Motivic Equivalence of Quadratic Forms

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Abstract. Let $X_\phi$ and $X_\psi$ be projective quadrics corresponding to quadratic forms $\phi$ and $\psi$ over a field $F$. If $X_\phi$ is isomorphic to $X_\psi$ in the category of Chow motives, we say that $\phi$ and $\psi$ are motivic isomorphic and write $\phi \sim_m \psi$. We show that in the case of odd-dimensional forms the condition $\phi \sim_m \psi$ is equivalent to the similarity of $\phi$ and $\psi$. After this, we discuss the case of even-dimensional forms. In particular, we construct examples of generalized Albert forms $q_1$ and $q_2$ such that $q_1 \sim_m q_2$ and $q_1 \neq q_2$.

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Let $F$ be a field of characteristic $\neq 2$ and $\phi$ be a quadratic form of dimension $\geq 3$ over $F$. By $X_\phi$ we denote the projective variety given by the equation $\phi = 0$. It is well known that the variety $X_\phi$ determines the form $\phi$ uniquely up to similarity. More precisely, the condition $X_\phi \cong X_\psi$ holds if and only if $\phi \cong k\psi$ for a suitable element $k \in F^*$. Now, let $\mathcal{M}: \mathcal{V}_F \to \mathcal{C}$ be an arbitrary functor from the category $\mathcal{V}_F$ of smooth projective $F$-varieties to a category $\mathcal{C}$. Is it possible to say anything specific about $\phi$ and $\psi$ if we know that $\mathcal{M}(X_\phi) \cong \mathcal{M}(X_\psi)$? Clearly, the answer depends on the category $\mathcal{C}$ and the functor $\mathcal{M}$. In the present paper, we mainly consider the example of the category $\mathcal{C} = \mathcal{MV}_F$ of Chow motives. In this particular case, we set $\mathcal{M}(X) = \mathcal{M}(X)$, where $\mathcal{M}(X)$ denotes the motive of $X$ in the category of Chow motives. If $\mathcal{M}(X_\phi) \cong \mathcal{M}(X_\psi)$, we say that $\phi$ is motivic equivalent to $\psi$ (and we write $\phi \sim_m \psi$).

Recently, Alexander Vishik has proved that $\phi \sim_m \psi$ iff $\operatorname{dim} \phi = \operatorname{dim} \psi$ and $i_W(\phi_L) = i_W(\psi_L)$ for all extensions $L/F$ (see [27]). His proof uses deep results concerning the Voevodsky motivic category. In [10], Nikita Karpenko found a new, more elementary, proof that, in contrast to Vishik’s proof, deals only with Chow motives. In §2, we give an elementary proof of Vishik’s theorem in the case of odd-dimensional forms. In fact, we prove a more precise result. Namely, we show that, in the case of odd-dimensional forms, the condition $\phi \sim_m \psi$ is equivalent to the similarity of the forms $\phi$ and $\psi$ (here we do not use any results of the paper of Vishik). In other words, we prove that the condition $\mathcal{M}(X_\phi) \cong \mathcal{M}(X_\psi)$ is equivalent to the condition

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$X_\phi \simeq X_\psi$ for the odd-dimensional quadrics $X_\phi$ and $X_\psi$. In the proof we use some results of §1 concerning low dimensional forms belonging to $W(F(\phi)/F)$.

In §3, we show that the condition $\phi \nsim \psi$ is equivalent to the condition $\phi \sim \psi$ for all forms of dimension $\leq 7$. Besides, we discuss the case of even-dimensional forms of dimension $\geq 8$. This case is much more complicated. For instance, for all $n \geq 3$, there exists an example of anisotropic $2^n$-dimensional forms $\phi$ and $\psi$ such that $\phi \nsim \psi$ but $\phi \not\sim \psi$. In §4, for any $n$ and $m$ such that $0 \leq m \leq n - 3$, we construct generalized Albert forms $q_1$ and $q_2$ such that $\dim(q_1) = \dim(q_2) = 2(2^n - 2^m)$, $q_1 \nsim q_2$, but $q_1 \not\sim q_2$. This example gives a negative answer to a question stated by T. Y. Lam [18].

Some words about terminology and notation. Mainly we use the same terminology and notation as in the book of T. Y. Lam [17], W. Scharlau [23], and the fundamental papers of M. Knebusch [11, 12]. However, there exist several differences. We use the notation $\langle a_1, \ldots, a_n \rangle$ for the Pfister form $(1, -a_1) \otimes \cdots \otimes (1, -a_n)$ (in [17] and [23], $\langle a_1, \ldots, a_n \rangle = (1, a_1) \otimes \cdots \otimes (1, a_n)$). We write $\phi \sim \psi$ if there exists an element $k \in F$ such that $k\phi \simeq \psi$ (i.e., if $\phi$ is similar to $\psi$). We say that $\phi$ and $\psi$ are half-neighbors if $\dim \phi = \dim \psi$ and there exist $s, r \in F$ such that $\pi = s\phi \perp r\psi$ is a Pfister form (see, e.g., [6]). In this case, we will write $\phi \nsim \psi$ and we say that $\phi$ and $\psi$ are half-neighbors of $\pi$. Our definition differs from the original definition of Knebusch [12]. However, we prefer to use the new definition since we want to regard any pair $\phi, \psi$ of $2^n$-dimensional similar forms as half-neighbors. We denote by $P_n(F)$ the set of all $n$-fold Pfister forms. The set of all forms similar to $n$-fold Pfister forms is denoted by $GP_n(F)$. We also use the notation $P_\ast(F) = \bigcup_n P_n(F)$ and $GP_\ast(F) = \bigcup_n GP_n(F)$.

