We give a complete picture regarding the behavior of positive solutions of the following important difference equation: $x_n = 1 + \sum_{i=1}^{k} \alpha_i x_{n-p_i}/\sum_{j=1}^{m} \beta_j x_{n-q_j}$, $n \in \mathbb{N}_0$, where $\alpha_i$, $i \in \{1, \ldots, k\}$, and $\beta_j$, $j \in \{1, \ldots, m\}$, are positive numbers such that $\sum_{i=1}^{k} \alpha_i = \sum_{j=1}^{m} \beta_j = 1$, and $p_i$, $i \in \{1, \ldots, k\}$, and $q_j$, $j \in \{1, \ldots, m\}$, are natural numbers such that $p_1 < p_2 < \cdots < p_k$ and $q_1 < q_2 < \cdots < q_m$. The case when $\gcd(p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_m) = 1$ is the most important. For the case we prove that if all $p_i$, $i \in \{1, \ldots, k\}$, are even and all $q_j$, $j \in \{1, \ldots, m\}$, are odd, then every positive solution of this equation converges to a periodic solution of period two, otherwise, every positive solution of the equation converges to a unique positive equilibrium.

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1. Introduction and preliminaries

In [1], we studied the behavior of positive solutions of the recursive equation

$$y_n = 1 + \frac{y_{n-k}}{y_{n-m}}, \quad n \in \mathbb{N}_0,$$

with $y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty)$ and $k, m \in \{1, 2, 3, 4, \ldots\}$, where $s = \max\{k, m\}$. We proved that if $2^i$ is the highest power of 2 which divides $m$, then if $2^{i+1} \nmid k$, $y_n$ tends to 2, exponentially, and otherwise every solution tends to a period $t$ solution, with $t = 2\gcd(k, m)$. The method we used in [1] is a little bit complicated and its idea essentially stems from the theory of nonexpansive metrics. Since the above result is formulated in number theoretic language, we expect that the result is a particular case of a more general result, which
motivates us to investigate the following somewhat natural generalization of (1.1):

\[ x_n = 1 + \frac{\sum_{i=1}^{k} \alpha_i x_{n-p_i}}{\sum_{j=1}^{m} \beta_j x_{n-q_j}}, \quad n \in \mathbb{N}_0, \]  

(1.2)

where \( \alpha_i, i \in \{1, \ldots, k\} \), and \( \beta_j, j \in \{1, \ldots, m\} \), are positive numbers such that \( \sum_{i=1}^{k} \alpha_i = \sum_{j=1}^{m} \beta_j = 1 \), and \( p_i, i \in \{1, \ldots, k\} \), and \( q_j, j \in \{1, \ldots, m\} \), are natural numbers such that \( p_1 < p_2 < \cdots < p_k \) and \( q_1 < q_2 < \cdots < q_m \).

Here, we give a complete picture regarding the asymptotic behavior of positive solutions of (1.2). For closely related results, see, for example, [1–16] and the references therein.

In the proof of the main result of this paper, we need the following result by Karakostas (see [8, 9]).

**Theorem 1.1.** Let \( J \) be some interval of real numbers, let \( f \in C[J^2, J] \), and let \( (x_n)_{n=-1}^{\infty} \) be a bounded solution of the difference equation

\[ x_{n+1} = f(x_n, x_{n-1}), \quad n \in \mathbb{N}_0, \]  

(1.3)

with \( I = \liminf_{n \to -\infty} I_n, S = \limsup_{n \to -\infty} x_n \) and with \( I, S \in J \). Then there exist two solutions \( (I_n)_{n=-\infty}^{\infty} \) and \( (S_n)_{n=-\infty}^{\infty} \) of the difference equation

\[ x_{n+1} = f(x_n, x_{n-1}) \]  

(1.4)

which satisfy the equation for all \( n \in \mathbb{Z} \), with \( I_0 = I, S_0 = S, I_n, S_n \in [I, S] \) for all \( n \in \mathbb{Z} \) and such that for every \( N \in \mathbb{Z}, I_N \) and \( S_N \) are limit points of \( (x_n)_{n=-1}^{\infty} \). Furthermore, for every \( m \leq -1 \), there exist two subsequences \( (x_{r_n}) \) and \( (x_{l_n}) \) of the solution \( (x_n)_{n=-1}^{\infty} \) such that the following are true:

\[ \lim_{n \to -\infty} x_{r_n+N} = I_N, \quad \lim_{n \to -\infty} x_{l_n+N} = S_N \quad \text{for every } N \geq m. \]  

(1.5)

The solutions \( (I_n)_{n=-\infty}^{\infty} \) and \( (S_n)_{n=-\infty}^{\infty} \) of (1.4) are called full limiting solutions of (1.4) associated with the solution \( (x_n)_{n=-1}^{\infty} \) of (1.3).

