\textbf{\textit{L}-IN Variant FOR \textit{SIEGEL–HILBERT FORMS}}

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\textsc{Abstract.} We prove a formula for the Greenberg–Benois \textit{L}\textsuperscript{-}invariant of the spin, standard and adjoint Galois representations associated with Siegel–Hilbert modular forms. In order to simplify the calculation, we give a new definition of the \textit{L}\textsuperscript{-}invariant for a Galois representation \(V\) of a number field \(F \neq \mathbb{Q}\); we also check that it is compatible with Benois' definition for \(\text{Ind}_{\mathbb{Q}}^F(V)\).

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\section{Introduction}

Since the historical results of Kummer and Kubota–Leopold on congruences for Bernoulli numbers, people have been interested in studying the \(p\)-adic variation of special values of \(L\)-functions.

More precisely, fix a motive \(M\) over \(\mathbb{Q}\). We suppose that \(M\) is Deligne critical at \(s = 0\) and that there exists a Deligne's period \(\Omega(M)\) such that \(\frac{L(M,0)}{\Omega(M)}\) is algebraic. Fix a prime \(p\) and two embeddings

\[\mathbb{C}_p \hookrightarrow \mathbb{Q}_p \hookrightarrow \mathbb{C}.\]

Let \(V\) be the \(p\)-adic realization of \(M\) and suppose that \(V\) is semistable (à la Fontaine). Thanks to work of Coates and Perrin-Riou, we have precise conjectures on how the special values should behave \(p\)-adically; we fix a regular sub-module of \(V\). This corresponds to the choice of a sub-\((\varphi, N)\)-module of \(\mathcal{D}_{\text{st}}(V)\) which is a section of the exponential map

\[\mathcal{D}_{\text{st}}(V) \to t(V) \cong \frac{\mathcal{D}_{\text{st}}(V)}{\text{Fil}^p \mathcal{D}_{\text{st}}(V)}.\]

Let \(h\) be the valuation of the determinant of \(\varphi\) on \(D\). We can state the following conjecture:
Conjecture 1.1. There exists a formal series $L_p^D(V,T) \in \mathbb{C}_p[[T]]$ which grows as $\log_p^h$ such that for all non-trivial, finite-order characters $\varepsilon : 1 + p\mathbb{Z}_p \to \mu_{p^\infty}$ we have

$$L_p^D(V,\varepsilon(1 + p) - 1) = C_\varepsilon(D) \frac{L(M \otimes \varepsilon, 0)}{\Omega(M)}$$

Moreover, for $\varepsilon = 1$ we have

$$L_p^D(V,0) = E(D) \frac{L(M,0)}{\Omega(M)},$$

where $E(D)$ is an explicit product of Euler-type factors depending on $D$ and $(D_{\text{st}}(V)/D)^{N=0}$.

It may happen that one of the factors of $E(D)$ vanishes and then we say that trivial zeros appear. Since the seminal work of [MTT86], people have been interested in describing the $p$-adic derivative of $L_p^D(V,(1 + p)^s - 1)$ when trivial zeros appear.

We suppose for simplicity that $L(M,0)$ is not vanishing. We have the following conjecture by Greenberg and Benois;

Conjecture 1.2. Let $t$ be the number of vanishing factors of $E(D)$. Then

- $\text{ord}_{s=0} L_p^D(V,(1 + p)^s - 1) = t$,
- $L_p^D(V,0)^* = \mathcal{L}(V^*(1), D^*) E^*(D) \frac{L(M,0)}{\Omega(M)}$.

Here $E^*(D)$ is the product of non-vanishing factors of $E(D)$ and $\mathcal{L}(V^*(1), D^*)$ is a number, defined in purely Galois theoretical terms (see Section 3.1), for the dual Galois representation $V^*(1)$.

The error factor $\mathcal{L}(V,D)$ is quite mysterious. It has been calculated in only few cases for the symmetric square of a (Hilbert) modular form by Hida, Mok and Benois and for symmetric power of Hilbert modular forms by Hida and Harron–Jorza. Unless $V$ is an elliptic curve over $\mathbb{Q}$ with multiplicative reduction at $p$ we can not prove the non-vanishing of $\mathcal{L}(V,D)$.

The aim of this paper is to calculate it in some new cases; let $F$ be a totally real field (we make no assumptions on the ramification at $p$) and $\pi$ be an automorphic representation of $\text{GSp}_{2g}/F$ of weight $k = (k_\tau)_\tau$, where $\tau$ runs through the real embeddings of $F$ and $(k_\tau) = (k_1,\ldots,k_g; k_0)$ (note that $k_0$ does not depend on $\tau$). We say that $\pi$ is parallel of weight $k$, $k \in \mathbb{Z}_{\geq 0}$ if $k_{i,\tau} = k$ for all $\tau$ and $i = 1,\ldots,g$ and $k_0 = gk$.

We suppose that it has Iwahoric level at all $p | p$. We suppose moreover that $\pi_p$ is either Steinberg (see Definition 4.8) or spherical. We partition consequently the prime ideals of $F$ above $p$ in $S^{\text{Stb}} \cup S^{\text{Sph}}$.

We have conjecturally two Galois representations associated with $\pi$, namely
the spinorial one $V_{\text{spin}}$ and the standard one $V_{\text{sta}}$. Let $V$ be one of these two representations. We choose for each prime $p$ of $F$ dividing $p$ a regular sub module $D_p$ of $D_{\text{st}}(V_{\text{spin}})$.

Consider a family of Siegel–Hilbert modular forms as in [Urb11] passing through $\pi$. Let us denote by $\beta_p(k)$ the eigenvalue of the normalized Hecke operators $U_{1,p}$ (see Definition 4.9) on this family. Let $S^{\text{Sph},1} = S^{\text{Sph},1}(V, D)$ be the subset of $S^{\text{Sph}}$ for which $(D_{\text{st}}(V_p)/D_p)^{N=0}$ does contain the eigenvalue 1. Conjecturally, it is empty for the spin representation. The eigenvalues 1 always appears in $D_{\text{st}}(V_p)$ for $V$ the standard representation but it may appear in $D_p$ (this is already the case for the symmetric square of a modular form).

Let $t_{\text{Sph}}$ be the cardinality of $S^{\text{Sph}}$ and $t_{\text{Sph}}$ be the cardinality of $S^{\text{Sph},1}$. We define $f_p = [\mathbb{F}_p : \mathbb{Q}_p]$.

**Theorem 1.3.** Let $\pi$ be as above, of parallel weight $k$. Let $V = V_{\text{spin}}$ and suppose hypothesis LGP of Section 4.2, then the expected number of trivial zeros for $L_p^D(V(k-1), T)$ is $t_{\text{Sph}}$ and

$$L(V(k-1), D) = \prod_{p \in S^{\text{Sph}}} -\frac{1}{f_p} \frac{d \log \beta_p(k)}{dk} \bigg|_{k=2}.$$

Let $V = V_{\text{ad}}$, then the conjectural number of trivial zero for $L_p^D(V, T)$ is $t_{\text{Sph}} + t_{\text{Sph}}$ and

$$L(V, D) = L(V, D)^{\text{Sph}} \prod_{p \in S^{\text{Sph}}} -\frac{1}{f_p} \frac{d \log \beta_p(k)}{dk} \bigg|_{k=2},$$

where $L(V, D)^{\text{Sph}}$ is a priori global factor. It is 1 if $t_{\text{Sph}} = 0$.

In Section 4.2 we shall provide also a formula for the $L$-invariant of $V_{\text{ad}}(s)$ (min$(k-g-1, g-1) \geq s \geq 1$).

The proof of the theorem is not different from the one of [Ben10] Theorem 2 which in turn is similar to the original one of [GS93].

Let now $g = 2$. Let $t$ be the number of primes above $p$ in $F$. We consider the $2t$-dimensional eigenvariety for $\text{GSp}_{4/F}$ with variables $k = \{k_p, 1, k_p, 2\}$ (see Section 5) and let us denote by $F_{p,i}(k)$ ($i = 1, 2$) the first two graded pieces of $D^\dagger_{\text{rig}}(V_{\text{spin}})$. The 10-dimensional Galois representation $\text{Ad}(V_{\text{spin}})$ has a natural regular sub-($\varphi, N$)-module induced by the $p$-refinement of $D^\dagger_{\text{rig}}(V_{\text{spin}})$ and which we shall denote by $D_{\text{Ad}}$. With this choice of regular sub module, $\text{Ad}(V_{\text{spin}})$ presents $2t$ conjectural trivial zeros. In Section 5 we prove the following theorem;

**Theorem 1.4.** Let $\pi$ be an automorphic form of weight $k$. Suppose that hypothesis LGP of Section 4.2 is verified for $V_{\text{spin}}$ and the point corresponding
to \( \pi \) in the eigenvariety \( \mathcal{X}' \) (as defined in Section 5) is étale over the weight space. We have then
\[
L(\text{Ad}(V_{\text{spin}}(\pi)), D_{\text{Ad}}) = \prod_p \frac{2}{f_p^2} \det \begin{pmatrix}
\frac{\partial \log F_p^{i,j}(k)}{\partial k^{i,j}_{p,1}} & \frac{\partial \log F_p^{i,j}(k)}{\partial k^{i,j}_{p,2}} \\
\frac{\partial \log F_p^{i,j}(k)}{\partial k^{i,j}_{p,1}} & \frac{\partial \log F_p^{i,j}(k)}{\partial k^{i,j}_{p,2}}
\end{pmatrix}
\]

We remark that this theorem is the first to really go beyond GL\(_2\) and its representations \( \text{Sym}^{n} \).

The motivation for Theorem 1.3 lies in a generalization of [Ros15] to Siegel forms. In loc. cit. we use Greenberg–Stevens method to prove a formula for the derivative of the symmetric square \( p \)-adic \( L \)-function and calculate the analytic \( L \)-invariant and the same method of proof could possibly be generalized to finite slope Siegel forms thanks to the overconvergent Maß-Shimura operators and overconvergent projectors of Z. Liu’s thesis.

With some work, it could also be generalized to totally real field where \( p \) is inert, as already done for the symmetric square [Ros13].

