ||\nabla(av)||_{L^2(\Omega)} \leq C ||a(\cdot)v_0(\cdot)||_{L^2(\Omega)},$$

where $C = C(\alpha, \beta)$ is a constant independent of $T$.

Proof: The uniqueness is already given in Theorem 1 (i).

For the proof of solvability we make the change of variable $w(x,t) = a(x)v(x,t)$ and arrive to the following problem

$$\begin{cases}
\tilde{a}w_t = \Delta w, & (x,t) \in Q_T \\
\eta \cdot \nabla w = 0, & (x,t) \in \partial \Omega \times (0,T) \\
w(x,0) = w_0(x) := a(x)v_0(x), & x \in \Omega,
\end{cases}$$

where $\tilde{a} = 1/a$. It is clear that $\tilde{a} \in L^\infty(\Omega)$.

Now we take a fundamental system $\{\varphi_k(x)\}$ in $W^1_0(\Omega)$. Since $\tilde{a}(x) \geq 1/\beta > 0$, for a.e. $x \in \Omega$, we can choose $\varphi_k(x)$ such that $\int_\Omega \tilde{a}(x)\varphi_k(x)\varphi_l(x) \, dx = 0$ for $k \neq l$. We shall look for approximate solutions

$$w^N(x,t) = \sum_{k=1}^N C^N_k(t)\varphi_k(x)$$

from the relation

$$(\tilde{a}w^N_t, \varphi_l) + (w^N_{x_i}, \varphi_{lx_i}) = 0, \quad l = 1, \ldots, N$$

(16)
and the equation
\[ C_N^l(0) = (w_0, \varphi_l), \quad l = 1, \ldots, N, \]  
(17)
where \((\cdot, \cdot)\) denotes the inner product in \(L^2(\Omega)\). Here and in what follows the terms of the form \((w_N^x, \varphi_{lx_i})\) mean \(\sum_{i=1}^{N}(w_N^x, \varphi_{lx_i})\).

The relation (16) is simply a system of \(N\) linear ordinary differential equations in the unknowns \(C_N^l(t) = C_N^l(t), \quad l = 1, \ldots, N\), whose principal terms are of the form \(dC_N^l(t)/dt\), the coefficients of \(C_N^l(t)\) being constant. By a well known theorem on the solvability of such systems, we see that (16) and (17) uniquely determine continuously differentiable functions \(C_N^l(t)\) on \([0, T]\).

Now we shall obtain bounds for \(w_N^x\) which do not depend on \(N\). To do this, let us multiply each equation of (16) by the appropriate \(C_N^l\), add then up from 1 to \(N\) and then integrate the result with respect to \(t\) from 0 to \(t \leq T\), to obtain:
\[ \int_{Q_t} \tilde{a}w_N^x w_N^x + \int_{Q_t} (w_N^x)^2 = 0. \]

From this we obtain
\[ \frac{1}{2} \int_{\Omega_1} \tilde{a}(w^N)^2 + \int_{Q_t} (w_N^x)^2 = \frac{1}{2} \int_{\Omega} \tilde{a}(w_0^N)^2, \]  
(18)
where \(w_0^N(x) = w_N^x(x, 0) = \sum_1^N C_N^K(0)\psi_K(x) = \sum_1^N (w_0, \psi_K)\psi_K\). Now, since \(1/\beta \leq \tilde{a} \leq 1/\alpha\), we obtain
\[ \frac{1}{2\beta} \|w^N(\cdot,t)\|^2_{\Omega} + \|w_N^x\|^2_{Q_t} \leq \frac{1}{2\alpha} \|w_N^0(\cdot,0)\|^2_{\Omega}, \]
where
\[ \|w_N^x\|_{Q_t} := \left( \int_{Q_t} \sum_{i=1}^{n} w_{x_i}^2 \right)^{1/2}. \]

We replace \(\|w_N^0\|^2_{\Omega}\) by \(y(t)\|w_0^N\|_{\Omega}\), where \(y(t) := \sup_{0 \leq \tau \leq t} \|w_N^N(\cdot,\tau)\|_{\Omega}\). This gives the inequality
\[ \|w_N^N(\cdot,t)\|^2_{\Omega} + \nu \|w_N^x\|^2_{Q_t} \leq \mu y(t)\|w_0^N\|_{\Omega} := j(t), \]
where \(\mu = \frac{\beta}{\alpha}, \quad \nu = 2\beta\). From this the two inequalities
\[ y^2(t) \leq j(t) \]  
(19)
and

$$\|w_N^x\|_{Q_t}^2 \leq v^{-1}j(t)$$ (20)

follow. We take the square root of each side of (19) and (20), add together the resulting inequalities, and then estimate the right-hand size in the following way:

$$|w_{N}|_{Q_t} := y(t) + \|w_N^x\|_{Q_t} \leq (1 + v^{-1/2})j^{1/2}(t) \leq (1 + v^{-1/2})\mu^{1/2}\|w_N^0\|_{\Omega}^{1/2}.$$ 