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1. Low dimensional forms in $W(F(\phi)/F)$

In this section, we give slight generalizations of some results of M. Knebusch. In fact, we modify some proofs of [12] by using Hoffman’s theorem [5]. We recall that Hoffman’s theorem asserts that for a pair of anisotropic quadratic forms $\phi$ and $\psi$ satisfying the condition $\dim \phi \leq 2^n < \dim \psi$, the form $\phi$ remains anisotropic over $F(\psi)$.

**Proposition 1.1.** Let $\phi$ and $\psi$ be anisotropic quadratic forms over $F$ such that $\dim \phi \geq \dim \psi$. Suppose that the form $\pi \overset{\mathrm{Def}}{=} \psi \perp \phi$ belongs to the group $W(F(\phi)/F)$. Then

1. If $\pi$ is isotropic, then $\pi$ is hyperbolic.
2. If $\pi$ is anisotropic, then $\pi$ is similar to a Pfister form.

**Proof.** (1) Assume that $\pi$ is isotropic but not hyperbolic. This means that $0 < \dim \pi_{an} < \dim \pi$. In the Witt ring $W(F)$, we have $\pi - \phi = \psi$. Therefore,

$$\dim(\pi_{an} \perp -\phi)_{an} = \dim \psi \leq \dim \phi < \dim \pi_{an} + \dim \phi = \dim(\pi_{an} \perp -\phi).$$

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2 see also [6, Prop. 2.4] and [3, Th. 1.6]

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Consequently, the form \( \pi_{an} \perp -\phi \) is isotropic. Hence the set \( D_F(\pi_{an}) \cap D_F(\phi) \) is nonempty.

Since \( \pi_F(\phi) \) is hyperbolic, it follows that \((\pi_{an})_F(\phi)\) is also hyperbolic. Since the set \( D_F(\pi_{an}) \cap D_F(\phi) \) is nonempty, the Cassels-Pfister subform theorem implies that \( \phi \subset \pi_{an} \). Therefore,

\[
\dim(\pi_{an} \perp -\phi)_{an} = \dim \pi_{an} - \dim \phi < \dim \pi - \dim \phi = \dim \psi.
\]

This contradicts to the relation \( \dim(\pi_{an} \perp -\phi)_{an} = \dim \psi \) proved above.

(2) Assume that \( \pi \) is not isotropic. To prove that \( \pi \) is similar to a Pfister form, it suffices to prove that \( \pi_{F(\phi)} \) is hyperbolic (see [12]).

Let \( \tilde{F} = F(\pi) \), \( \tilde{\pi} = \pi_{\tilde{F}} \), \( \tilde{\phi} = \phi_{\tilde{F}} \), and \( \tilde{\psi} = \psi_{\tilde{F}} \). Since \( \dim \psi \leq \frac{1}{2} \dim \pi \), Hoffmann's theorem implies that the form \( \tilde{\psi} = \psi_{\tilde{F}(\pi)} \) is anisotropic. If we assume that \( \tilde{\phi} \) is anisotropic, then we can apply item (1) of Proposition 1.1 to the \( \tilde{F} \)-forms \( \tilde{\phi}, \tilde{\psi} \), and \( \tilde{\pi} \). Then we conclude that \( \tilde{\pi} \) is hyperbolic. Now, we assume that \( \tilde{\phi} = \phi_{F(\pi)} \) is isotropic. Since \( \pi_{F(\phi)} \) is hyperbolic and \( \phi_{F(\phi)} \) is isotropic, it follows that \( \pi_{F(\phi)} \) is hyperbolic. Thus, the form \( \pi_{F(\pi)} \) is hyperbolic in any case and the proposition is proved.

**Corollary 1.2.** (Fitzgerald, [3, Th. 1.6].) Let \( \phi \) be an \( F \)-form, and let \( \pi \in W(F(\phi)/F) \) be an anisotropic nonzero form of dimension \( \leq 2 \dim \phi \). Then \( \pi \in GP_1(F) \) and one of the following conditions holds:

- \( \phi \) is a Pfister neighbor of \( \pi \).
- \( \phi \) is a half-neighbor of \( \pi \).

**Proof.** Since \( \pi \) is anisotropic and \( \pi_{F(\phi)} \) is hyperbolic, the form \( \phi \) is similar to a subform of \( \pi \). Multiplying \( \phi \) by a scalar, we may assume that \( \phi \subset \pi \). Let \( \psi \) be the complement of \( \phi \) in \( \pi \). Then all hypotheses of Proposition 1.1 hold. Since \( \pi \) is anisotropic, Proposition 1.1 implies \( \pi \in GP_1(F) \). The rest of the proof is an immediate consequence of the definitions of Pfister neighbors and half-neighbors, and the Cassels-Pfister subform theorem.

**Corollary 1.3.** (cf. [12, Th. 8.9].) Let \( \phi \) and \( \eta \) be anisotropic forms such that \( \dim \phi \geq \dim \eta \) and \((\phi_{F(\phi)})_{an} \simeq (\eta_{F(\phi)})_{an} \). Then either \( \phi \sim \eta \) or \( \phi \perp -\eta \in GP_1(F) \).