We hope to calculate the \( L \)-invariant for \( V_{\text{std}} \) and \( \text{Ad}(V_{\text{spin}}) \) for more general forms in a future work.

In Section 2 we recall the theory of \( (\varphi, \Gamma) \)-module over a finite extension of \( \mathbb{Q}_p \). It will be used in Section 3 to generalize the definition of the \( L \)-invariant à la Greenberg–Benois to Galois representations \( V \) over general number field \( F \) (note that we do not suppose \( p \) split or unramified). This definition does not require one to pass through \( \text{Ind}_{\mathbb{Q}}^{F}(V) \) to calculate the \( L \)-invariant which in turn simplifies explicit calculation. We shall check that this definition coincides with Benois’ definition for \( \text{Ind}_{\mathbb{Q}}^{F}(V) \).

We prove the above-mentioned theorems in Section 4 and 5, inspired mainly by the methods of [Hid07].

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2 Some results on rank one \((\varphi, \Gamma)\)-module

Let \(L\) be a finite extension of \(\mathbb{Q}_p\). The aim of this section is to recall certain results concerning \((\varphi, \Gamma)\)-modules over the Robba ring \(\mathcal{R}_L\). Let \(L_0\) be the maximal unramified extension contained in \(L\). Let \(L'_0\) be the maximal unramified extension contained in \(L_\infty := L(\mu_{p^\infty})\) and \(L' = L \cdot L'_0\). Let \(e_L := [L(\mu_{p^\infty}) : L_0(\mu_{p^\infty})] = [\Gamma_{\mathbb{Q}_p} : \Gamma_L]\), where \(\Gamma_L := \text{Gal}(L_\infty/L)\). We define

\[
B^\dagger_{L, \text{rig}} = \left\{ f = \sum_{n \in \mathbb{Z}} a_n \pi^n_L \mid a_n \in L'_0, \text{ such that } f(X) = \sum_{n \in \mathbb{Z}} a_n X^n \text{ is holomorphic on } p^{-e_L} \leq |X|_p < 1 \right\},
\]

where \(\pi_L\) is a certain uniformizer coming from the theory of field of norms. Note that \(B^\dagger_{L, \text{rig}}\) is classically called the Robba ring of \(L'_0\). For sake of notation, we shall denote write \(\mathcal{R}_L := B^\dagger_{L, \text{rig}}\). We hope that this will cause no confusion in what follows.

We have an action of \(\varphi\) on \(\mathcal{R}_L\). If \(L = L_0\), there is no ambiguity and we have:

\[
\varphi(\pi_L) = (1 + \pi_L)^{p - 1}, \quad \varphi(a_n) = \varphi(L'_0)(a_n).
\]

Otherwise the action on \(\pi_L\) is more complicated. Similarly, we have a \(\Gamma_L\)-action. If \(L = L_0\) we have

\[
\gamma(\pi_L) = (1 + \pi_L)^{\chi_{\text{cycl}}(\gamma)} - 1,
\]

where \(\chi_{\text{cycl}}\) is the cyclotomic character. If \(L\) is ramified we also have an action of \(\Gamma_L\) on the coefficients given by

\[
\gamma(a_n) = \sigma_\gamma(a_n)
\]

where \(\sigma_\gamma\) is the image of \(\gamma\) via

\[
\Gamma_L \rightarrow \Gamma/L \rightarrow \Gamma/L \rightarrow \text{Gal}(L'_0/L_0).
\]

If \(a_n\) is fixed by \(\varphi\) and \(\Gamma_L\), then it is in \(\mathbb{Q}_p\). We have \(\text{rk}_{\mathbb{Q}_p} \mathcal{R}_L = [L_\infty : \mathbb{Q}_{p,\infty}]\).

Let \(\delta : L^\times \rightarrow E^\times\) be a continuous character. Let \(\mathcal{R}_L(\delta)\) be the rank one \((\varphi, \Gamma_L)\)-module defined as follows: fix a uniformizer \(\varpi_L\) of \(L\) and write \(\delta = \delta_0\delta_1\) with \(\delta_0|_{\mathcal{O}_L^\times} = \delta|_{\mathcal{O}_L^\times}\), \(\delta_0(\varpi_L) := 1\) and \(\delta_1\) is trivial on \(\mathcal{O}_L^\times\) and \(\delta_1(\varpi_L) := \delta(\varpi_L)\). As \(\delta_0\) is a unitary character, it defines by class field theory a unique one dimensional Galois representation \(\hat{\delta}_0\). Fontaine’s theorem on the equivalence of category between \((\varphi, \Gamma_L)\)-modules and Galois representations \([\text{Fon90}]\) gives
us a one dimensional $(\varphi, \Gamma_L)$-module $\mathbf{D}_{\text{rig}}^\dagger(\delta_0)$.

We define $\mathcal{R}_L(\delta_i) := \mathcal{R}_L \otimes_{\mathbb{Q}_p} E e_{\delta_i}$ so that $\varphi^{f_\tau}(e_{\delta_i}) = \delta_i(\varphi_L) e_{\delta_i}$ (here $f_\tau$ is the degree of $L_\tau$ over $\mathbb{Q}_p$), $\gamma(e_{\delta_i}) = e_{\delta}$ and $\varphi$ does not act on the $E$-coefficient. Finally, we define $\mathcal{R}_L(\delta) = \mathbf{D}_{\text{rig}}^\dagger(\delta_0) \otimes_{\mathcal{R}_L} \mathcal{R}_L(\delta_1)$.

We now classify the cohomology of such a $(\varphi, \Gamma_L)$-modules. It will be useful to calculate it explicitly in terms of $C_{\varphi, \gamma}$-complexes [Ben11 §1.1.5]. We fix then a generator $\gamma_L$ of $\Gamma_L$; if clear from the context, we shall drop the subscript $L$ and write simply $\gamma$.

**Proposition 2.1.** We have $H^0(\mathcal{R}_L(\delta)) = 0$ unless $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$ for all $\tau$; in this case we have $H^0(\mathcal{R}_L(\delta)) \cong E$. We shall denote its basis by $t^{-m_\tau} \otimes e_\delta$, where

$$t^{-m_\tau} = (t^{-m_\tau}) \in \prod_\tau B^+_{\text{dr}} \otimes_{L, \tau} E.$$

If $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$, then

$$\dim_E H^1(\mathcal{R}_L(\delta)) = [L : \mathbb{Q}_p] + 1.$$

If $\delta(z) = |N_{L/\mathbb{Q}_p}(z)|_p \prod_\tau \tau(z)^{k_\tau}$ with $k_\tau \geq 1$, then

$$\dim_E H^1(\mathcal{R}_L(\delta)) = [L : \mathbb{Q}_p] + 1.$$

Otherwise

$$\dim_E H^1(\mathcal{R}_L(\delta)) = [L : \mathbb{Q}_p].$$

We have $H^2(\mathcal{R}_L(\delta)) = 0$ unless $\delta(z) = |N_{L/\mathbb{Q}_p}(z)|_p \prod_\tau \tau(z)^{k_\tau}$ with $k_\tau \geq 1$; in this case we have $H^2(\mathcal{R}_L(\delta)) \cong E$.

Note that when we choose $t^{-m_\tau}$ as a basis we are implicitly using the fact that we can embed certain sub-rings of $\mathcal{R}_L$ into $B^+_{\text{dr}}$ (see [Ben11 §1.2.1]).

**Proof.** The same results is stated in [Nak09 Proposition 2.14, 2.15, Lemma 2.16] for $E - B$-pairs, but the proof for $(\varphi, \Gamma)$-modules is the same. Recall that have a canonical duality [Liu08] given by cup product

$$H^i(D) \times H^{2-i}(D^*(\chi_{\text{cycl}})) \to H^2(\chi_{\text{cycl}}).$$

The last fact is then a direct consequence. $\square$

This allows us to define a canonical basis of $H^2(\mathcal{R}_L(|N_{L/\mathbb{Q}_p}(z)|_p \prod_\tau \tau(z)^{k_\tau}))$. We define $H^1_t(D)$ as the $H^1$ of the complex

$$\mathcal{D}_{\text{cris}}(D) \to t_D \otimes \mathcal{D}_{\text{cris}}(D)$$

and we have immediately [Nak09 Proposition 2.7]

$$\dim_E H^1_t(D) = \dim_E (H^0(D)) + \dim_E t_D. \quad (2.2)$$

Hence
Lemma 2.3. If $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$, then
\[
\dim_E H^1_F(R_L(\delta)) = 1.
\]
If $\delta(z) = |N_{L/Q_p}(z)|_p \prod_\tau \tau(z)^{k_\tau}$ with $k_\tau \geq 1$, then
\[
\dim_E H^1_F(R_L(\delta)) = d.
\]

Proposition 2.4. Let $D$ be a semi-stable $(\varphi, \Gamma)$-module over $R_L$ with non-negative Hodge–Tate weight. Suppose that $D_{st}(D) = D_{st}(D)^{\varphi=1}$. Then $D$ is crystalline,
\[
D \cong \oplus R_L(\delta_i)
\]
with $\delta_i(z) = \prod_\tau \tau(z)^{m_i, \tau}$, $m_i, \tau \leq 0$ and $D_{st}(D) = D_{cris}(D) = H^0(D)$.

Proof. We follow closely the proof [Ben11] Proposition 1.5.8. As $N, \varphi = p, \varphi N$ we obtain immediately that $N = 0$, hence $D$ is crystalline.

Let $r$ be the rank of $D$ over $R_L$. We write the Hodge–Tate weight as $(m_i)_{i=1}^r$ where $m_i = (m_i, \tau)$.

We prove the proposition by induction; the case $r = 1$ is easy.

If $D$ is not split, for $r = 2$, we can suppose, as $D$ is de Rham, that for each $\tau$ we have $-m_{1, \tau} \leq -m_{2, \tau}$, hence $m_1 = 0$ by twisting. Let $\delta$ be defined by $\prod_\tau \tau(z)^{m_{\tau}}$. So we have an extension of $R_L(\delta)$ by $R_L$. Let $d_2$ be a lift of $R_L$ of a basis of $R_L$. As $\varphi = 1$ we have $\varphi d_2 = d_2$. As the extension is crystalline we know that $\gamma$ acts trivially too, hence the extension splits.