From this we obtain the following bound for $|w_{N}|_{Q_t}$:

$$|w_{N}|_{Q_t} \leq (1 + v^{-1/2})^{2}\mu\|w_N^0\|_{\Omega}.$$ (21)

Thus, we obtain the inequality

$$|w_{N}|_{Q_t} \leq c\|w_N^0\|_{\Omega},$$ (22)

which holds for any $t$ in $[0,T]$, with $c = c(\alpha, \beta)$ independent of $t$ and $T$. But $\|w_N^0\|_{\Omega} \leq \|w_0\|_{\Omega}$, so we have the bound

$$|w_{N}|_{Q_t} \leq C_1,$$ (23)

with a constant $C_1$ independent of $N$. Because of (23), we can choose a subsequence $\{w_{N_k}\}$ ($k = 1, 2, \ldots$) from the sequence $\{w_N\}$ ($N = 1, 2, \ldots$) which converges weakly in $L_2(Q_T)$, together with the derivatives $w_{N_k}^x$, to some element $w \in W^{1/2}_2(Q_T)$ (as a result of subsequent arguments, we shall show that the entire sequence $\{w_N\}$ converges to $w$). This element $w(x,t)$ is the desired generalized solution of the problem (15).

Indeed, let us multiply (16) by an arbitrary absolutely continuous function $d_i(t)$ with $dd_i(t)/dt \in L_2(0,T)$, add up the equations thus obtained from 1 to $N$, and then integrate the result from 0 to $t \leq T$. If we integrate the first term by parts with respect to $t$, we obtain an identity:

$$\int_{\Omega} \tilde{\alpha}w^N\Phi dx - \int_{Q_t} [\tilde{\alpha}w^N\Phi_t + w^N_x\Phi_{x_i}] dx dt = \int_{\Omega} \tilde{\alpha}w_0^N\Phi(x,0) dx$$ (24)

in which $\Phi = \sum_{i=1}^{2} d_i(t)\varphi_1(x)$. Let us denote by $\mathcal{M}_N$ the set function $\Phi$ with $d_i(t)$ having the properties indicated above. The totality $\cup_{i=1}^{\infty} \mathcal{M}_i$ is dense in $W^{1/2}_2(Q_T)$. 


For a fixed $\Phi \in \mathcal{M}_p$ in (24) we can take the limit of the subsequence $\{w^{N_k}\}$ chosen above, starting with $N_k \geq p$. As a result, we obtain (24) for $w$. But since $\cup_{p=1}^{\infty} \mathcal{M}_p$ is dense in $W^1_2(Q_T)$, it is not hard to obtain that $w$ satisfies (ii) in the corresponding definition of solution of problem (15).

Finally it can be easily seen that the difference, $w^{N_k} - w^{N_l}$ satisfies the inequality (22):

$$|w^{N_k} - w^{N_l}|_{Q_T} \leq C(T)\|w^{N_k}_0 - w^{N_l}_0\|_{\Omega}.$$ 

This implies that $w^{N_k}$ converges to $w$ in the norm $|\cdot|_{Q_T}$, showing that $w \in C([0,T]:L^2(\Omega)) \cap W^{1,0}_2(Q_T)$. Now, applying (18) to the subsequence $w^{N_k}$ and taking limits we obtain (13). From this, following the same argument that led to (22), we obtain (14). This finishes the proof of the Theorem.

\[ \square \]

The following result is a consequence of the previous Theorem. The proof is given in [9].

**Theorem 3.** Any solution $v(x,t)$ of (5)-(7) satisfies

$$\lim_{t \to \infty} \|v(\cdot,t) - v_\infty\|_{L^2(\Omega)} = 0,$$

where $v_\infty := \frac{1}{a(x)} \left( \int_\Omega v_0 \right) \left( \int_\Omega \frac{1}{a} \right)^{-1}$.

Now we are ready to prove the main result of this section. A global solution for the problem (1)-(3) is a function $u(x,t)$, $x \in \Omega$, $t > 0$, such that $u$ is a solution of the problem (5)-(7) in $\Omega \times (k\tau,(k+1)\tau]$, $k = 0,1,2,\ldots$, with $a(x) = \varphi(u(x,[t/\tau]\tau))$.