**Proof.** Let \( \psi = -\eta \) and \( \pi = \phi \perp -\eta = \phi \perp \psi \). All the hypotheses of Proposition 1.1 hold. In the case where \( \pi \) is isotropic, Proposition 1.1 implies that \( \pi \) is hyperbolic. Then \( \phi = \eta \) in the Witt ring. Since \( \phi \) and \( \eta \) are anisotropic, we have \( \phi \simeq \eta \). If \( \pi \) is anisotropic, Proposition 1.1 implies that \( \phi \perp -\eta = \pi \in GP_1(F) \).

2. **Motivic equivalence of odd-dimensional forms**

**Definition 2.1.** To any field \( F \), let be assigned an equivalence relation \( \sim_F \) on the set of all quadratic forms over \( F \) such that the following conditions hold:

(i) If \( \phi \) and \( \psi \) are forms over \( F \) such that \( \phi \sim \psi \), then \( \phi \sim_F \psi \).

(ii) If \( \phi \) and \( \psi \) are forms over \( F \) such that \( \phi \sim_F \psi \), then, for any extension \( E/F \), we have \( \phi_E \sim_E \psi_E \).

(iii) If \( \phi \) and \( \psi \) are forms over a field \( F \) such that \( \phi \sim_F \psi \), then \( \dim \phi = \dim \psi \) and \( i_W(\phi) = i_W(\psi) \).

A collection of equivalence relations \( \sim_F \) satisfying properties (i)-(iii) will be called a good equivalence relation on quadratic forms (over all fields).
Below we will drop the index $F$ at $\sim_F$ and write simply $\sim$.

**Definition 2.2.** Let $\phi$ and $\psi$ be $F$-forms. We say that the quadratic form $\phi$ is equivalent to the quadratic form $\psi$ in the sense of Vishik if $\dim \phi = \dim \psi$ and for any field extension $E/F$ we have $i_E(\phi_E) = i_E(\psi_E)$. In this case, we write $\phi \sim \psi$.

The following lemma is obvious.

**Lemma 2.3.** The equivalence relation $\sim$ is a minimal good equivalence relation. More precisely,

- The equivalence relation $\sim$ is a good relation.
- For any good relation $\tilde{\sim}$, the condition $\phi \sim \psi$ implies $\phi \tilde{\sim} \psi$.

**Example 2.4.** Let $X$ be a smooth variety over $F$. By $M(X)$ we denote the motive of $X$ in the category of Chow motives. Let us define the equivalence $\sim$ of quadratic forms $\phi$ and $\psi$ as follows:

$$\phi \sim \psi \text{ if } M(X) \simeq M(X).$$

Then $\sim$ is a good equivalence relation.

**Proof.** Clearly, conditions (i) and (ii) in Definition 2.1 are fulfilled. We need to verify only condition (iii). Let $X = X_0$ and let $\bar{F}$ denote the algebraic closure of $F$. By [9, Item (2.2) and Prop. 2.6] 3

- $\dim \phi$ coincides with the largest integer $m$ such that $\text{CH}_{m-2}(X) \neq 0$.
- The integer $i_W(\phi)$ coincides with the largest integer $m$ satisfying the conditions $m \leq \frac{1}{2} \dim \phi$ and $\text{coker}(\text{CH}_{m-1}(X) \to \text{CH}_{m-1}(X_{\bar{F}})) = 0$.

Thus, it suffices to show that the groups $\text{coker}(\text{CH}^j(X) \to \text{CH}^j(X_{\bar{F}}))$ and $\text{CH}^j(X)$ depend only on the motive of $X$. This can easily be proved if we observe that the functor $\text{CH}^j$ is representable in the category of Chow motives. Namely, $\text{CH}^j(X) = \text{Hom}_{MV}(M(pt_{\bar{F}})(j), M(X))$, where $M(pt_{\bar{F}})$ is the motive of $pt_{\bar{F}} = \text{Spec}(\bar{F})$ and the object $M(pt_{\bar{F}})(j)$ is defined, e.g., in [24]. Thus, $\text{CH}^j(X)$ depends only on the motive of $X$. Now, we consider the base change functor $\Phi : MV_F \to MV_{\bar{F}}$. Since the homomorphism $\text{CH}^j(X) \to \text{CH}^j(X_{\bar{F}})$ coincides with the homomorphism

$$\Phi : \text{Hom}_{MV}(M(pt_{\bar{F}})(j), M(X)) \to \text{Hom}_{MV}(\Phi(M(pt_{\bar{F}})(j)), \Phi(M(X))),$$

it follows that the group $\text{coker}(\text{CH}^j(X) \to \text{CH}^j(X_{\bar{F}}))$ also depends only on $M(X)$.  

**Theorem 2.5.** Let $\sim$ be a good equivalence relation. Let $\phi$ and $\psi$ be odd-dimensional quadratic forms over a field. Then the condition $\phi \sim \psi$ is equivalent to the condition $\phi \sim \psi$.

**Proof.** We start the proof with three lemmas

**Lemma 2.6.** Let $\phi$ and $\psi$ be odd-dimensional anisotropic forms of dimension $\geq 3$ such that $\dim \phi = \dim \psi$ and $(\phi_{F(\phi)})_{an} \simeq (\psi_{F(\psi)})_{an}$. Then $\phi \sim \psi$.