Suppose now $r > 2$. Take $v$ in the Fil $^{2m}D_{st}(D)$, the smallest filtered piece of $D_{st}(D)$. We can associate to it $R_L(\delta)$, where $\delta(z) = \prod_\tau \tau(z)^{m_{\tau}}$. We have
\[
0 \to R_L(\delta) \to D \to D' \to 0.
\]
By inductive hypothesis $D' \cong \oplus_{i=1}^{d-1} R_L(\delta_i)$. We can write
\[
\text{Ext}(D', R_L(\delta)) = \oplus_{i=1}^{d-1} \text{Ext}(R_L(\delta_i), R_L(\delta))
\]
and we are reduced to the case $r = 2$ which has already been dealt. 

We now want to calculate $H^1_F(R_L(\delta))$ for $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$. We recall the following lemma [Ben11] Lemma 1.4.3

Lemma 2.5. The extension $\text{cl}(a, b)$ in $H^1(R_L(\delta))$ corresponding to the couple $(a, b)$ is crystalline if and only if the equation $(1 - \gamma)x = b$ has a solution in $R_L(\delta)$ [4].

The following proposition in an immediate consequence of the above lemma [Ben11, Theorem 1.5.7 (i)] (see also the construction of [Nak09] at page 900)

Proposition 2.6. Let $e_\delta$ be a basis for $R_L(\delta)$. Then $x_\delta = \text{cl}(t^{-m}, 0)e_\delta$ is a basis of $H^1_F(R_L(\delta))$. 

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We recall that for a $\mathbb{Z}_p$-extension of $\text{Hom}(G_{L, E}) \cong H^1(G_{L, E})$.

We now have to cut out another “canonical” one-dimensional subspace in $H^1(\mathcal{R}_L(\delta))$ which trivially intersects $H^1(\mathcal{R}_L(\delta))$ (and reduces to the cyclotomic $\mathbb{Z}_p$-extension in the sense of the previous remark). We recall that for $L = \mathbb{Q}_p$ Benois has defined in [Benois Proposition 1.5.9] a canonical complement $H^1(\mathcal{R}_{Q_p}(\mathbb{Z}_m))$ of $H^1(\mathcal{R}_{Q_p}(\mathbb{Z}_m))$ inside $H^1(\mathcal{R}_{Q_p}(\mathbb{Z}_m))$. He has also defined a canonical basis $y_m$ of $H^1(\mathcal{R}_{Q_p}(\mathbb{Z}_m))$.

We hence define the extension

$$y_m := \frac{\log_p(\chi_{\text{cycl}}(\gamma_L)))}{\text{cl}(0, t^{-2m})_{\mathfrak{e}_3}}.$$

When $L = \mathbb{Q}_p$, this is the same element $y_m$ as defined by Benois.

We can calculate cohomology of induced $(\varphi, \Gamma)$-modules. Indeed, we now consider two $p$-adic fields $K$ and $L$, $L$ a finite extension of $K$. The main reference for this part is [Liu08 §2.2]. Let $D$ be a $(\varphi, \Gamma)$-module, we define

$$\text{Ind}_{\mathcal{R}_L}^{\mathcal{R}_K}(D) = \{ f : \Gamma_K \to D | f(hg) = hf(g) \forall h \in \Gamma_L \}.$$

It has rank $[L : K][\text{rk}_{\mathcal{R}_K}(D) \text{ over } \mathcal{R}_K]$; indeed, $\mathcal{R}_L$ is a $\mathcal{R}_K$-module of rank $[L : K]/|\Gamma_K/\Gamma_L| = [L_0 : K_0]$. (The unramified part of $L/K$ plus the ramified part which is disjoint by $K_\infty$. See after [Liu08 Theorem 2.2].) If $D$ comes from a $G_L$-representation $V$ we have

$$D(\text{Ind}_{\mathcal{R}_L}^{\mathcal{R}_K}(V)) = \text{Ind}_{\mathcal{R}_L}^{\mathcal{R}_K}(D(\text{Ind}_{\mathcal{R}_L}^{\mathcal{R}_K}(V))).$$

We have then the equivalent of Shapiro’s lemma

$$H^i(D) \cong H^i(\text{Ind}_{\mathcal{R}_L}^{\mathcal{R}_K}(D)).$$

Moreover, the aforementioned duality for $(\varphi, \Gamma)$-modules is compatible with induction [Liu08 Theorem 2.2].

If $D \cong \mathcal{R}_L(\delta)$ is free of rank one, then we have an explicit description of $\text{Ind}_{\mathcal{R}_L}^{\mathcal{R}_K}(D)$. Let $e_\infty = |\Gamma_K/\Gamma_L|$, we write $\{\omega^i\}_{i=0}^{e_\infty - 1}$ for $(\Gamma_K/\Gamma_L)^\wedge$. The $\text{Ind}_{\mathcal{R}_L}^{\mathcal{R}_K}(D)$ is the $\mathcal{R}_L$-span of $f_i$, where $f_i(g) = \omega^i(g)\delta(\chi_{\text{cycl}}(g)e_3)$.

We go back to the previous setting where $K = \mathbb{Q}_p$ (hence $e_\infty = e_L$). Suppose $\delta(z) = \prod \tau(z)^{m_r}$ with $m_r \leq 0$ and let $D = \text{Ind}_{\mathcal{R}_L}^{\mathcal{R}_K}(\mathcal{R}_L(\delta))$. Note that in this case $D_{\text{st}}(D) \cong E^{(\omega)}$ is a filtered $\varphi$-module where $\varphi$ acts as a permutation of length $f_L$. To $\mathcal{D}_{\text{st}}(D)^{\omega=1}$ corresponds (by Proposition 2.3) over $\mathbb{Q}_p$ a rank-one $(\varphi, \Gamma)$-module $\mathcal{R}_{Q_p}(\mathbb{Z}_m)$, for $m_0$ the minimum of the $m_r$’s (hence $-m_0$ is the greatest Hodge–Tate weight of $D$).

The identifications

$$H^0(\mathcal{R}_{Q_p}(\mathbb{Z}_m)) = \mathcal{D}_{\text{st}}(\mathcal{R}_{Q_p}(\mathbb{Z}_m))^{\omega=1} = \mathcal{D}_{\text{st}}(D)^{\omega=1} = H^0(D) = H^0(\mathcal{R}_L(\delta))$$
induces (via the maps $\text{cl}(0,)$ and $\text{cl}(0,0)$) an injection
\[ H^1(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0})) \hookrightarrow H^1(\text{Ind}_{L}^{G}(\mathcal{R}_L(\delta))). \] (2.8)
which sends $x_{m_0}$ to $x_m$ and $y_{m_n}$ to $y_m$.

We consider a $(\varphi, \Gamma)$-module $M$ which sits in the non-split exact sequence
\[ 0 \to M_0 := \bigoplus_{i=1}^{r_1} \mathcal{R}_L(\delta_i) \to M \to M_1 := \bigoplus_{i=1}^{r_1} \mathcal{R}_L(\delta'_i) \to 0, \] (2.9)
where $\delta_i(z) = |N_L/\mathbb{Q}_p(z)|_p \prod \tau(z)^{m_i, \tau}$ with $m_{i, \tau} \geq 1$ for all $\tau$ and $\delta'_i(z) = \prod \tau(z)^{k_{i, \tau}}$ with $k_{i, \tau} \leq 0$ for all $\tau$. We say that $M$ is of type $U_{m,k}$ if the image of $M$ in $H^1(M_1)$ is crystalline.

**Proposition 2.10.** Suppose that $M$ as above is not of type $U_{m,k}$. Then we have $\dim_E(H^1(M)) = 2[L: \mathbb{Q}_p]r$ and $H^2(M) = H^0(M) = 0$. Moreover, if we write
\[ 0 \to H^0(M_1) \xrightarrow{\Delta} H^1(M_0) \xrightarrow{f_1} H^1(M) \xrightarrow{g_1} H^2(M_1) \to 0 \]
we have $H^1(M_0) = \text{Im}(\Delta_1) \oplus H^1_1(M_0)$, $\text{Im}(f_1) = H^1_1(M)$ and $H^1(M_1) = \text{Im}(g_1) \oplus H^1_1(M_1)$.

**Proof.** We have $H^0(M) = 0$ by definition of $M$. Note that $M^*(\chi_{\text{cycl}})$ is a module of the same type, hence $H^2(M) = H^0(M^*(\chi_{\text{cycl}})) = 0$. We can write
\[ 0 \to H^0(M_1) \to H^1(M_0) \xrightarrow{f_1} H^1(M) \xrightarrow{g_1} H^1(M_1) \to H^2(M_0) \to 0 \]
and conclude by Proposition\textsuperscript{2.11}.

Note that $\dim_E H^1_1(M_1) = rd$ by \textsuperscript{2.2}.

By hypothesis, we have that $\text{Im}(\Delta_1) \cap H^1_1(M_0) = 0$ and the first statement follows from dimension counting.

The third statement follows from duality.

For the second statement $H^1_1(M_0)$ injects into $H^1_1(M)$. As both have the same dimension, we conclude. \qed

We give the following key lemma for the definition of the $\mathcal{L}$-invariant

**Lemma 2.11.** The intersection of $T := \text{Im}(H^1(M))$ and $\text{Im}(H^1(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0})))$ in $\text{Im}(H^1(M_1))$ is one dimensional.

**Proof.** The intersection is non-empty as the sum of their dimension is $d+2$ and $\text{Im}(H^1(M_1))$ has dimension $d+1$. We have that $H^1_1(M_1)$ is contained in the image of $H^1(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0}))$ via \textsuperscript{2.8} and by the previous proposition the former is not in the image of $g_1$ and we are done. \qed

In particular, we deduce that $T$ surjects into the image of $H^1_1(\mathcal{R}_{\mathbb{Q}_p}(z^{m_0}))$.  

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3 \textit{L}-invariant over number fields

Let $F$ be a number field. We consider a global Galois representation

$V : G_F \rightarrow \text{GL}_n(E)$

where $E$ is $p$-adic field. We suppose that it is unramified outside a finite number of places $S$ containing all the $p$-adic places. We suppose moreover that it is semistable at all places above $p$ (i.e. $D_{\text{rig}}(V_{|_F^p})$ is of rank $n$ over $F_p^{\text{ur}} \otimes_{F_p} E$, being $F_p^{\text{ur}}$ the maximal unramified extension of $F_p$ contained in $F_p^{\text{ur}}$).