**Theorem 4.** If $u_0 \in L^\infty(\Omega)$ then the problem (1)-(3) has a unique global solution.

**Proof:** The proof is by induction in $k$. The case $k = 0$ is obtained directly from Theorem 2. Assuming the case $k$ and using the Remark 2 after the proof of Theorem 1 we obtain that $u(x,(k+1)\tau) \in L^\infty(\Omega)$ and $a(x)u(x,(k+1)\tau) \leq W^2_2(\Omega)$. From this it follows that we can apply Theorem 2 to solve (5)-(7) in $\Omega \times ((k+1)\tau,(k+2)\tau]$ with $a(x) = \varphi(u(x,(k+1)\tau))$ and $u_0(x) = u(x,(k+1)\tau)$. This finishes the proof. \[ \square \]

## 2 Aggregation

In this section we consider some results that show the aggregating behavior that the solutions of (1)-(3) exhibit. The first result is a direct consequence of Theorem 3. For any $u_0 \in L^\infty(\Omega)$ and $\tau > 0$, let $u(x,t;u_0,\tau)$ denote the solution of (1)-(3).
Theorem 5. Suppose that \( \varphi(u) \) and \( f(u) = u\varphi(u) \) satisfy Hypothesis 1. For any \( \epsilon > 0 \) there exists \( \tau > 0 \) such that

\[
\|u(\cdot, \tau; u_0, \tau) - u_\infty(\cdot)\|_{L^2(\Omega)} < \epsilon,
\]

where \( u_\infty := \frac{1}{\varphi(u_0(x))} \left( \int_\Omega u_0(x) \, dx \right) \left( \int_\Omega \frac{dx}{\varphi(u_0(x))} \right)^{-1} \).

Since \( \varphi(u) \) is a non-increasing function this result states that, for large enough \( \tau \), the solutions of (1)-(3) concentrate its mass around the points of higher density of the initial data \( u_0(x) \), thus showing the kind of aggregating behavior that we were expecting.

Another way to look at this result is to notice that, by the change of variable \( s = t/\tau \) and the definition \( w(x, s) := u(x, s\tau) \), problem (1)-(3) is transformed into the equivalent problem

\[
egin{align*}
&w_s = \Delta [\tau \varphi(w(x, [s]))w(x, s)], \quad x \in \Omega, \ s > 0 \\
&\eta \cdot \nabla [\tau \varphi(w(x, [s]))w(x, s)] = 0, \quad x \in \partial \Omega, \ s > 0 \\
&w(x, 0) = u_0(x) := u_0(x), \quad x \in \Omega.
\end{align*}
\]

Hence, taking \( \tau > 0 \) big accounts for multiplying \( \varphi \) by a large constant. Therefore, Theorem 5 states that for any \( \epsilon > 0 \) we can choose \( \tilde{\varphi} := \tau \varphi \), multiplying the original \( \varphi \) by a large constant \( \tau \), such that the solution \( w \) of (25) satisfies

\[
\|w(\cdot, 1) - u_\infty(\cdot)\|_{L^2(\Omega)} < \epsilon.
\]

That is, given an initial data \( u_0 \), we can generate aggregation around the points of higher density of \( u_0 \), at a prescribed time, by an adequate choice of \( \varphi \).

Proof of Theorem 5: We consider the problem

\[
egin{align*}
v_t &= \Delta [a(x)v(x, t)], \quad x \in \Omega, \ t > 0 \\
&\eta \cdot \nabla [a(x)v(x, t)] = 0, \quad x \in \partial \Omega, \ t > 0 \\
v(x, 0) &= v_0(x) := u_0(x), \quad x \in \Omega,
\end{align*}
\]

with \( a(x) := \varphi(u_0(x)) \). By Theorem 3, for any \( \epsilon > 0 \) there exists \( \tau > 0 \) such that

\[
\|v(\cdot, t) - v_\infty(\cdot)\|_{L^2(\Omega)} < \epsilon,
\]
for any $t > \tau$, where $v_\infty := \frac{1}{a(x)} \left( \int_\Omega v_0(x) \, dx \right) \left( \int_\Omega \frac{dx}{a(x)} \right)^{-1}$. By uniqueness of the solutions of (1)-(3) it follows that $u(\cdot, \tau; u_0, \tau) = v(\cdot, \tau)$. This finishes the proof.

