Lecture 5&6:
Kernel methods for modeling geophysical fluids: shallow water equations on a sphere and mantle convection in a spherical shell.

Grady B. Wright
Boise State University
Shallow water wave equations on a rotating sphere
Shallow water equations (SWE) on a rotating sphere

- Model for the nonlinear dynamics of a shallow, hydrostatic, homogeneous, and inviscid fluid layer.

- Idealized test-bed for the horizontal dynamics of all 3-D global climate models.

### Equations

<table>
<thead>
<tr>
<th>Equations</th>
<th>Momentum</th>
<th>Transport</th>
</tr>
</thead>
</table>
| Spherical coordinates | \[
\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s + \hat{f} \mathbf{k} \times \mathbf{u}_s + g \nabla h \mathbf{u}_s = 0
\] | \[
\frac{\partial h^*}{\partial t} + \nabla \cdot (h^* \mathbf{u}_s) = 0
\] |
| Cartesian coordinates | \[
\frac{\partial \mathbf{u}_c}{\partial t} + \mathbf{P} \left[ (\mathbf{u}_c \cdot P \nabla_c) \mathbf{u}_c + f(\mathbf{x} \times \mathbf{u}_c) \cdot \hat{i} + g(P \hat{\mathbf{p}} \cdot \nabla_c) h \right] = 0
\] | \[
\frac{\partial h^*}{\partial t} + (P \nabla_c) \cdot (h^* \mathbf{u}_c) = 0
\] |

Singularity at poles!

Smooth over entire sphere!
Numerical Example I: Global RBF collocation method

Forcing terms added to the shallow water equations to generate a flow that mimics a short wave trough embedded in a westerly jet. (Test case 4 of Williamson et. al. 1992)
Errors after trough travels once around the sphere

- Results of the RBF Shallow Water Model:
  (Flyer & W, 2009)

**Error as a function of time and $N$**

- $N = 784$
- $N = 1849$
- $N = 3136$
- $N = 4096$
- $N = 5041$

**Error height field, $t = 5$ days**

$N = 3136$, white $< 10^{-5}$

Error (exact - numerical)
Comparison with commonly used methods

<table>
<thead>
<tr>
<th>Method</th>
<th>$N$</th>
<th>Time step</th>
<th>Relative $\ell_2$ error</th>
</tr>
</thead>
<tbody>
<tr>
<td>RBF</td>
<td>4,096</td>
<td>8 minutes</td>
<td>$2.5 \times 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td>5,041</td>
<td>6 minutes</td>
<td>$1.0 \times 10^{-8}$</td>
</tr>
<tr>
<td>Sph. Harmonic</td>
<td>8,192</td>
<td>3 minutes</td>
<td>$2.0 \times 10^{-3}$</td>
</tr>
<tr>
<td>Double Fourier</td>
<td>32,768</td>
<td>90 seconds</td>
<td>$4.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>Spect. Element</td>
<td>24,576</td>
<td>45 seconds</td>
<td>$4.0 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Time-step for RBF method: Temporal Errors = Spatial Errors
Time-step for other methods: Limited by numerical stability

- RBF method runtime in MATLAB using 2.66 GHz Xeon Processor

<table>
<thead>
<tr>
<th>$N$</th>
<th>Runtime per time step (sec)</th>
<th>Total Runtime</th>
</tr>
</thead>
<tbody>
<tr>
<td>4,096</td>
<td>0.41</td>
<td>6 minutes</td>
</tr>
<tr>
<td>5,041</td>
<td>0.60</td>
<td>12 minutes</td>
</tr>
</tbody>
</table>

For much higher numerical accuracy, RBFs uses less nodes & larger time steps
Numerical Example II: RBF-FD method

(Flyer, Lehto, Blaise, Wright, and St-Cyr. 2012)

Flow over a conical mountain (Test case 5 of Williamson et. al. 1992)

Remarks:
• The mountain is only continuous, not differentiable.