In this section we generalize Greenberg–Benois definition of the $\textit{L}$-invariant for $V$ whenever it presents trivial zeros. Note that we do not require $p$ split or unramified in $F$.

Let $t$ be the number of trivial zeros. The classical definition by Greenberg \cite{Gre94} describes the $\textit{L}$-invariant as the “slope” of a certain $t$-dimensional subspace of $H^1(G_{Q_p}, Q_p^t)$ which is a $2t$-dimensional space with a canonical basis given by $\text{ord}_p$ and $\log_p$.

In our setting, the main obstacle is that the cohomology of the $(\varphi, \Gamma)$-module $R_{F, p}$ is no longer two-dimensional and it is not immediate to find a suitable subspace. Inspired by Hida’s work for symmetric powers of Hilbert forms \cite{Hid07}, we consider the image of $H^1(R_{Q_p})$ inside $H^1(R_{F, p})$.

If $t$ denotes the number of expected trivial zeros, we show that we can define, similarly to \cite{Ben11}, a $t$-dimensional subspace of $H^1(G_{F,S}, V)$ whose image in $H^1(R_{Q_p})$ has trivial intersection with the crystalline cocycle. This is enough to define the $\textit{L}$-invariant; we further check that our definition is compatible with Benois’.

3.1 Definition of the $\textit{L}$-invariant

We define local cohomological conditions $L_v$ in order to define a Selmer group; we denote by $G_v$ a fixed decomposition group at $v$ in $G_{F,S}$ and by $I_v$ the inertia.

For $v \nmid p$ we define

$L_v := \text{Ker} \left( H^1(G_v, V) \rightarrow H^1(I_v, V) \right)$.

If $v \mid p$ we define

$L_v := H^1_i(F_v, V) = \text{Ker}(H^1(G_v, V) \rightarrow H^1(G_v, V \otimes_E B_{\text{cris}}))$.

If $D^1_{\text{rig}}(V)$ denotes the $(\varphi, \Gamma)$-module associated with $V$ we also have $L_p = H^1_i(D^1_{\text{rig}}(V))$. We define then the Bloch-Kato Selmer group

$H^1_i(V) := \text{Ker} \left( H^1(G_{F,S}, V) \rightarrow \prod_{v \in S} \frac{H^1(G_v, V)}{L_v} \right)$.

We make the following additional hypotheses:
$C_1$) $H^1_0(V) = H^1_0(V^*(1)) = 0,$

$C_2$) $H^0(G_{FS}, V) = H^0(G_{FS}, V^*(1)) = 0,$

$C_3$) $\varphi$ on $D_{sat}(V_{|p})$ is semisimple at $1 \in F_{p}^{ur} \otimes_{\mathbb{Q}_p} E$ and $p^{-1} \in F_{p}^{ur} \otimes_{\mathbb{Q}_p} E$ for all $p \mid p$.

$C_4$) $D^\dagger_{rig}(V_{|p})$ has no saturated sub-quotient of type $U_{m,k}$ for all $p \mid p$.

Note that if $V$ satisfies the previous four conditions, so does $V^*(1)$.

The first two conditions tell us that the Poitou–Tate sequence reduces to

$$H^1(G_{FS}, V) \cong \bigoplus_{v \in \mathcal{S}} H^1(G_v, V)$$

(3.1)

For each $p \mid p$ we denote by $V_p$ the restriction to $G_{F_p}$ of $V$. We choose a regular sub-module $D_p \subset D_{sat}(V_p)$ and define a filtration $(D_{p,i})$ of $D_{sat}(V_p)$.

$$D_{p,i} = \begin{cases} 0 & i = -2, \\ (1 - p^{-1}\varphi)D_p + N(D_p^{s=1}) & i = -1, \\ D_p & i = 0, \\ D_p + D_{sat}(V_p)^{s=1} \cap N^{-1}(D_p^{s=p^{-1}}) & i = 1, \\ D_{sat}(V_p) & i = 2. \end{cases}$$

(3.2)

We have that $D_{p,1}/D_{p,-1}$ coincides with the eigenvectors of $\varphi$ on $D_{sat}(V_p)$ of eigenvalue 1 (resp. $p^{-1}$) and which are in the kernel (resp. in the image) of $N$.

This filtration induces a filtration on $D^\dagger_{rig}(V_p)$. Namely, we pose

$$F_1D^\dagger_{rig}(V_p) = D^\dagger_{rig}(V_p) \cap (D_{p,1} \otimes R_{F_p, \log} [t^{-1}]).$$

We define

$$W_p := F_1D^\dagger_{rig}(V_p)/F_{-1}D^\dagger_{rig}(V_p).$$

The same proof as [Ben11, Proposition 2.1.7] tells us that we can find a unique decomposition

$$W_p = W_{p,0} \bigoplus W_{p,1} \bigoplus M_p$$

such that $t_{p,0} = \dim_E H^0(W_p^*(1)) = \text{rank}_{R_{F_p}} W_{p,0}$, $t_{p,1} = \dim_E H^0(W_p) = \text{rank}_{R_{F_p}} W_{p,1}$ and $M_p$ sits in a sequence

$$0 \rightarrow M_{p,0} \xrightarrow{t} M_p \xrightarrow{\beta} M_{p,1} \rightarrow 0$$

such that $\text{gr}^0(D^\dagger_{rig}(V_p)) = W_{p,0} \oplus M_{p,0}$ and $\text{gr}^1(D^\dagger_{rig}(V_p)) = W_{p,1} \oplus M_{p,1}$. Moreover $M_p$ is non-split; by construction we have $H^0(M_p) = H^2(M_p) = 0$ and if the exact sequence were split we would have $H^0(M_p) \neq 0$ and $H^2(M_p) \neq 0$. 

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We can prove exactly in the same way as [Ben11 Proposition 2.1.7 (i)] that C4 implies \( \text{rank}_{\mathcal{R}_p} M_{p,1} = \text{rank}_{\mathcal{R}_p} M_{p,0} \).

In order to define the \( L \)-invariant we shall follow verbatim Benois’ construction. For sake of notation, we write \( D^\dagger_p \) for \( D^\dagger_{i_{1g}}(V_p) \). We obtain from [Ben11 Proposition 1.4.4 (i)]

\[
H^1_f(\text{gr}^2(D^\dagger_p)) = H^0(\text{gr}^2(D^\dagger_p)) = 0.
\]

We deduce the following isomorphism

\[
H^1_f(F^0_1 D^\dagger_p) = H^1_f(D^\dagger_p) = H^1_f(F_p, V).
\]

As the Hodge–Tate weights of \( F^{-1}_1 D^\dagger_p \) are \(< 0\), we obtain from [Ben11, Proposition 1.5.3 (i)] and Poiteau–Tate duality

\[
H^2(F^{-1}_1 D^\dagger_p) = 0.
\]

Using the long exact sequence associated with

\[
0 \to F^{-1}_1 D^\dagger_p \to F_1 D^\dagger_p \to W_p \to 0
\]

we see that

\[
\frac{H^1(W_p)}{H^1(F^{-1}_1 D^\dagger_p)} = \frac{H^1(F_1 D^\dagger_p)}{H^1(F_p, V)}
\]

As Greenberg and Benois do, we make the extra assumption that C5) \( W_{p,0} = 0 \) for all \( p \mid p \).

Using Proposition [2.4] we can write \( \text{gr}^1(D^\dagger_p) = \bigoplus_{i=1}^{t_p+1+r_p} \mathcal{R}_p (\prod_{\tau_p} \tau_p(z)^{m_{i,\tau_p}}) \).

We define the \( 2(t_p+1+r_p) \)-dimensional subspace obtained as the image of

\[
\text{Ind}_p := \left\{ \sum_{i=0}^{t_p+1+r_p} E x_{m_i} + E y_{m_i} \right\} \subset H^1(\text{gr}^1(D^\dagger_p)). \quad (3.4)
\]

We define

\[
T_p = (H^1(F_1 D^\dagger_p) \cap \text{Ind}_p)/H^1_f(F_p, V).
\]

It has dimension \( t_p + 1 + r_p \).

Write \( t = \sum_p t_p + r_p \). We have a unique \( t \)-dimensional subspace \( H^1(D, V) \) of \( H^1(G_{F,S}, V) \) projecting via \( (3.1) \) to \( \bigoplus_p T_p \). We have an isomorphism (cfr. [Ben11, Proposition 1.5.9])

\[
\text{Ind}_p \cong \mathcal{D}_{\text{cris}}(W_{p,1} \oplus M_{p,1}) \oplus \mathcal{D}_{\text{cris}}(W_{p,1} \oplus M_{p,1}) \cong E^{t_p+1+r_p} \oplus E^{t_p+1+r_p},
\]

where the first (resp. second) factor is identified with \( E^{t_p+1+r_p} \) via the basis \( \{ x_{m_i} \} \) (resp. \( \{ y_{m_i} \} \)). We shall denote the two projections by \( \iota_{t_p} \) and \( \iota_{c_p} \).

We denote by \( \iota_t \) (resp. \( \iota_c \)) the projection of \( H^1(D, V) \) to \( E^t \) via \( \oplus \iota_{t_p} \) (resp. \( \oplus \iota_{c_p} \)). By the remark after Lemma [2.11] and the definition of \( T_p \), we have that \( \iota_c \) is surjective.

Summing up, we can give the following definition;
Definition 3.5. The $\mathcal{L}$-invariant of the pair $(V,D)$ is

$$\mathcal{L}(V,D) := \det(\iota_1 \circ \iota_c^{-1}),$$

where the determinant is calculated w.r.t. the basis $(x_{m_i}, y_{m_j})_{1 \leq i,j \leq t}$.

Remark 3.6. There is no a priori reason for which $\mathcal{L}(V,D)$ should be non-zero.

In the case $W_p = M_p$ we see from the description of $H^1(F_pD^\dagger_p)$ that the space $T_p$ depends only on $V|_{F_p}$ exactly as in the classical case.