In what follows we restrict ourselves to a more specific function $\varphi$. Let $\varphi(u) = \begin{cases} k_1, & 0 \leq u \leq \alpha_1 \\ \psi(u), & \alpha_1 \leq u \leq \alpha_2 \\ k_2, & \alpha_2 \leq u \end{cases}$, where $\psi(u)$ is a non-increasing function, $k_1, k_2, \alpha_1$ and $\alpha_2$ are positive constants such that $k_2 < k_1$, $\alpha_1 < \alpha_2$ and $k_2 \alpha_2 < k_1 \alpha_1$. For example, we can choose $\psi$ to be linear, that is $\psi(u) = k_1 + k_2 - k_1 \alpha_2 - \alpha_1 (u - \alpha_1)$.

The following result shows that, under certain restrictions on the initial data, the solutions of (1)-(3) converge to a steady state. It is not difficult to show that a function $u \in L^\infty(\Omega)$ is a steady state solution of (1)-(2) if and only if $f(u(x)) = \text{constant}$ for a.e. $x \in \Omega$. Let $\beta_i$ be such that $\beta_2 < \alpha_1 < \alpha_2 < \beta_1$, and $f(\beta_i) = f(\alpha_i), i = 1, 2$. That is, $k_1 \beta_2 = k_2 \alpha_2$ and $k_2 \beta_1 = k_1 \alpha_1$.

**Theorem 6.** Let $\tilde{\Omega} \subset \Omega$ be such that both $\tilde{\Omega}$ and $\Omega \setminus \tilde{\Omega}$ have positive measure. Suppose that $u_0$ satisfies $\beta_2 \leq u_0(x) \leq \alpha_1$, for a.e. $x \in \tilde{\Omega}$ and $\alpha_2 \leq u_0(x) \leq \beta_1$ for a.e. $x \in \Omega \setminus \tilde{\Omega}$. Then, the solution $u(x, t)$ of (1)-(3) satisfies

$$\lim_{t \to \infty} \|u(\cdot, t) - u_\infty\|_{L^2(\Omega)} = 0,$$

where $u_\infty$ is a steady solution of (1)-(2). Moreover,

$$u_\infty = \begin{cases} \gamma_2, & x \in \tilde{\Omega} \\ \gamma_1, & x \in \Omega \setminus \tilde{\Omega} \end{cases},$$

where

$$\gamma_i = \frac{k_i}{k_2|\tilde{\Omega}| + k_1|\Omega \setminus \tilde{\Omega}|} \int_\Omega u_0(x) \, dx, \quad i = 1, 2$$

and $\beta_2 \leq \gamma_2 \leq \alpha_1 < \alpha_2 \leq \gamma_1 \leq \beta_1$.

**Proof:** First, we will show that $u(x, t)$ satisfies

$$\beta_2 \leq u(x, t) \leq \alpha_1, \quad x \in \tilde{\Omega}, \quad t \geq 0$$

and

$$\alpha_2 \leq u(x, t) \leq \beta_1, \quad x \in \Omega \setminus \tilde{\Omega}, \quad t \geq 0.$$
Let
\[ v_1(x) = \begin{cases} \alpha_1, & x \in \tilde{\Omega} \\ \beta_1, & x \in \Omega \setminus \tilde{\Omega}, \end{cases} \]
and
\[ v_2(x) = \begin{cases} \beta_2, & x \in \tilde{\Omega} \\ \alpha_2, & x \in \Omega \setminus \tilde{\Omega}. \end{cases} \]
Then \( v_i \) \((i = 1, 2)\) are steady solutions of (1)-(2). For \( 0 \leq t \leq \tau \) let \( a(x) := \varphi(u_0(x)) \); then
\[ a(x) = \begin{cases} k_1, & x \in \tilde{\Omega} \\ k_2, & x \in \Omega \setminus \tilde{\Omega}. \end{cases} \]
Then, \( v(x, t) := u(x, t), 0 \leq t \leq \tau, \) is the solution of (5)-(7) in \([0, \tau]\). Moreover, since
\[ v_2(x) = \beta_2 \leq v_0(x) \leq \alpha_1 = v_1(x), \ x \in \tilde{\Omega} \]
and
\[ v_2(x) = \alpha_2 \leq v_0(x) \leq \beta_1 = v_1(x), \ x \in \Omega \setminus \tilde{\Omega}, \]
we have that
\[ v_2(x) \leq v_0(x) \leq v_1(x) \]
for almost every \( x \in \Omega \). Since \( v_i \) \((i = 1, 2)\) are steady solutions of (5)-(6), it follows from Theorem 1 that
\[ v_2(x) \leq v(x, t) \leq v_1(x) \]
for almost every \( x \in \Omega \) and \( 0 \leq t \leq \tau \). That is, (27) and (28) hold for \( 0 \leq t \leq \tau \). Repeating the same argument inductively we obtain that (27) and (28) hold for any \( t \geq 0 \), as we wanted to show.