**Proof.** If $\phi \not\sim \psi$, Corollary 1.3 shows that $\phi \perp -\psi \in G_P(F)$. Since $\dim \phi = \dim \psi$, we conclude that $\dim \psi$ is a power of 2. Since $\dim \psi \geq 3$, we see that $\dim \psi$ is even. We get a contradiction to the assumption of the lemma.

3 see also [22, Prop. 2] and [25].
The following lemma is obvious.

**Lemma 2.7.** Let \( \phi \) and \( \psi \) be odd-dimensional forms such that \( \dim \phi = \dim \psi \) and \( \det \phi = \det \psi \). Then the condition \( \phi \sim \psi \) is equivalent to the condition \( \phi \simeq \phi \).

**Lemma 2.8.** Let \( \phi \) and \( \psi \) be odd-dimensional forms such that \( \dim \phi_{an} = \dim \psi_{an} \geq 3 \). Suppose that \( \phi_{F(\phi_{an})} \sim \psi_{F(\phi_{an})} \). Then \( \phi \sim \psi \).

*Proof.* Replacing first \( \phi \) and \( \psi \) by \( \phi_{an} \) and \( \psi_{an} \), respectively, we may assume that \( \phi \) and \( \psi \) are anisotropic. Replacing then \( \phi \) by \( \frac{\det \phi}{\det \psi} \phi \) and \( \psi \) by \( \frac{\det \psi}{\det \phi} \psi \), we may assume that \( \det \phi = 1 = \det \psi \). Since \( \phi_{F(\phi)} \sim \psi_{F(\phi)} \), Lemma 2.7 implies that \( \phi_{F(\phi)} \simeq \psi_{F(\phi)} \). By Lemma 2.6, we have \( \phi \simeq \psi \).

Now, we return to the proof of Theorem 2.5. We use induction on \( n = \dim \phi_{an} = \dim \psi_{an} \). The case where \( n = 1 \) is obvious. So we may assume that \( n \geq 3 \). Since \( \phi \sim \psi \), we have \( \phi_{F(\phi_{an})} \sim \psi_{F(\phi_{an})} \). By the induction assumption, we have \( \phi_{F(\phi_{an})} \sim \psi_{F(\phi_{an})} \). Now, Lemma 2.8 implies that \( \phi \sim \psi \).

**Corollary 2.9.** Let \( \phi \) and \( \psi \) be odd-dimensional quadratic forms over a field. Then \( \phi \sim \psi \iff \phi \simeq \psi \iff \phi \sim \psi \).

### 3. Even-Dimensional Forms

In this section, we study the relation \( \simeq \) in the case of even-dimensional forms. If quadratic forms \( \phi \) and \( \psi \) of dimension \( \geq 2 \) satisfy the condition \( \phi \sim \psi \), then \( \phi_{F(\psi)} \) and \( \psi_{F(\phi)} \) are isotropic (because \( \phi_{F(\phi)} \) and \( \psi_{F(\psi)} \) are isotropic).

**Proposition 3.1.** Let \( \phi \) and \( \psi \) be quadratic forms of dimension \( \leq 5 \). Then \( \phi \sim \psi \iff \phi \simeq \psi \iff \phi \sim \psi \).

*Proof.* In view of Corollary 2.9, we may assume that \( n = \dim \phi = \dim \psi \) is even. Thus, it suffices to consider the cases \( n = 2, 4 \), and \( 6 \). The implications \( \phi \sim \psi \Rightarrow \phi \simeq \psi \Rightarrow \phi \sim \psi \) are obvious. Therefore, we must verify only that \( \phi \sim \psi \) implies \( \phi \sim \psi \). Since \( \phi \sim \psi \), the forms \( \phi_{F(\psi)} \) and \( \psi_{F(\phi)} \) are isotropic. In the case \( n = 2 \), this obviously means that \( \phi \sim \psi \). If \( n = 4 \), then \( \phi \sim \psi \) by Wadsworth’s theorem [28]. Thus, we may assume that \( n = 6 \). We need the following assertion concerning the isotropy of 6-dimensional forms.

**Lemma 3.2.** (see [4, 13, 16, 21]). Let \( \phi \) and \( \psi \) be anisotropic 6-dimensional forms such that \( \phi_{F(\psi)} \) is isotropic. Then either \( \phi \sim \psi \) or \( \psi \) is a 3-fold Pfister neighbor.

In view of this lemma, we may assume that \( \psi \) is a Pfister neighbor of a 3-fold Pfister form \( \pi \). Since \( \psi_{F(\phi)} \) is isotropic, it follows that \( \pi_{F(\phi)} \) is isotropic. Hence \( \phi \) is a Pfister neighbor of \( \pi \). Therefore, \( \phi \sim (\pi - \langle d_{\phi} \rangle)_{an} \) and \( \psi \sim (\pi \perp - \langle d_{\psi} \rangle)_{an} \). Thus, it suffices to verify that \( d_{\psi} \phi = d_{\psi} \psi \). This is a consequence of the following chain of equivalent conditions

\[
a = d_{\psi} \phi \iff \imath_{W}(\phi_{F(\psi)}) = 3 \iff \imath_{W}(\psi_{F(\phi)}) = 3 \iff a = d_{\psi} \psi
\]

The proof is complete.

Now, we begin to study even-dimensional forms of dimension \( \geq 8 \).