3.2 Comparison with Benois’ definition

Fix a global field $F$ and let $\{p\}$ be the set of primes above $p$. Let $G_p$ denote a fixed decomposition group at $p$ in $G_Q$ and let $p_0$ be the corresponding place of $F$. Let $G_{p_0,F}$ be the decomposition group at $p_0$ in $G_F$. For each other place $p$ above $p$ in $F$, we have $G_p = \tau_p G_{p_0,F} \tau_p^{-1}$. We shall denote by $G_{p,F}$ the corresponding decomposition group in $G_F$. Consider a $p$-adic Galois representation

$$V : G_F \rightarrow \text{GL}_n(E).$$

We shall suppose $E$ big enough to contain the Galois closure of $F_p$, for all $p$. We have then

$$\text{Ind}_F^Q(V)|_{G_p} \cong \bigoplus_p \tau_p^{-1} \text{Ind}_{G_{p,F}}^{G_p} V|_{G_{p,F}}.$$

where $\tau_p \in G_p \setminus \text{Hom}(F, \overline{Q})$.

Consider the $(\varphi, \Gamma)$-module

$$D^\dagger := D^\dagger_{rig} \left( \text{Ind}_F^Q V \right).$$

We let $D$ be the regular $(\varphi, N)$-module of $D_{rig}(D^\dagger)$ induced by $\{D_p\}_p$. As before we have a filtration $(F_i D^\dagger)$ on $D^\dagger$ induced by the filtration on $D$. We denote by $W$ the quotient $F_1 D^\dagger / F_0 D^\dagger$. Note that it is semistable. We write $W = W_0 \oplus M \oplus W_1$. We suppose that $V$ satisfies the hypotheses C1-C5 of the previous section.

Lemma 3.7. Let $M$ be as in (2.9). We have

$$0 \rightarrow \text{Ind}(M_0) \rightarrow \text{Ind}(M) \rightarrow \text{Ind}(M_1) \rightarrow 0.$$
Proposition 3.8. We have a commutative diagram

\[
\begin{array}{ccc}
H^1(G,Q,\text{Ind}(V)) & \xrightarrow{\text{Res}_p} & H^1(F_1D^\dagger,\text{Ind}(V)) \\
\downarrow & & \downarrow \\
H^1(G,F,S) & \xrightarrow{\oplus_p \text{Res}_p} & \bigoplus_p H^1(D^\dagger)
\end{array}
\]

whose vertical arrows are isomorphism.

Proof. We follow [Hid06, §3.4.4]. Recall that we wrote $D^\dagger_p$ for $D^\dagger_{\text{rig}}(V_p)$. Shapiro's lemma tells us that

\[
H^1(G_p,\text{Ind}(V)) \cong \bigoplus_p H^1(D^\dagger_p).
\]

We are left to show that $H^1(F_1D^\dagger(\text{Ind}(V)))$ is sent by $\iota_p$ into $(H^1(F_1D^\dagger_p) \cap \text{Inv}_p)$ and we shall conclude by dimension counting.

We have then an injection

\[
F_1D^\dagger(\text{Ind}(V)) \hookrightarrow \bigoplus_p \text{Ind}(F_1(D^\dagger_{\text{rig}}(V_p))).
\]

Then clearly the image of $\iota_p$ lands in $H^1(F_1D^\dagger_p)$. But we have also the injection

\[
gr^1(D^\dagger_{\text{rig}}(\text{Ind}(V))) \hookrightarrow \bigoplus_p \text{Ind}(gr^1(D^\dagger_{\text{rig}}(V_p)))
\]

which by (2.8) tells us that the image of $\iota_p$ lands in $\text{Inv}_p$ and we are done. 

Corollary 3.9. We have $L(V,D) = L(\text{Ind}^F_{\text{Qp}}(V),\text{Ind}^F_{\text{p}}(D))$.

4 Siegel–Hilbert modular forms, the local case

The calculation of the $L$-invariant requires to produce explicit cocycles in $H^1(D,V)$; when $V$ appears in $\text{Ad}(V')$ for a certain representation $V'$ we can sometimes use the method of Mazur and Tilouine [MT90] to produce these cocycles. This has been done in many cases for the symmetric square [Hid04, Mok12] and generalized to symmetric powers of the Galois representation associated with Hilbert modular forms in [Hid07, HJ13]. The main limit of this approach is that for most representations $V$ it is computationally heavy to obtain $V$ as the quotient of an adjoint representation.

In the case $D^\dagger_{\text{rig}}(V) = W = M$ the situation is way simpler; if $t = 1$ it has been proved in [Ben10] that to produce the cocycle in $H^1(V,D)$ it is enough to find deformations of $V_{\text{Qp}}$.

We shall generalized the method of Benois to our situation in the case $W_p = M_p$ and $r_p = 1$. This will allow us to give a complete formula for the $L$-invariant of the Galois representations associated with a Siegel–Hilbert modular form which is Steinberg at all primes above $p$. 

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4.1 The case $t_p = r_p = 1$

We now suppose that $W_p = M_p$ and $r_p = 1$. For sake of notation, in this section we shall drop the index $p$. In particular, in this subsection $F = F_p$.

All that we have to do is to check that the calculation of [Ben11, Theorem 2] works in our setting.

We write as before

$$0 \to M_0 \to M \to M_1 \to 0$$

and, only in this subsection, we shall write $\delta$ for the character defining $M_0$ and $\psi$ for the character defining $M_1$. We suppose $\delta = \delta' \circ N_{F/Q_p}$ for $\delta'(z) = |\gamma|_p z^k$ with $k \geq 1$ and $\psi = \psi' \circ N_{F/Q_p}$ with $\psi'(z) = z^m$ with $m \leq 0$. We consider an infinitesimal deformation

$$0 \to M_{0,A} \to M_A \to M_{1,A} \to 0,$$

over $A = E[T]/(T^2)$. We suppose that $M_{0,A}$ (resp. $M_{1,A}$) is an infinitesimal deformation of $M_0$ (resp. $M_1$) which still factors through $N_{F/Q_p}$.

We shall write $\delta_A, \delta'_A, \psi_A$ and $\psi'_A$ for the corresponding one-dimensional character.

**Theorem 4.1.** Suppose that $d \log_p(\delta_A(\psi^{-1}_A)(\chi_{cycl}(\gamma_{Q_p}))) \neq 0$; then

$$L(M, M_0) = -\log_p(\chi_{cycl}(\gamma_{Q_p})) \frac{f^{-1}d \log_p(\delta_A(\psi^{-1}_A)(\chi_{cycl}(\gamma_{Q_p})))}{d \log_p(\delta'_A(\psi^{-1}_A)(\chi_{cycl}(\gamma_{Q_p})))}.$$ 

**Proof.** Recall the definition of Ind in [3.2]. We have a vector $v = ax_m + by_m$ in $H^1(F, D^1) \cap \text{Ind}$. By definition $L(M, M_0) = ab^{-1}$. The extension $M_{j,A}$ provides us with connecting morphisms $B_{ij} : H^i(M_j) \to H^{i+1}(M_j)$. We have by definition

$$B_{ij}(t^{-m}e_m) = (d \log(\psi_A'(p))t^{-m}e_m, d \log(\psi_A'(\chi_{cycl}(\gamma_{Q_p})))t^{-m}e_m)$$

$$= d \log(\delta_A'(p)x_m + d \log(\delta_A'(\chi_{cycl}(\gamma_{Q_p})))y_m). \quad (4.2)$$

As in [Ben10, §3.2] we consider the dual extension

$$0 \to M^*_1(\chi_{cycl}) \to M^*(\chi_{cycl}) \to M^*_0(\chi_{cycl}) \to 0,$$

and we shall denote with a $^*$ the corresponding map in the long exact sequence of cohomology.

We have hence $\text{ker}(\Delta_1) \perp \text{Im}(\Delta_0^*)$ under duality, and a map

$$H^1(M^*_1) \to H^1(R_{Q_p}(z^{1-m})).$$

By duality again, we deduce that the image of $\Delta_0^*$ inside the target of the above arrow is

$$a\alpha_{1-m} + b\beta_{1-m},$$
where $\alpha_{1-m}$ (resp. $\beta_{1-m}$) is the dual of $x_m$ (resp. $y_m$) as in [Ben10] Proposition 1.1.5.

We now consider the map

$$B_1^{1*} : H^1(M_1^*(\chi_{\text{cycl}})) \to H^2(M_1^*(\chi_{\text{cycl}})) = H^2(R_{\mathbb{Q}_p}(|z|^m)) \cong E.$$  

We can use [Ben10, Proposition 2.4] to see that after the above identification of $H^2$ with $E$ we have

$$B_1^{1*}(\alpha_{1-m}) = c \log_p(\chi_{\text{cycl}}(\gamma_{\mathbb{Q}_p}))-1\, d \log_p(\psi_A^{-1}(\chi_{\text{cycl}}(\gamma_{\mathbb{Q}_p})))$$  

(4.3)

$$B_1^{1*}(\beta_{1-m}) = cd \log_p(\psi_A^{-1}(p)),$$  

(4.4)

where $c \in E^\times$. We consider the following anti-commutative diagram

$$
\begin{array}{ccc}
H^0(M_0^*(\chi_{\text{cycl}})) & \xrightarrow{\Delta^*_1} & H^1(M_1^*(\chi_{\text{cycl}})) \\
\downarrow B_0^{1*} & & \downarrow B_1^{1*} \\
H^1(M_0^*(\chi_{\text{cycl}})) & \xrightarrow{\Delta^*_1} & H^2(M_1^*(\chi_{\text{cycl}}))
\end{array}
$$

which means

$$B_1^{1*} \Delta^*_0 = - \Delta^*_1 B_0^{1*}.$$

We calculate this identity on $t^{1-k} c_{1-k}$. Applying (4.3) and (4.4) to $\psi_A^{-1} \chi_{\text{cycl}},$ $\delta_A^{-1} \chi_{\text{cycl}}$ and using [Ben10, (3.6)] which says

$$\Delta_1 B_0^{1*}(t^{1-k}) = c (bd \log_p(\delta_A^{-1}(p)) + ad \log_p(\delta_A^{-1}(\chi_{\text{cycl}}(\gamma_{\mathbb{Q}_p}))))$$

we get

$$b^{-1}a = - \log_p(\chi_{\text{cycl}}(\gamma_{\mathbb{Q}_p}))-d \log_p(\delta_A^{-1}(\chi_{\text{cycl}}(\gamma_{\mathbb{Q}_p}))).$$

We conclude as $\delta_A(p)^f = \delta_A(\infty)$.