### 2. Main results

First, we study the boundedness character of positive solutions of (1.2). For closely related results, see, for example, [4, 6, 12–14].

**Theorem 2.1.** Every positive solution of (1.2) is bounded.

*Proof.* Assume that \( (x_n) \) is a positive solution of (1.2). Note that \( x_n > 1 \) for \( n \geq 0 \). Hence, it is possible to choose positive numbers \( l \) and \( L \) greater than one such that \( ll = L + l \) and \( l \leq x_i \leq L \) for \( i \in \{0, 1, \ldots, s - 1\} \), where \( s = \max\{p, q\} \). Employing (1.2), we obtain

\[ l = 1 + \frac{l}{L} \leq x_i = 1 + \frac{\sum_{i=1}^{k} \alpha_i x_{s-p_i}}{\sum_{j=1}^{m} \beta_j x_{s-q_j}} \leq 1 + \frac{L}{l} = L. \]  

(2.1)
By the induction, we obtain that $x_n \in [l, L]$ for every $n \in \mathbb{N}_0$, finishing the proof of the theorem.

We are now in a position to formulate and prove the main result of this paper.

**Theorem 2.2.** Consider (1.2). Assume that

$$G := \gcd(p_1, \ldots, p_k, q_1, \ldots, q_m) = 1.$$  \hfill (2.2)

Then if all $p_i$, $i \in \{1, \ldots, k\}$, are even and all $q_j$, $j \in \{1, \ldots, m\}$, are odd, every positive solution of (1.2) converges to a periodic solution of period two. Otherwise, every positive solution of (1.2) converges to a unique positive equilibrium.

**Proof.** Let

$$\mathcal{P} = \{ p_i \mid i = 1, \ldots, k \}, \quad \mathcal{Q} = \{ q_j \mid j = 1, \ldots, m \}. \hfill (2.3)$$

Assume first that $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$. In view of Theorem 2.1, every positive solution $(x_n)$ of (1.2) is bounded which implies that there are finite $\liminf_{n \to \infty} x_n = I$ and $\limsup_{n \to \infty} x_n = S$. Letting $n \to \infty$ in (1.2), we obtain

$$1 + \frac{I}{S} \leq I \leq S \leq 1 + \frac{S}{I}, \hfill (2.4)$$

from which it follows that

$$SI = I + S. \hfill (2.5)$$

Let $(L_{-i})_{i \in \mathbb{Z}}$ be a full limiting sequence of a solution $(x_n)$ of (1.2), such that $L_0 = S$. Since $(L_{-i})_{i \in \mathbb{Z}}$ is a solution of (1.2) belonging to the interval $[I, S]$, we have that

$$S = L_0 = 1 + \frac{\sum_{i=1}^{k} \alpha_i L_{-p_i}}{\sum_{j=1}^{m} \beta_j L_{-q_j}} \leq 1 + \frac{S}{I} = S. \hfill (2.6)$$

From (2.6), it follows that $L_{-p_i} = S$ for every $i \in \{1, \ldots, k\}$ and $L_{-q_j} = I$ for every $j \in \{1, \ldots, m\}$. Employing assumption $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$, we obtain $I = S$, from which the result follows in this case.

Now we assume that $\mathcal{P} \cap \mathcal{Q} = \emptyset$. Further, assume that there is $p_{i_0} \in \mathcal{P}$ which is odd. Let $p_{i_0} = 2s + 1$ and let $q_{j_0}$ be an arbitrary element of $\mathcal{Q}$. Then, (1.2) can be written in the form

$$x_n = 1 + \frac{\alpha_{i_0} x_{n-(2s+1)}}{\beta_{j_0} x_{n-q_{j_0}}} + \sum_{i=1, i \neq i_0}^{k} \frac{\alpha_i x_{n-p_i}}{\beta_{j} x_{n-q_{j}}}. \hfill (2.7)$$