**Lemma 3.3.** (see, e.g., [27]). Let \( \phi \) and \( \psi \) be half-neighbors. Then \( \phi \sim \psi \).
For the reader’s convenience, we cite the proof (which, in fact, is trivial).

**Proof.** The condition \( \phi \cong \psi \) means that \( \dim \phi = \dim \psi \) and there exist \( s, r \in F^* \) such that \( s\phi \perp r\psi = \pi \in P_4(F) \). Let \( L/F \) be a field extension. If both \( \phi_L \) and \( \psi_L \) are anisotropic, then \( i_W(\phi_L) = 0 = i_W(\psi_L) \). If at least one of the forms \( \phi_L \) or \( \psi_L \) is isotropic, then \( \pi_L \) is also isotropic. Taking into account the condition \( \pi \in P_4(F) \), we conclude that \( \pi_L \) is hyperbolic. Therefore, \( s\phi_L = -r\psi_L \) in the Witt ring. Since \( \dim \phi = \dim \psi \), we have \( s\phi_L = -r\psi_L \). Hence \( i_W(\phi_L) = i_W(\psi_L) \).

The following lemma shows that there exist examples of nonsimilar half-neighbors.

**Lemma 3.4.** (see [6], [8]). For any \( n \geq 3 \), there exists a field \( F \) and \( 2^n \)-dimensional half-neighbors \( \phi \) and \( \psi \) such that \( \phi \not\sim \psi \).

As a consequence of this result, we see that, for any \( n \geq 3 \), there exists a pair of \( 2^n \)-dimensional forms \( \phi \) and \( \psi \) such that \( \phi \not\sim \psi \) and \( \phi \not\sim \psi \). In particular, Proposition 3.1 cannot always be generalized for \( 8 \)-dimensional forms.

Nevertheless, for \( 8 \)-dimensional forms with trivial determinant, we have the following

**Proposition 3.5.** Let \( \phi \) and \( \psi \) be \( 8 \)-dimensional forms with trivial determinant. Then the following conditions are equivalent:

1. \( \phi \not\sim \psi \);
2. \( \phi_{F(\psi)} \) and \( \psi_{F(\phi)} \) are isotropic;
3. \( \phi \) and \( \psi \) are half-neighbors.

**Proof.** The implications \( (3) \Rightarrow (1) \Rightarrow (2) \) are obvious. The implication \( (2) \Rightarrow (3) \) follows immediately from the results of A. Laghribi [16], [15], [14].

4. **Generalized Albert forms**

In this section, we construct examples of nonsimilar \( \sim \)-equivalent forms based on the so-called generalized Albert forms.

**Definition 4.1.** A generalized Albert form (or \( n \)-Albert form) is a form of type \( q = \pi \perp -r' \), where \( \pi' \) and \( \pi' \) are pure parts of \( n \)-fold Pfister forms \( \pi \) and \( \tau \).

**Remark 4.2.**

- Any \( n \)-Albert form has dimension \( 2(2^n - 1) \).
- Suppose that \( q \) is an \( n \)-Albert form. By [2, Proof of Prop. 4.4], the anisotropic part \( q_{an} \) looks like \( q_{an} = \langle a_1, \ldots, a_m \rangle q' \), where \( q' \) is an anisotropic \( (n - m) \)-Albert form. In particular, \( \dim q_{an} \) has dimension \( 2^m - 2(2^m - 1) = 2(2^n - 2^m) \), where \( 0 \leq m \leq n \). We say that \( m \) is the linkage number of the \( n \)-Albert form \( q \).
- Every 1-Albert form has the form \( q = \langle a \rangle' \perp -\langle b \rangle = \langle -a, b \rangle \). Hence any 2-dimensional form is a 1-Albert form.
- Every 2-Albert form has the form

\[
q = \langle a_1, a_2 \rangle' \perp -\langle b_1, b_2 \rangle = \langle -a_1, -a_2, a_1, a_2, b_1, b_2, -b_1 b_2 \rangle.
\]

Thus, a 2-Albert form is the “classical” 6-dimensional Albert form.
Our interest in \(n\)-Albert forms is motivated by the following observation of A. Vishik (see [27]): if \(q_1\) and \(q_2\) are \(n\)-Albert forms such that \(q_1 \equiv q_2 \pmod{I^{n+1}(F)}\), then \(q_1 \sim q_2\).

The following question is due to Lam [18, Item (6.6), Page 28].

**Question 4.3.** Let \(q_1\) and \(q_2\) be \(n\)-Albert forms such that \(q_1 \equiv q_2 \pmod{I^{n+1}(F)}\). Is it always true that \(q_1 \sim q_2\)?

The answer to this question is obviously positive in the case \(n = 1\). In the case \(n = 2\), the answer is also positive. This is a version of a Jacobson’s theorem (see, e.g., [26, Th. 3]). In this section, we construct a counterexample to this question for any \(n \geq 3\).

**Theorem 4.4.** There exists a field \(F\) and anisotropic \(3\)-Albert forms \(q_1\) and \(q_2\) over \(F\) such that \(q_1 \equiv q_2 \pmod{I^4(F)}\) and \(q_1 \not\sim q_2\). In particular, the answer to Question 4.3 is negative in the case \(n = 3\).