**Remark 4.5.** In particular, this theorem proves that this definition of $\mathcal{L}$-invariant is compatible with the Fontaine-Mazur one [Pot14, Zha14].

### 4.2 Calculation of the $\mathcal{L}$-invariant for Steinberg forms

We fix a totally real field $F$. Let $I$ be the set of real embeddings. Fix two embeddings

$$\mathbb{C}_p \leftrightarrow \mathbb{Q} \leftrightarrow \mathbb{C}$$

as before. We partition $I = \sqcup_p I_p$ according to the $p$-adic place which each embedding induces. We shall denote by $q_p = p^{l_p}$ the residual cardinality for
each prime ideal \( p \). We consider an irreducible representation \( \pi \) of \( \text{GSp}_{2g/F} \):
algebraic of weight \( k = (k_\tau) \), where \( (k_\tau) = (k_{\tau,1}, \ldots, k_{\tau,g}; k_0) \) (\( k_0 \) is a parallel weight for \( \text{Res}_F^G(\mathbb{G}_m) \)) with \( k_{\tau,1} \leq k_{\tau,2} \leq \cdots \leq k_{\tau,g} \). If \( k_{\tau,1} \geq g + 1 \) for all \( \tau \), then the weight is cohomological. The cohomological weight of \( \pi \) is then

\[
(\mu_\tau)_p = (k_\tau)_p - (g + 1, \ldots, g + 1; 0)_p.
\]

For parallel weights \( k \), we shall choose \( k_0 = g k \).

We now describe the conjectural Galois representation associated with \( \pi \). We have a spin Galois representation \( V_{\text{spin}} \) (whose image is contained in \( \text{GL}_{2g} \)) and a standard Galois representation \( V_{\text{sta}} \) (whose image is contained in \( \text{GL}_{2g+1} \)) given respectively by the spinorial and the standard representation of \( \text{GSpin}_{2g+1} = L^{\infty} \text{GSp}_{2g} \).

Thanks to the work of Scholze [Sch15] we now dispose of the standard Galois representation (see for example [HJ13, Theorem 18]). We also know the existence of the spin representation in many cases [KS14].

We now recall some expected properties of these Galois representations. Our main reference is [HJ13, §3.3]. We will make the following assumption on \( \pi \) at \( p \):

for each \( p \mid p \) either \( \pi_p \) is spherical or Steinberg.

We explain what we mean by Steinberg. Consider the Satake parameters at \( p \), normalized as in [BS00, Corollary 3.2], \((\alpha_{p,1}, \ldots, \alpha_{p,g})\). We have the following theorem on Iwahori-spherical representation of \( \text{GSp}_{2g}(F_p) \) [Tad94, Theorem 7.9].

**Theorem 4.6.** Let \( \alpha_1, \ldots, \alpha_g, \alpha \) be \( g + 1 \) character of \( F_p^\times \). Let \( B_{\text{GSp}_{2g}} \) be the Borel subgroup of \( \text{Sp}_{2g}(F_p) \). Then \( \text{Ind}_{B_{\text{GSp}_{2g}}}^{\text{GSp}_{2g}}(\alpha_1 \times \cdots \times \alpha_g \times \alpha) \) is not irreducible if and only if one of the following conditions is satisfied:

i) There exist at least three indexes \( i \) such that \( \alpha_i \) has exact order two and the \( \alpha_i \)'s are mutually distinct;

ii) There exists \( i \) such that \( \alpha_i = |N(\ )|_p^{-1} \); 

iii) There exist \( i \) and \( j \) such that \( \alpha_i = |N(\ )|_p^{-1} \alpha_j \).

**Remark 4.7.** As shown in [HJ13, Lemma 19], such a points are contained in a proper subset of the Hecke eigenvariety for \( \text{GSp}_{2g} \).

**Definition 4.8.** We say that \( \pi_p \) is Steinberg if \( \alpha_i = |N(\ )|_p^{-1} \alpha_i \).

If \( \pi_p \) is Steinberg at \( p \), then \( \alpha_{p,1}(\varpi_p) = q_{p}^{-1}\alpha_{p,1}(\varpi_p) \).

Trivial zeros appear also for automorphic forms which are only partially Steinberg at \( p \) and can be dealt exactly at the same way as the parallel one but for the sake of notation we prefer not to deal with them.

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To each $g + 1$ non-zero elements $(t_1, \ldots, t_g; t_0) \in (A^\times)^{g+1}$ we associate the diagonal matrix

$$u(t_1, \ldots, t_g; t_0) := (t_1, \ldots, t_g, t_0 t_g^{-1}, \ldots, t_0 t_1^{-1})$$

of $\text{GSp}_{2g}(A)$.

For $1 \leq i \leq g - 1$ we denote by $u_{p,i}$ the diagonal matrix associated with $(1, \ldots, 1, \mathfrak{w}_p^{-1}, \ldots, \mathfrak{w}_p^{-1}, \mathfrak{w}_p^{-2})$, where $\mathfrak{w}_p$ appears $i$ times; we also denote by $u_{p,0}$ the diagonal matrix corresponding to $(1, \ldots, 1; \mathfrak{w}_p^{-1})$.

**Definition 4.9.** The Hecke operators $U_{p,i}$, for $1 \leq i \leq g$ are defined as the double coset operator $[Iw_{u_{p,g-i}}]$. We have that $U_{p,g}$ is the “classical” $U_p$ operator [BS00] §0. We shall say then that $\pi$ is of finite slope for $U_{p,g}$ if $U_{p,g}$ has eigenvalue $\alpha_{p,0} \neq 0$ on $\pi_p$.

We are interested to study the possible $p$-stabilization of $\pi$ (i.e., Iwahori fixed vectors). If $\pi_p$ is unramified at $p$, we have then $2^g g!$ choices (see [HL13 Lemma 16] or [BS00 Proposition 9.1]). If $\pi_p$ is Steinberg, we have instead only one possible choice, as the monodromy $N$ has maximal rank.

Suppose that we can lift $\pi$ to an automorphic representation $\pi^{(2^g)}$ of $GL_{2^g}$. We suppose also that we can lift $\pi$ to an automorphic representation $\pi^{(2g+1)}$ of $GL_{2g+1}$.

Let $V = V_{\text{spin}}$ (resp. $V_{\text{sta}}$) be the Galois representation associated with $\pi^{(2^g)}$ (resp. $\pi^{(2g+1)}$). We make the following assumption

**LGP**) $V$ is semistable at all $p \mid p$ and strong local-global compatibility at $l = p$ holds.

These hypotheses are conjectured to be always true for $f$ as above. Arthur’s transfer from $\text{GSp}_{2g}$ to $GL_{2g+1}$ has been proven in [Xu] (note that it is now unconditional [MW]) and for $V = V_{\text{sta}}$ this hypothesis is then verified thanks to [Car14 Theorem 1.1]. These hypotheses are also satisfied in many cases for $V = V_{\text{spin}}$ in genus 2 (see [AS06, PSS14]).

Roughly speaking, we require that

$$\text{WD}(V|_{F_p})^{gg} \cong \iota_n^{-1} \pi_p^{(n)}$$

where $\text{WD}(V|_{F_p})$ is the Weil-Deligne representation associated with $V|_{F_p}$ à la Berger, $\pi_p^{(n)}$ is the component at $p$ of $\pi^{(n)}$, and $\iota_n$ is the local Langlands correspondence for $GL_n(F_p)$ geometrically normalized ($n = 2g + 1$ when $V$ is the standard representation and $n = 2^g$ when $V$ is the spinorial representation).

When $\pi_p$ is an irreducible quotient of $\text{Ind}^G_{\text{GSp}_{2g}}(\alpha_{p,1} \otimes \cdots \otimes \alpha_{p,g})$ we have that
the Frobenius eigenvalues on $\text{WD}(V_{\text{spin}|F_p})^{ss}$ are the $2^g$ numbers

$$
\left( \alpha_{p,0} \prod_{0 \leq r \leq g} \alpha_{p,i_1}(\varpi_p) \cdots \alpha_{p,i_r}(\varpi_p) \right).
$$

The ones on $\text{WD}(V_{\text{sta}|F_p})^{ss}$ are

$$
(\alpha_{p,g}(\varpi_p), \ldots, \alpha_{p,1}(\varpi_p), 1, \alpha_{p,1}(\varpi_p), \ldots, \alpha_{p,g}(\varpi_p)).
$$

Moreover, the monodromy operator should have maximal rank (i.e. one-dimensional kernel) if we are Steinberg or be trivial otherwise. (This is also a consequence of the weight-monodromy conjecture for $V$.)

Let $p$ be a $p$-adic place of $F$ and let $\tau$ be a complex place in $I_p$. The Hodge–Tate weights of $V_{\text{spin}|F_p}$ at $\tau$ are then

$$
\left( k_0 + \frac{1}{2} \sum_{i=1}^{g} \varepsilon(i)(k_{i,\tau} - i) \right),
$$

where $\varepsilon$ ranges among the $2^g$ maps from $\{1, \ldots, g\}$ to $\{\pm 1\}$. The one of $V_{\text{sta}|F_p}$ are $(1-k_{\tau,g}, \ldots, g-k_{\tau,1}, 0, k_{\tau,1}-g, \ldots, k_{\tau,g}-1)$.

Thanks to work of Tilouine-Urban [TU99], Urban [Urb11], Andreatta-Iovita-Pilloni [AIP15] we have families of Siegel modular forms;

**Theorem 4.10.** Let $W = \text{Hom}_{\text{cont}} \left( \mathbb{Z}_p^g \times \left( (\mathcal{O}_F \otimes \mathbb{Z}_p)^g \right)^{\text{ss}}, \mathcal{C}_p \right)$ be the weight space. There exist an affinoid neighborhood $U$ of $\kappa_0 = (z, (z_i)^g_{i=1}) \mapsto z^{k_0} \prod_{\tau \in I} \prod_{i} \tau(z_i)^{k_{i,\tau}}$ in $W$, an equidimensional rigid variety $X = X_\tau$ of dimension $dg + 1$, a finite surjective map $w : X \to U$, a character $\Theta : H^{NP} \to \mathcal{O}(X)$, and a point $x$ in $X$ above $\frac{1}{2}$ such that $x \circ \Theta$ corresponds to the Hecke eigensystem of $\pi$.