• No analytical solution.

• Comparisons in numerical solutions are done against some reference numerical solutions at a high resolution.
Convergence comparison: 3 reference solutions

- Standard Literature/Comparison: NCAR's Sph. Har. T426, Resolution $\approx 30$ km at equator
- New Model at NCAR Discontinuous Galerkin – Spectral Element, Resolution $\approx 30$ km
- RBF-FD model, Resolution $\approx 60$ km

Convergence plot RBF-FD with stencil size of $m=31$
Error vs. runtime comparison

Machine: MacBook Pro, Intel i7 2.2 GHz, 8 GB Memory

- Further improvements for both methods may be possible using local mesh/node refinement near the mountain.
Numerical Example III: RBF-FD method

- Evolution of a highly non-linear wave: (Test case from Galewsky et. al. *Tellus*, 2004)
  Rapid cascade of energy from large to small scales resulting in sharp vorticity gradients

- RBF-FD method with $N=163,842$ nodes and $m=31$ point stencil.

Visualization of the relative vorticity
Thermal convection in a 3D spherical shell with applications to the Earth's mantle.
Simulating convection in the Earth's mantle


- **Model assumptions:**
  1. Fluid is incompressible
  2. Viscosity of the fluid is constant
  3. Boussinesq approximation
  4. Infinite Prandtl number, \( \text{Pr} = \frac{\text{kinematic viscosity}}{\text{thermal diffusivity}} \rightarrow \infty \)

- **Non-dimensional Equations:**

  \[
  \nabla \cdot \mathbf{u} = 0 \quad \text{(continuity)},
  \]
  \[
  \nabla^2 \mathbf{u} + \text{Ra} \, T \, \hat{\mathbf{r}} - \nabla p = 0 \quad \text{(momentum)},
  \]
  \[
  \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T - \nabla^2 T = 0 \quad \text{(energy)}.
  \]

- **Boundary conditions:**

  Velocity: impermeable and shear-stress free
  Temperature (isothermal): \( T = 1 \) at core mantle bdry., \( T = 0 \) at crust mantle bdry.

- Rayleigh, Ra, number governs the dynamics.  
  - Model for Rayleigh-Bénard convection
Global method for discretizing the equations

- Use a hybrid RBF-Pseudospectral method
- Collocation procedure using a 2+1 approach with
  - \( N \) RBF nodes on each spherical surface (angular directions) and
  - \( M \) Chebyshev nodes in the radial direction.

\[ N \text{ RBF nodes (ME) on a spherical surface} \]

\[ 3-D \text{ node layout showing } M \text{ Chebyshev nodes in radial direction} \]
Global method for discretizing the equations

- Rewrite the momentum equation using poloidal potential $\Phi$:

$$
\Delta_S \Omega + \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Omega}{\partial r} \right) = \text{Ra} \, r \, T
$$

$$
\Delta_S \Phi + \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) = r^2 \Omega,
$$

$$
u = \nabla \times \nabla \times ((\Phi r) \hat{r})
$$

$$
\frac{\partial T}{\partial t} = - \left( u_r \frac{\partial T}{\partial r} + u_s \cdot (P \nabla T) \right) + \frac{1}{r^2} \Delta_S T + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right)
$$

- We have seen how to create a discrete representation for $P \nabla$ using RBFs.

- Need a method to create a discrete representation of $\Delta_S$:

- A similar procedure can be used to $P \nabla$, by noting that

$$
\Delta_S \phi(||x - x_j||) = \frac{1}{4} \left[ (4 - ||x - x_j||^2) \phi''(||x - x_j||) + \frac{4 - 3||x - x_j||^2}{||x - x_j||} \phi'(||x - x_j||) \right]
$$
Steps of computational algorithm

\[ \Delta_S \Omega + \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Omega}{\partial r} \right) = \text{Ra} r T \]

\[ \Delta_S \Phi + \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) = r^2 \Omega, \]

\[ \mathbf{u} = \nabla \times \nabla \times ((\Phi r) \tilde{r}) \]

\[ \frac{\partial T}{\partial t} = - \left( u_r \frac{\partial T}{\partial r} + u_s \cdot (P \nabla T) \right) + \frac{1}{r^2} \Delta_S T + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) \]