Let $(L_{-i})_{i \in \mathbb{Z}}$ be a full limiting sequence of a solution $(x_n)$ of (1.2), such that $L_0 = S = \limsup_{n \to \infty} x_n$. From

$$S = L_0 = 1 + \frac{\alpha_{i_0} L_{-(2s+1)}}{\beta_{j_0} L_{-q_{j_0}}} + \sum_{j=1, j \neq j_0}^{m} \frac{\beta_j L_{-q_j}}{\beta_{j} L_{-q_{j}}}, \hfill (2.8)$$
similar to (2.6), we obtain

\[ L_{-(2s+1)} = S, \quad L_{-q_{j_0}} = I. \] (2.9)

From (2.9) and since \((L_{-i})_{i \in \mathbb{Z}}\) is a solution of (2.7), it follows that

\[ L_{-2(2s+1)} = S, \quad L_{-2q_{j_0}} = S. \] (2.10)

Indeed, since

\[ S = L_{-(2s+1)} = 1 + \alpha_i L_{-2(2s+1)} + \sum_{i=1, i \neq i_0}^{k} \alpha_i L_{-p_i - (2s+1)} \]
\[ \leq 1 + S \frac{I}{I} = S, \] (2.11)

we obtain the first equality in (2.10). On the other hand, from

\[ I = L_{-q_{j_0}} = 1 + \frac{\alpha_i L_{-q_{j_0} - (2s+1)} + \sum_{i=1, i \neq i_0}^{k} \alpha_i L_{-q_{j_0} - p_i}}{\beta_{j_0} L_{-2q_{j_0}} + \sum_{j=1, j \neq j_0}^{m} \beta_j L_{-q_{j_0} - q_j}} \geq 1 + \frac{I}{S} = I, \] (2.12)

the second equality in (2.10) follows.

By induction we obtain

\[ L_{-(2s+1)i} = S, \quad i \in \mathbb{N}, \] (2.13)

\[ L_{-q_{j_0}j} = \begin{cases} I, & j \text{ odd}, \\ S, & j \text{ even}. \end{cases} \] (2.14)

If we take \(i = q_{j_0}\) in (2.13) and \(j = 2s + 1\) in (2.14), we obtain \(I = L_{-(2s+1)q_{j_0}} = S\), as desired.

Now, assume that all \(p_i \in \mathcal{P}\) are even, and \(\mathcal{P}\) has odd as well as even elements. Then, (1.2) can be written in the form

\[ x_n = 1 + \frac{\sum_{i=1}^{k} \alpha_i x_{n-p_i}}{\beta_{j_0} x_{n-q_{j_0}} + \beta_{j_1} x_{n-q_{j_1}} + \sum_{j=1, j \neq j_0, j_1}^{m} \beta_j x_{n-q_j}}, \] (2.15)

where \(q_{j_0} = 2s\) and \(q_{j_1} = 2t + 1\).

From a result in number theory [11], we know that the condition \(G = 1\) implies that for each sufficiently large \(n\), say, \(n \geq n_0\), there are nonnegative numbers \(d_i \in \mathbb{N}_0, i \in \{1, \ldots, k + m\}\), such that

\[ \sum_{i=1}^{k} p_i d_i + \sum_{j=1}^{m} q_j d_{k+j} = n. \] (2.16)

From condition \(G = 1\), by using (2.15) and (2.16), and employing the procedure described above for getting formulae (2.13) and (2.14), we obtain that the subsequence \((L_{-i})_{i \geq n_0}\) of the full limiting sequence \((L_i)_{i \in \mathbb{Z}}\) with \(L_0 = S\) takes values \(I\) and \(S\).
Now we prove that the sequence \((L_{-i})_{i \in \mathbb{N}}\) is eventually periodic with periods \(p_1, p_2, \ldots, p_k\) and also with periods \(2q_1, \ldots, 2q_m\). Indeed, if we replace \(n\) in (2.15) by \(-n_0 - l, l \in \{0, 1, \ldots, p_1 - 1\}\), we obtain that \(L_{-n_0-l} = L_{-n_0-l-p_i}\) for every \(i \in \mathbb{N}\) and each \(l \in \{0, 1, \ldots, p_1 - 1\}\), that is, \((L_{-i})_{i \in \mathbb{N}}\) is eventually periodic with period \(p_1\). Similarly it can be proven that \((L_{-i})_{i \in \mathbb{N}}\) is eventually periodic with periods \(p_2, \ldots, p_k\). The periodicity with periods \(2q_1, \ldots, 2q_m\) can be proven similar to (2.9) and (2.10) and by using induction.