Moreover, there exists a dense set of points $x$ of $X$ coming from classical cuspidal Siegel–Hilbert automorphic forms of weight $(k_{i,\tau}; k_0)$ which are regular and spherical at $p$.

**Remark 4.11.** Assuming Leopoldt’s conjecture, the multiplicative group appearing in the definition of $W$ is, up to a finite subgroup, $((\mathcal{O}_F \otimes \mathbb{Z}_p)^g)^{\text{ss}}/\mathcal{O}_F^*$ (i.e. the $\mathbb{Z}_p$-points of the torus of $\text{Res}_{F}^{\mathbb{Q}}(\text{GSp}_{2g})$ modulo the $\mathbb{Z}_p$-points of the center).

This allows us to define two pseudo-representations $R_? : G_{\mathbb{Q}} \to \mathcal{O}(X)$, for $? = \text{spin}, \text{sta}$, interpolating the trace of the representations associated with classical Siegel forms [BC09, Proposition 7.5.4]. Suppose now that $V_{?}$ is absolutely irreducible (this is conjectured to hold when $\pi$ is Steinberg at least at
one prime); we have then, shrinking \( \mathcal{U} \) around \( k \) if necessary, a big Galois representation \( \rho_\tau \) with value in \( \text{GL}_n(\mathcal{O}(\mathcal{X})) \) such that \( \text{Tr}(\rho_\tau) = \mathcal{R}_\tau \) [BC09 page 214].

For \( 1 \leq j < g \) we define
\[
\lambda_p(u_{p,g-j}) = \sum_{r \in I_p} k_{r,1} + \cdots + k_{r,j} - k_0
\]
and
\[
\lambda_p(u_{p,0}) = \sum_{r \in I_p} (k_{r,1} + \cdots + k_{r,g} - k_0)/2.
\]
We have analytic functions \( \beta_{p,j} := \Theta(U_{p,j}\lambda_p(u_{p,g-j}))(p) \in \mathcal{O}(\mathcal{X}) \). We proceed now as in [HJ13]. We recall the following theorem [Liu13 Theorem 0.3.4]:

**Theorem 4.12.** Let \( \rho : G_{F_p} \to \text{GL}_n(\mathcal{O}(\mathcal{X})) \) be a continuous representation. Suppose that there exist \( \kappa_1(x), \ldots, \kappa_n(x) \) in \( F_p \otimes \mathbb{Q}_p \mathcal{O}(\mathcal{X}), F_1(x), \ldots, F_n(x) \) in \( \mathcal{O}(\mathcal{X}) \), and a Zariski dense set of points \( Z \subset \mathcal{X} \) such that

- for any \( x \in \mathcal{X} \), the Hodge–Tate weights of \( \rho_x \) are \( \kappa_1(x), \ldots, \kappa_n(x) \);
- for any \( z \in Z \), \( \rho_z \) is crystalline;
- for any \( z \in Z \), \( \kappa_{r,1}(z) < \ldots < \kappa_{r,n}(z) \), for all \( r \in I_p \);
- for any \( z \in Z \), the eigenvalues of \( \varphi^{c_{r}} \) on \( \mathcal{D}_{c_{r}}(V_x) \) are \( \prod_{r \in I_p} \tau(\varphi^{c_{r}})^{\kappa_{r,1}(z)} F_1(z), \ldots, \prod_{r \in I_p} \tau(\varphi^{c_{r}})^{\kappa_{r,n}(z)} F_n(z) \);
- for any \( C \in \mathbb{R} \), defines \( Z_C \subset Z \) as the set of points \( z \) such that for all \( I, J \subset \{1, \ldots, n\} \) such that \( |\sum_{i \in I} \kappa_{i,r}(z) - \sum_{j \in J} \kappa_{j,r}(z)| > C \) for all \( r \in I_p \). We require that for all \( z \in Z \) and \( C \in \mathbb{R} \), \( Z_C \) accumulates at \( z \).
- for \( 1 \leq i \leq n \) there exist character \( \chi_i : O_{F_p}^\times \to \mathcal{O}(\mathcal{X})^\times \) such that its derivative at 1 is \( \kappa_i \) and at each \( z \in Z \) we have \( \chi(z) = \prod_r \tau(z)^{\kappa_{i,r}(z)} \).

Then, for all \( x \in \mathcal{X} \) non-critical and regular (\( \kappa_1(x) < \ldots < \kappa_n(x) \) and the eigenvalues of \( \varphi \) on \( \mathcal{D}_{c_{r}}(V_x) \) are distinct for all \( i \)) there exists a Zariski neighbourhood \( U \) of \( x \) such that \( \rho_U \) is trianguline and its graded pieces are \( \mathcal{R}_U(\chi_i) \).

Here the rank one \( (\varphi, \Gamma) \)-module \( \mathcal{R}_U(\chi_i) \) over \( U \) is defined similarly as in Section 2 following [Liu13 §0.2].

We can apply this theorem and show that the \( (\varphi, \Gamma) \)-module associated with \( \rho_\tau|_{G_{F_p}} \) is trianguline. We now explicit the triangulation, given in [HJ13 §3.3].

As seen before, a \( p \)-stabilization of \( \pi_p \) corresponds to a permutation \( \nu \) and a map \( \varepsilon \).

The eigenvalues of \( \varphi^{c_{r}} \) are given by
\[
\prod_{r \in I_p} \tau(\varphi^{c_{r}})^{\varepsilon_{r,\nu,1}} \beta_{p,1},
\]
\[
\prod_{r \in I_p} \tau(\varphi^{c_{r}})^{\varepsilon_{r,\nu,1}} \beta_{p,1}^{-1},
\]
\[
\prod_{r \in I_p} \tau(\varphi^{c_{r}})^{\varepsilon_{r,\nu,1}} \beta_{p,2}^{-1}.
\]
where $c_i$’s are a positive integer independent of the weight.

We define the following characters of $F_p$ with value in $O(X)$:

\[
\chi_{p,1}(\varpi_p) = \beta_{p,1}, \\
\chi_{p,i}(\varpi_p) = \frac{\beta_{p,i-1}}{\beta_{p,i}}, \\
\chi_{p,g}(\varpi_p) = \frac{\beta_{p,g-1}}{\beta_{p,g}},
\]

and $\chi_{p,i}(u) = \prod_{\tau \in I_p} \tau(u)^{c_i + \mu_i, \tau}$.

From [HJ13, Lemma 19] we have that the graded pieces of $D_{\text{rig}}^\dagger(V_{\text{sta}}|_p)$ are then given by the characters $\chi_{p,g}, \ldots, \chi_{p,1}, \ldots, \chi_{p,g}$.

Concerning $V_{\text{spin}}$, we number the subsets of $\{1, \ldots, g\}$ as $I_1, I_2, \ldots, I_{2g}$. Each $I_j$ corresponds to a map $\varepsilon_j : \{1, \ldots, g\} \to \pm 1$.

We have then the graded pieces $\delta_{p,j}$ are given by the characters

\[
\delta_{p,\varepsilon_j}(u) = \prod_{\tau \in I_p} \tau(u)^{d_j + \mu_j \chi_j(\varpi_p)},
\]

\[
\delta_{p,\varepsilon_j}(\varpi_p) = \beta_{p,g} \prod_{i \in I_g} \chi_{p,i}(\varpi_p).
\]

Let $V$ be either $V_{\text{sta}}$ or $V_{\text{spin}}$. If $\pi_p$ is Steinberg, there is only one choice of a regular $(\varphi, N)$-sub-module $D_p$ of $D_{\text{st}}(V_{G,F_p})$, where $V$ is one of the two representations associated with $\pi$ described above. If the form is not Steinberg at $p$ many different regular sub-module can be chosen.

In any case, we expect (and we shall assume in the follow) that there is at most one trivial zero for each $p$. Consider now the representation $\pi$ of parallel weight $k$ (i.e. associated with $N_{E/Q}(\det \xi), k \in \mathbb{Z}$) as in the introduction.

We give a preliminary proposition on the factorization of the $L$-invariant. Recall the set $S^{\text{Sph}}$ and $S^{\text{Stb}}$ defined in the introduction, we have the following:

**Proposition 4.13.** We have the following factorization

\[
\mathcal{L}(V, D) = \mathcal{L}(V, D)^{\text{Sph}} \prod_{p \in S^{\text{Stb}}} \mathcal{L}(V, D)_p,
\]

where $\mathcal{L}(V, D)^{\text{Sph}}$ comes from the prime in $S^{\text{Sph}}$ and the factors $\mathcal{L}(V, D)_p$ are local.

**Proof.** We follow [Hid07, §1.3]. In the notation of Section 3, we write $W_1 = \oplus_{p \in S^{\text{Sph}}} W_{p,1}$ and $M_1 = \oplus_{p \in S^{\text{Sph}}} M_{p,1}$. We are left to show that the endomorphism $\varepsilon_1 \circ \varepsilon_1^{-1}$ of $D_{\text{cris}}(W_1 \oplus M_1) \cong E'$ keeps stable $D_{\text{cris}}(M_1)$ and on the quotient it respects the direct sum decomposition $\oplus_{p \in S^{\text{Sph}}} D_{\text{cris}}(W_{p,1})$.
Consider a prime \( p_0 \in S^{Stb} \) and a cocycle \( c \in H^1(D, V) \) such that \( \text{res}_p(c) = 0 \) for all \( p \neq p_0 \). This means that \( \text{res}_p(c) \in H^1_D(F_p, V) = H^1_D(F_p, M_p) \) (by (3.3)). We have hence \( \text{tr}_c \text{res}_p(c) = 0 \) for all primes \( p \neq p_0 \) as \( H^1_D^{Stb} \) is the direct sum complement of \( H^1_D \) (see [Ben11 Proposition 1.5.9]).