1. Discretize \( \Delta_S \) and \( P \nabla \) for the unit sphere using \( N \) RBFs.
2. Discretize \( \frac{\partial}{\partial r} \) and \( \frac{\partial^2}{\partial r^2} \) using \( M \) Chebyshev polynomials.
3. Use \( T \) initial condition to solve for \( \Omega \).
4. Use \( \Omega \) solution to solve for \( \Phi \).
5. Use \( \Phi \) to compute the velocity \( \mathbf{u} \).
6. Discretize energy equation in time using an implicit/explicit scheme
   (a) Use trapezoidal rule for diffusion operator.
   (b) Use 3rd order Adams-Bashforth for the advection operator.
7. Time-step the energy equation to get a new \( T \), go back to step 3.
Ra=7000 benchmark: validation of method

Perturbation initial condition: \[ 0.01 \left[ Y_4^0(\theta, \lambda) + \frac{5}{7} Y_4^4(\theta, \lambda) \right] \]

Steady solution:

\[ N = 1600 \text{ nodes on each spherical shell} \]
\[ M = 23 \text{ shells} \]
\[ \text{Blue=downwelling, Yellow=upwelling, Red=core} \]

Comparisons against main previous results from the literature:

<table>
<thead>
<tr>
<th>Method</th>
<th>No of nodes</th>
<th>( \text{Nu}_{\text{outer}} )</th>
<th>( \text{Nu}_{\text{inner}} )</th>
<th>( \langle V_{\text{RMS}} \rangle )</th>
<th>( \langle T \rangle )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite volume</td>
<td>663,552</td>
<td>3.5983</td>
<td>3.5984</td>
<td>31.0226</td>
<td>0.21594</td>
</tr>
<tr>
<td>Finite elements (CitCom)</td>
<td>393,216</td>
<td>3.6254</td>
<td>3.6016</td>
<td>31.09</td>
<td>0.2176</td>
</tr>
<tr>
<td>Finite differences (Japan)</td>
<td>12,582,912</td>
<td>3.6083</td>
<td></td>
<td>31.0741</td>
<td>0.21639</td>
</tr>
<tr>
<td>Spherical harmonics -FD</td>
<td>552,960</td>
<td>3.6086</td>
<td></td>
<td>31.0765</td>
<td>0.21582</td>
</tr>
<tr>
<td>Spherical harmonics -FD</td>
<td>Extrapolated</td>
<td>3.6096</td>
<td></td>
<td>31.0821</td>
<td>0.21577</td>
</tr>
<tr>
<td>RBF-Chebyshev</td>
<td>36,800</td>
<td>3.6096</td>
<td>3.6096</td>
<td>31.0820</td>
<td>0.21578</td>
</tr>
</tbody>
</table>

\( \text{Nu} \) = ratio of convective to conductive heat transfer across a boundary
Fully convective simulation: $Ra=10^6$

**Model setup:**
- Convection dominated flow
- $N = 6561$ RBF nodes, $M = 81$ Chebyshev nodes
- Time-step $O(10^{-7})$, which is about 34,000 years
- Simulation time to $t=0.08$ (4.5 times the age of the earth)

**Results:**

![Simulation](image)

Blue=downwelling, Yellow= upwelling, Red=core

G. B. Wright, N. Flyer, and D. A. Yuen, 2010
Current focus

- Improving computational efficiency using RBF-FD.

- First step is to do RBF-FD on each spherical surface instead of global RBFs.

- Ultimate goal is to go to fully 3D RBF-FD formulas (no tensor-product structure):

Flyer, W, & Fornberg (2013)