**Proof.** We need the following theorem of Hoffmann.

**Theorem 4.5.** (see [6, Th. 4.3]). There exists a field \(k\) and anisotropic 8-dimensional quadratic forms over \(k\),

\[
\phi_1 = s_1 \langle a_1, b_1 \rangle - k_1 \langle c_1, d_1 \rangle,
\]

\[
\phi_2 = s_2 \langle a_2, b_2 \rangle - k_2 \langle c_2, d_2 \rangle
\]

such that \(\phi_1 \equiv \phi_2 \pmod{I^4(k)}\), \(\text{ind } C(\phi_1) = \text{ind } C(\phi_2) = 4\) and \(\phi_1 \not\sim \phi_2\). \(\square\)

**Remark 4.6.** In fact, the formulation of Theorem 4.3 in [6] differs from the one presented above. In his theorem, Hoffmann has constructed a pair \(\psi, \psi' \in I^7(k)\) of 8-dimension quadratic forms such that \(\psi \not\sim \psi'\) and \(\psi \not\sim \psi\). Clearly, changing \(\psi\) by a scalar, we may always assume that \(\psi = (\text{mod } I^4(k))\). To obtain Theorem 4.5, it suffices to show that we may always take \(\phi\) and \(\psi\) in the form of direct sums of forms belonging to \(GP_2(k)\). In the proof of [6, Theorem 4.3] it is so for the form \(\phi\) (the explicit formula for \(\phi\) in [6] shows that \(\phi\) contains a subform \(f_1(x, y, x_2)\)). The required statement concerning \(\psi\) is obvious since \(I_{k, \sqrt{-2}}(\psi_{k, \sqrt{-2}}) = I_{k, \sqrt{-2}}(\psi_{k, \sqrt{-2}}) \geq 2\).

Now we return to the proof of Theorem 4.4. Under the conditions of this theorem, we obviously have \((a_1, b_1) + (c_1, d_1) = c(\phi_1) = c(\phi_2) = (a_2, b_2) + (c_2, d_2)\). Hence there exists an Albert form \(\rho\) (of dimension 6) such that \(c(\phi_1) = c(\phi_2) = c(\rho)\). Hence \(\text{ind } C(\rho) = \text{ind } C(\phi_1) = 4\). By an Albert’s theorem, \(\rho\) is anisotropic (see [1, Th. 3] or [26, Th. 3]). Since \((a, b) + (c, d) = c(\rho)\) for \(i = 1, 2\), there exist \(r_1\) and \(r_2\) such that

\[
\langle a_1, b_1 \rangle' \perp \langle c_1, d_1 \rangle' \simeq r_1 \rho,
\]

\[
\langle a_2, b_2 \rangle' \perp \langle c_2, d_2 \rangle' \simeq r_2 \rho.
\]

In the Witt ring \(W(k(t))\), we have

\[
tp - \phi_i = tr_{i, t}(\langle a_i, b_i \rangle - \langle c_i, d_i \rangle) = (s_i, \langle a_i, b_i \rangle - k_i \langle c_i, d_i \rangle)
\]

\[
= tr_{i, t}(\langle a_i, b_i \rangle - tr_{i, t} s_i, \langle a_i, b_i \rangle) - tr_{i, t}(\langle c_i, d_i \rangle - tr_{i, t} k_i, \langle c_i, d_i \rangle)
\]

\[
= tr_i(\langle a_i, b_i, tr_i s_i \rangle - \langle c_i, d_i, tr_i k_i \rangle).
\]

We set \(q_i = \langle a_i, b_i, tr_i s_i \rangle \perp \langle c_i, d_i, tr_i k_i \rangle\) and \(F = k(t)\). Since \(tp - \phi_i = tr_t q_i\) in the Witt ring \(W(F)\) and \(\dim(tp \perp -\phi_i) = 6 + 8 = 14 = \dim q_i\), we have \(tp \perp -\phi_i \simeq tr_t q_i\).\(\)
Since $\rho$ and $\phi$ are anisotropic, $q_1$ is also anisotropic by Springer’s theorem [see [17, Ch. 6, Th. 1.4] or [23, Ch. 6, Cor. 2.6]].

Now, we need the following obvious assertion.

**Lemma 4.7.** (see, e.g., [6, Lemma 3.1]). Let $\mu_1, \mu_2, \nu_1, \nu_2$ be anisotropic quadratic forms over $k$. Suppose that the form $\mu_1 \perp tv_1$ is similar to $\mu_2 \perp tv_2$ over the field of rational functions $k(t)$. Then

- either $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$,
- or $\mu_1 \sim \nu_2$ and $\nu_1 \sim \mu_2$.

Thus, we have proved that $q_1$ and $q_2$ are anisotropic 3-Albert forms such that $q_1 \equiv q_2 \pmod{I^4(F)}$ and $q_1 \not\equiv q_2$. The theorem is proved.

**Corollary 4.8.** For any $n \geq 3$, there exists a field $E$ and $n$-Albert forms $\gamma_1$ and $\gamma_2$ over $E$ such that $\gamma_1 \equiv \gamma_2 \pmod{I^{n+1}(E)}$ and $\gamma_1 \not\equiv \gamma_2$. In other words, the answer to Question 4.3 is negative for any $n \geq 3$.