If \( p \) in \( S^{Stb} \) by Proposition 2.10 we also have \( \text{tr}_c \text{res}_p(c) = 0 \).

The proposition then follows from standard linear algebra as in [Hid07 Corollary 1.9].

**Remark 4.14.** A key ingredient in the proof of the factorization at Steinberg places is that each prime ideal brings a single trivial zero.

We now consider the case \( V = V_{sta} \). We have a contribution to trivial zeros from the \( \pi_p \)'s which are Steinberg and possibly from the \( \pi_p \) which are spherical. In particular, if we choose the regular sub-module coming from an ordinary filtration, we always have a trivial zero coming from each place.

For all \( 1 \leq s \leq \min(k-g-1, g-1) \) we have also \( c^{Stb} \) trivial zeros for \( V(s) \).

**Theorem 4.15.** For \( \pi_p \) Steinberg we have

\[
\mathcal{L}(V, D)_p = \frac{1}{f_p} \frac{d \log p \beta_{p,1}(k)}{dk} \bigg|_{k=\frac{1}{2}},
\]

where \( k \) is the parallel weight variable.

For \( 1 \leq s \leq \min(k-g-1, g-2) \) we also have

\[
\mathcal{L}(V(s), D(s))_p = \frac{1}{f_p} \frac{d \log p (\beta_{p,s-1}\beta_{p,1}(k))}{dk} \bigg|_{k=\frac{1}{2}}
\]

and if \( g-1 \leq k-g-1 \) we have

\[
\mathcal{L}(V(g-1), D(g-1))_p = \frac{1}{f_p} \frac{d \log p (\beta_{p,1}\beta_{p,1}^{-1}(k))}{dk} \bigg|_{k=\frac{1}{2}}.
\]

**Proof.** We note that we can specialize to a parallel family, so that no contribution from the denominator appears. We can apply Theorem 4.1 for \( \delta_A v_{\lambda}^{-1}(\pi_p) = \chi_{p,1}(\pi_p) \). The factor \( \log_p(u) \) disappears because of the change of variable \( T \mapsto u^{k-1} \) (\( u \) any topological generator of \( \mathbb{Z}^*_p \)).

**Remark 4.16.** The presence of \( f_p \) in the denominator can be explained in terms of the \( p \)-adic \( L \)-function for the induced representation, its missing Euler factors at \( p \) and Conjecture 1.2. See [Hid09 pag. 1348].

From now on, \( V = V_{\text{spin}}(k-1) \) (\( s = k-1 \) is the only critical integer); if \( \pi_p \) is spherical it should not give any trivial zeros (as the corresponding \( p \)-adic representation is conjectured to be crystalline and consequently the \( \beta_i's \) are Weil numbers of non-zero weight).

So we are left to see what happen at the primes Steinberg at \( p \). Twisting by \( \beta_{p,g} \) the triangulated \((\psi, \Gamma)\)-module of \( \rho_{\text{spin}} \) we are in the hypothesis of Theorem 4.1 and we have

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Theorem 4.17. For \( \pi_p \) Steinberg we have

\[
L(V, D) = \frac{1}{f_p} \left( \frac{d \log \beta_{p,1}(k)}{dk} \right) |_{k= \frac{4}{4.17}},
\]

where \( k \) is a parallel weight variable.

5 The case of the adjoint representation

We prove Theorem 1.4 of the introduction. We consider only the case \( g = 2. \)

Fix an automorphic representation \( \pi \) of weight \( k = (k_{r-1}, \ldots, k_r, g); k_0) \), and let \( V = V_{\text{spin}} \) be the spin representation associated with \( \pi \). Let \( \rho = \rho_{\text{spin}} \) be the corresponding big Galois representation.

We specialize the eigenvariety \( X \) of Theorem 4.1.10 to the subspace of the weight space given by the equations \( k_{i,\tau} = k_{i,\tau'} \) if \( \tau \) and \( \tau' \) induce the same \( p \)-adic place \( p \) and \( k_0 = k_{0,0} \). We shall denote the new variable by \( k_{p,i} \) and this eigenvariety by \( X' \). For simplicity, we rewrite the graded pieces of \( V \) as

\[
\delta_{p,1}(w_p) = F_{p,1}^{-1}(k), \quad \delta_{p,2}(u) = N_{F_p/Q_p}(u) \frac{k_{p,1} + k_{p,2} - 3}{2},
\]

\[
\delta_{p,2}(w_p) = F_{p,2}^{-1}(k), \quad \delta_{p,3}(u) = N_{F_p/Q_p}(u) \frac{k_{p,2} + k_{p,1} + 1}{2},
\]

\[
\delta_{p,3}(w_p) = F_{p,3}(k), \quad \delta_{p,4}(u) = N_{F_p/Q_p}(u) \frac{k_{p,3} - k_{p,1} - 1}{2},
\]

where \( k = (k_{p,1}, k_{p,2}; k_0)_p \).

The representation space of \( \text{Ad}(V) \) is given by the matrices

\[
\mathcal{S}_{p_4} = \{ X \in \mathcal{S}_{p_4} | X'J + JX = 0 \}.
\]

The \( p \)-stabilization on \( V \) induces a natural \( p \)-stabilization and consequently a regular sub-module \( D_{\text{Ad}} \) on \( \text{Ad}(V_{\text{spin}}) \). We have

\[
D_{\text{Ad}}^{-1} = \{ \text{nilpotent } X \},
\]

\[
D_{\text{Ad}}^0 = \{ \text{unipotent } X \}.
\]

The basis for the space \( D_{\text{Ad}}^{-1} / D_{\text{Ad}}^{-2} \) is given by the two diagonal matrices \( d_1 = [-1, 0, 0, 1] \) and \( d_2 = [0, -1, 1, 0] \). We shall denote by \( d_{p,1} \) these matrices when seen as a vector for \( \text{Ad}(V_p) \).

Proposition 5.1. Suppose that C1–C4 holds for \( V \). Suppose that the classical \( E \)-point \( x \) in the eigenvariety \( X' \) corresponding to \( \pi \) is \( \acute{e}tale \) above the weight space. Then, the space \( L(D_{\text{Ad}}, V) \) is generated by the image of

\[
\left( \frac{d \log \beta_{p,1}(k)}{dk_{p,1}} \right)_{p, j=1,2}.
\]
Proof. The proof is standard and goes back to [MT90], so we shall only sketch it. Let \( A = E[T]/(T^2) \). Consider an infinitesimal deformation of \( \rho \) given by
\[
\rho_A = V \oplus \rho';
\]
ote that \( \rho' \) can be written as the first order truncation of \( \frac{\partial \rho}{\partial v} \), where \( v \) is any direction in the weight space.

From \( \rho_A \) we can construct a cocyle \( c_{x,A} \) defined by
\[
G_F \ni \sigma \mapsto \rho'(\sigma)V^{-1}(\sigma).
\]
It is easy to check that this defines a cocycle with values in \( V \otimes V^* \). Moreover its image lands in \( \text{Ad}(V) \subset V \otimes V^* \) as the determinant is fixed (by our choice of the Hodge–Tate weight on \( X' \)). Writing explicitly the matrix for the \((\varphi, \Gamma)\)-module associated with \( \rho_A \) we obtain
\[
\begin{pmatrix}
\frac{\partial \rho_{p,1}}{\partial v} & \ast & \ast & \ast \\
\frac{\partial \rho_{p,2}}{\partial v} & \ast & \ast & \ast \\
\frac{\partial \rho_{p,3}}{\partial v} & \ast & \ast & \ast \\
\frac{\partial \rho_{p,4}}{\partial v} & \ast & \ast & \ast
\end{pmatrix}
\begin{pmatrix}
\delta_{p,1}^{-1} & \ast & \ast & \ast \\
\delta_{p,2}^{-1} & \ast & \ast & \ast \\
\delta_{p,3}^{-1} & \ast & \ast & \ast \\
\delta_{p,4}^{-1} & \ast & \ast & \ast
\end{pmatrix}
\]
In particular, they are upper triangular and their projection via \( \iota_f \) onto the vector \( d_{p,1} \) is
\[
\frac{\partial \log F_{p,1}(k)}{\partial v} |_{k=\Lambda}.
\]
Similarly for \( d_{p,2} \).

We also have that the projection via \( \iota_c \) onto \( d_{p,1} \) is
\[
\frac{\partial (k_{p,1} + k_{p,2})/2}{\partial v} |_{k=\Lambda}.
\]
By hypothesis, the projection to the weight space is étale at \( x \) and hence \( \left\{ \frac{\partial (k_{p,1} + k_{p,2})/2}{\partial v} \right\}_{p,j=1,2} \) is a base of the tangent space at \( x \) in \( X' \) and we are done.

We can now prove Theorem 5.4 which we recall;

THEOREM 5.2. Let \( \pi \) be an automorphic form of weight \( k \). Suppose that hypothesis LGp is verified for \( V_{\text{spin}} \) and the point corresponding to \( \pi \) in the eigenvariety \( X' \) is étale over the weight space. We have then
\[
L(\text{Ad}(V_{\text{spin}}), D_{\text{Ad}}) = \prod_p \frac{2}{f_p^2} \det \left( \frac{\partial \log F_{p,1}(k)}{\partial k_{p,j}} \right) \left( \frac{\partial \log F_{p,2}(k)}{\partial k_{p,j}} \right)_{1 \leq i,j \leq t, k=\Lambda}.
\]

Proof. By hypothesis we can use Proposition 5.1 so we just have to follow the proof of [Hid06, Theorem 3.73]. The matrix of \( \iota_c \) is exactly what appears in the Theorem, while the matrix of \( \iota_f \) can be directly calculated using the formula
\[
\frac{\partial \log_F(u_{x+y}^p x)}{\partial k_{p,j}} = \pm \delta_{p, p'} \delta_{i,j} \quad \text{(where } \delta_{a,b} \text{ here is Kronecker delta and gives a contribution of } 2^{-1} \text{ for each prime ideal } p) \]

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Proof. The proof is standard and goes back to [MT90], so we shall only sketch it. Let \( A = E[T]/(T^2) \). Consider an infinitesimal deformation of \( \rho \) given by
\[
\rho_A = V \oplus \rho';
\]


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