This implies, in particular, that \( u(x, t) \) is a solution of (5)-(7) on \([0, \infty)\) with \( a(x) = \varphi(u_0(x)) \). Therefore, it follows from Theorem 3 that
\[ \lim_{t \to \infty} \| u(\cdot, t) - u_\infty \|_{L^2(\Omega)} = 0, \]
where
\[ u_\infty := \frac{1}{a(x)} \left( \int_\Omega u_0 \right) \left( \int_\Omega \frac{1}{a} \right)^{-1}. \]
Then (26) follows by noticing that
\[
\int_{\Omega} dx \ a(x) = \frac{k_2|\tilde{\Omega}| + k_1|\Omega \setminus \tilde{\Omega}|}{k_1k_2}.
\]

Now, by using the hypothesis that \(\beta_2 \leq u_0(x) \leq \alpha_1\), for a.e. \(x \in \tilde{\Omega}\) and \(\alpha_2 \leq u_0(x) \leq \beta_1\) for a.e. \(x \in \Omega \setminus \tilde{\Omega}\), we obtain
\[
\beta_2|\tilde{\Omega}| + \alpha_2|\Omega \setminus \tilde{\Omega}| \leq \int_{\Omega} u_0 \leq \alpha_1|\tilde{\Omega}| + \beta_1|\Omega \setminus \tilde{\Omega}|.
\]
Hence,
\[
\beta_2 = \frac{k_2\beta_2|\tilde{\Omega}| + k_2\alpha_2|\Omega \setminus \tilde{\Omega}|}{k_2|\tilde{\Omega}| + k_1|\Omega \setminus \tilde{\Omega}|} \leq \frac{k_2\alpha_1|\tilde{\Omega}| + k_2\beta_1|\Omega \setminus \tilde{\Omega}|}{k_2|\tilde{\Omega}| + k_1|\Omega \setminus \tilde{\Omega}|}.
\]
Here we have used the fact that \(k_1\beta_2 = k_2\alpha_2\) and \(k_2\beta_1 = k_1\alpha_1\). Therefore, \(\beta_2 \leq \gamma \leq \alpha_1\). Similarly, we obtain that \(\alpha_2 \leq \gamma \leq \beta_1\). Hence, \(f(\gamma_2) = k_1\gamma_2\) and \(f(\gamma_1) = k_2\gamma_1\). Therefore, since \(k_1\gamma_2 = k_2\gamma_1\), \(f(\gamma_1) = f(\gamma_2)\). That is, \(u_\infty\) is a steady solution of (1)-(2). This finishes the proof.

\]
Aggregation on a Nonlinear Parabolic Functional Diff. Equation

per unit volume, directly at \( x \) by births and deaths. The product \( u(x,t)\gamma(x,t) \) gives the number of individuals supplied at \( x \).

The functions \( u, \varphi \) and \( \gamma \) must be consistent with the following “Law of population balance”: For every regular subregion \( R \) of \( \Omega \) and for all \( t \),

\[
\frac{d}{dt} \int_R u(x,t)dx = \int_{\partial R} \eta \cdot \nabla [u(x,t)\varphi(x,t)]ds_x + \int_R u(x,t)\gamma(x,t)dx,
\]

where \( \eta \) is the outward unit normal to the boundary \( \partial R \) of \( R \). This equation asserts that the rate of change of population of \( R \) must equal the rate at which individuals leave \( R \) across its boundary plus the rate at which individuals are supplied directly to \( R \).

Using the well known Divergence Theorem we obtain

\[
\frac{d}{dt} \int_R u(x,t)dx = \int_R \Delta [u(x,t)\varphi(x,t)]dx + \int_R u(x,t)\gamma(x,t)dx
\]

Since \( R \) is an arbitrary region in \( \Omega \) we obtain the following local counterpart

\[
\frac{\partial u}{\partial t} = \Delta [\varphi(x,t)u(x,t)] + u(x,t)\gamma(x,t).
\]

In this paper we are only concerned with migration mechanisms. Therefore we assume that \( \varphi \) is not explicitly dependent upon the position and time but on the population density \( u \) at times \( t = k\tau \), \( k = 0,1,2,\ldots \), for a given \( \tau > 0 \). That is, \( \varphi(x,t) = \varphi(u(x,[t/\tau]\tau)) \) where \([\theta]\) denotes the greatest integer less than or equal to \( \theta \).

Introducing this in the previous equation we arrive at the following nonlinear functional differential equation for the density \( u \):

\[
\frac{\partial u}{\partial t} = \Delta [\varphi(u(x,[t/\tau]\tau))u(x,t)] + \Gamma(u(x,[t/\tau]\tau))u(x,t)
\]

References