**Proof.** Let $q_1, q_2$ and $F$ be as in Theorem 4.4. We write $q_1$ and $q_2$ in the form $q_1 = x_1^2 + \ldots + x_n^2 - 2x_{n-1}^2$ and $q_2 = x_1^2 + \ldots + x_n^2 - 2x_{n-1}^2$ with $x_1, x_2, x_3, x_4 \in P_3(F)$ and put $E = F[x_1, \ldots, x_{n-3}]$ and

$$
\gamma_1 = (x_1 \langle x_1, \ldots, x_{n-3} \rangle \rangle)^t \perp -(x_1 \langle x_1, \ldots, x_{n-3} \rangle \rangle)^t,
$$

$$
\gamma_2 = (x_2 \langle x_1, \ldots, x_{n-3} \rangle \rangle)^t \perp -(x_2 \langle x_1, \ldots, x_{n-3} \rangle \rangle)^t.
$$

Obviously, $\gamma_1 = q_1 \langle x_1, \ldots, x_{n-3} \rangle \rangle$ in the Witt ring $W(E)$. Since $q_1 \equiv q_2 \pmod{I^4(F)}$, we have $\gamma_1 \equiv \gamma_2 \pmod{I^{n+1}(E)}$. Since $q_1 \not\equiv q_2$, we have $q_1 \langle x_1, \ldots, x_{n-3} \rangle \rangle \not\equiv q_2 \langle x_1, \ldots, x_{n-3} \rangle \rangle$ (see, e.g., Lemma 4.7). Hence $\gamma_1 \not\equiv \gamma_2$. 

We have constructed a pair of $n$-Albert forms $\gamma_1$ and $\gamma_2$ such that $\gamma_1 \equiv \gamma_2$ and $\gamma_1 \not\equiv \gamma_2$. Obviously, in our example, we have $\dim(\gamma_1)_{an} = 2^{m-3} + 1 = 2^{n-3}(2^{n-3} - 2) = 2(2^n - 2^{n-3})$. In other words, both $n$-Albert forms $\gamma_1$ and $\gamma_2$ are $(n-3)$-linked. We can generalize this example as follows.

**Theorem 4.9.** For any $n \geq 3$ and $m$ such that $0 \leq m \leq n - 3$, there exists a field $F$ and $n$-Albert forms $q_1$ and $q_2$ over $F$ such that $q_1 \equiv q_2 \pmod{I^{n+1}(F)}$, $q_1 \not\equiv q_2$, and dim($q_1$)$_{an} = \dim(q_2)_{an} = 2(2^n - 2^m)$.

Here we only outline the proof of the theorem.

**Step 1.** It suffices to prove this theorem only in the case $m = 0$ (this means that $q_1$ and $q_2$ are anisotropic). After this, the general case can be obtained in the same way as Corollary 4.8.

**Step 2.** Consider a field $E$ and $n$-Albert forms $\gamma_1$ and $\gamma_2$ as in Corollary 4.8. Since $\gamma_1 \equiv \gamma_2 \pmod{I^{n+1}(E)}$, there exist $x_1, \ldots, x_N \in P_{n+1}(E)$ for some integer $N$...
such that $\gamma_1 - \gamma_2 = \sum_{i=1}^{N} \pi_i$. We consider the quadratic forms
\[
\bar{q}_1 = \langle x_1, \ldots, x_n \rangle' \setminus \langle y_1, \ldots, y_n \rangle',
\]
\[
\bar{q}_2 = \langle z_1, \ldots, z_n \rangle' \setminus \langle t_1, \ldots, t_n \rangle',
\]
\[
\tau = \sum_{i=1}^{N} \langle u_{i,1}, \ldots, u_{i,n+1} \rangle.
\]
over the field of rational functions
\[
\bar{E} = E(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n, t_1, \ldots, t_n, u_{i,1}, \ldots, u_{N,n+1}).
\]
Obviously there exists a place $\bar{s} : \bar{E} \to E$ such that $\bar{q}_1 \mapsto \gamma_1$, $\bar{q}_2 \mapsto \gamma_2$, and $\langle u_{i,1}, \ldots, u_{i,n+1} \rangle \mapsto \pi_i$ for all $i = 1, \ldots, N$. Since $\gamma_1 - \gamma_2 = \sum_{i=1}^{N} \pi_i$, the form $s_\tau(\bar{q}_1 \setminus -\bar{q}_2 \setminus -\tau)$ is hyperbolic.

**Step 3.** We define the field $F$ as a “generic” extension $F/\bar{E}$ such that $(\bar{q}_1)_F = (\bar{q}_2)_F = \tau_F$. More precisely, we set $F = \bar{E}_h$, where $\bar{E}_0, \bar{E}_1, \ldots, \bar{E}_h$ is the generic splitting tower for the $E$-form $\bar{q}_1 \setminus -\bar{q}_2 \setminus -\tau$. We claim that the $F$-forms $q_1 \overset{\text{Def}}{=} (\bar{q}_1)_F$ and $q_2 \overset{\text{Def}}{=} (\bar{q}_2)_F$ satisfy the hypotheses of Theorem 4.9. Since $q_1 - q_2 = \tau_F$, we have $q_1 = q_2$ (mod $\mathcal{I}^{n+1}(F)$). Thus, it suffices to verify that $q_1$ and $q_2$ are anisotropic and $q_1 \not\sim q_2$.

**Step 4.** Using properties of generic splitting fields (see [23, Ch. 4, Cor. 6.10] or [11, Th. 5.1]), we can extend $s : E \to E$ to a place $s : F \to E$. Obviously, $s_\tau(q_1) = \gamma_1$ and $s_\tau(q_2) = \gamma_2$. Therefore, the condition $\gamma_1 \not\sim \gamma_2$ implies $q_1 \not\sim q_2$.

**Step 5.** To prove that $q_1$ and $q_2$ are anisotropic, it suffices to construct a field extension $K/\bar{E}$ with the same key property as $F$ (i.e., $(\bar{q}_1)_K = (\bar{q}_2)_K = \tau_K$) and such that $(\bar{q}_1)_K$ and $(\bar{q}_2)_K$ are anisotropic. Since $F/\bar{E}$ is a “generic” extension, we necessarily get that $q_1 = (\bar{q}_1)_F$ and $q_2 = (\bar{q}_2)_F$ are anisotropic. The following extension $K/\bar{E}$ has the required properties:
\[
K = \bar{E}(\sqrt{x_1/z_1}, \ldots, \sqrt{x_n/z_n}, \sqrt{y_1/t_1}, \ldots, \sqrt{y_n/t_n}, \sqrt{u_{1,1}}, \ldots, \sqrt{u_{N,1}}).
\]
The “sketch” of the proof is complete. In fact, Steps 4 and 5 are the most difficult points. We refer the reader to the paper [7, Proof of Lemma 2.2], where similar arguments (as in Step 5) are presented with complete proofs.

**Corollary 4.10.** For any $m$ and $n$ such that $0 \leq m \leq n - 3$, there exists a field $F$ and anisotropic $2(2^n - 2^m)$-dimensional forms $q_1$ and $q_2$ over $F$ such that $q_1 \not\sim q_2$ and $q_1 \not\sim q_2$.

5. Open questions

Obviously, Theorem 4.9 cannot be generalized to the cases $m = n - 1$ and $m = n$ because in these cases the anisotropic parts of $n$-Albert forms either belong to $GP_n(F)$ or are zero. There is only one case, where we cannot say anything definite. Namely, $m = n - 2$. For this reason, we propose the following modification of Lam’s Question 4.3.

**Conjecture 5.1.** Let $q_1$ and $q_2$ be Albert forms (i.e., $6$-dimensional forms with trivial discriminants). Let $\phi_1 = \langle a_1, \ldots, a_k \rangle q_1$ and $\phi_2 = \langle b_1, \ldots, b_k \rangle q_2$. Suppose that $\phi_1 \equiv \phi_2$ (mod $\mathcal{I}^{k+3}(F)$). Then $\phi_1 \sim \phi_2$.
We note that, in this conjecture, we always may assume that \( a_i = b_i \) for \( i = 1, \ldots, k \). Indeed, putting \( \pi = \langle (a_1, \ldots, a_k) \rangle \), we obtain \( (\phi_1 | F(\pi)) \equiv (\phi_1 | F(\pi) = 0 (\mod I^{k+1}(F(\pi))) \). By the Arason–Pfister theorem, we conclude that \( \phi_2 \) is hyperbolic over the field \( F(\pi) \). Hence \( \phi_2 \) has the form \( \phi_2 = \pi q_2 = \langle (a_1, \ldots, a_k) \rangle q_2' \). Comparing dimensions, we get \( \dim q_2 = 6 \). Let us write \( q_2' = \langle c_1, \ldots, c_6 \rangle \) and set \( q_2'' = \langle c_1, \ldots, c_6, c_6 \rangle \), where \( c_6 = -c_1 \ldots c_5 \). We have \( \pi (c_6, -c_6') = \pi q_2'' = \phi_2 - \pi q_2' \in I^{k+2}(F) + I^k(F) \cdot I^2(F) = I^{k+4}(F) \). Since \( \dim \pi (c_6, -c_6') = 2^i \cdot 2 < 2^{k+2} \), the Arason–Pfister theorem shows that \( \pi (c_6, -c_6') \) is hyperbolic. Hence \( \pi q_2'' = \pi q_2'' \). Therefore, \( \phi_2 = \pi q_2'' = \langle (a_1, \ldots, a_k) \rangle q_2'' \). Since \( q_2'' \) is an Albert form, we have proved that the conjecture reduces to the case where \( b_i = a_i \).

Another question concerning the \( \sim \)-equivalence is motivated by the results of §3 and §4. First of all, in view of Lemma 3.4 and Corollary 4.10, we have the following assertion.

**Proposition 5.2.** Let \( d \) be an integer belonging to the set

\[ \{ 2^a | n \geq 3 \} \cup \{ 2^i (2^j - 1) | i \geq 1, j \geq 3 \} \]

Then there exist anisotropic \( d \)-dimensional quadratic forms \( \phi \) and \( \psi \) over a suitable field such that \( \phi \sim \psi \) and \( \phi \not\sim \psi \).

Here we state the following

**Problem 5.3.** Describe the set \( \mathcal{V} \mathcal{E} \) of all integers \( d \) for which there exist anisotropic \( d \)-dimensional quadratic forms \( \phi \) and \( \psi \) over a suitable field such that \( \phi \sim \psi \) and \( \phi \not\sim \psi \).

We know almost the full answer to this problem. The results of the previous sections imply that \( \mathcal{V} \mathcal{E} \subseteq \{ 8, 10, 12, \ldots, 2i, \ldots \} \). Besides, we can prove that any even integer \( \geq 8 \) (except possibly 12) belongs to \( \mathcal{V} \mathcal{E} \).

**References**


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