Since all \(p_i \in \mathcal{P}\) are even and \(G = 1\), we have that

\[
2 \leq \gcd(p_1, p_2, \ldots, p_k, 2q_1, \ldots, 2q_m) = 2 \gcd\left(\frac{p_1}{2}, \frac{p_2}{2}, \ldots, \frac{p_k}{2}, q_1, \ldots, q_m\right) \leq 2G = 2,
\]

that is,

\[
\gcd(p_1, p_2, \ldots, p_k, 2q_1, \ldots, 2q_m) = 2. \tag{2.18}
\]

Hence, the sequence \((L_{-i})_{i \in \mathbb{N}}\) is eventually periodic with period two. Since \((L_i)_{i \in \mathbb{Z}}\) is a solution of (1.2), we obtain that \((L_i)_{i \in \mathbb{Z}}\) is also periodic with period two.

Assume now that

\[
\ldots, x, y, x, y, x, y, \ldots,
\]

is a two-periodic solution of (2.15). Then we have

\[
x = 1 + \frac{x}{cx + (1 - c)y}, \quad y = 1 + \frac{y}{cy + (1 - c)x},
\]

for some \(c \in (0, 1)\). Hence,

\[
(c - 1)xy = cx^2 - (c + 1)x - (1 - c)y = cy^2 - (c + 1)y - (1 - c)x,
\]

from which it follows that \(c(x - y)(x + y - 2) = 0\). If \(x + y = 2\) and \(x \neq y\), then we have that \(x\) and \(y\) are different positive solutions of the equation

\[
x = 1 + \frac{x}{cx + (1 - c)(2 - x)},
\]

which implies that \((2c - 1)(x - 1)^2 = 1\). Hence, if \(c \leq 1/2\), then this equation does not have real roots. If \(c > 1/2\), then \(x = 1 \pm (1/(2c - 1))^{1/2}\) are solutions. However, since \(c \in (1/2, 1)\), the number \(1 - (1/(2c - 1))^{1/2}\) is negative. Therefore, it follows that \(x = y\) as desired.

Assume now that the set \(\mathcal{P}\) contains only even elements while \(\mathcal{Q}\) contains only odd elements. Then, it is easy to see that (1.2) in this case has infinite prime two-periodic solutions of the form \(x, y, x, y, \ldots\), such that \(xy = x + y\). Similar to (2.18), it can be proven that, in this case, the full limiting sequence \((L_i)_{i \in \mathbb{Z}}\), \(L_0 = S\) is periodic with period two and that

\[
L_{2i} = S, \quad L_{2i-1} = I, \quad i \in \mathbb{Z}. \tag{2.23}
\]
Assume that $\varepsilon, \delta \in (0, S)$ are such that
\[
(S - \varepsilon)(I + \delta) = (S - \varepsilon) + (I + \delta).
\]
(2.24)

Then, for such chosen $\varepsilon$ and $\delta$, there is a $k_0 \in \mathbb{Z}$ such that
\[
(x_{k_0 + 2j} > S - \varepsilon, \quad x_{k_0 + 2j - 1} < I + \delta),
\]
(2.25)
for $j \in \{1, 2, \ldots, \lfloor s/2 \rfloor + 1\}$, where $s = \max\{p_k, q_m\}$.

From (1.2) and (2.25), we have that
\[
x_{k_0 + 2\lfloor s/2 \rfloor + 3} < 1 + \frac{I + \delta}{S - \varepsilon} = I + \delta,
\]
(2.26)
and
\[
x_{k_0 + 2\lfloor s/2 \rfloor + 4} > 1 + \frac{S - \varepsilon}{I + \delta} = S - \varepsilon.
\]

By induction, we obtain
\[
x_{k_0 + 2i + 1} < I + \delta, \quad x_{k_0 + 2i} > S - \varepsilon,
\]
(2.27)
for every $i \in \mathbb{N}$. From (2.27) and the fact that $\varepsilon \to 0$ implies $\delta \to 0$, it follows that $\lim_{n \to \infty} x_{2n} = S$ and $\lim_{n \to \infty} x_{2n-1} = I$, or $\lim_{n \to \infty} x_{2n} = I$ and $\lim_{n \to \infty} x_{2n-1} = S$, finishing the proof of the theorem.

Remark 2.3. Note that the case when all $p_i, i \in \{1, \ldots, k\},$ and $q_j, j \in \{1, \ldots, m\},$ are even is excluded from the consideration in Theorem 2.1 since we assume that $G = 1.$ However, this case is reduced to the cases considered in Theorem 2.1. Indeed, let $2^s$ be the highest power of 2 which divides $G$, then (1.2) can be separated into $2^s$ different equations of the form
\[
x_n^{(t)} = 1 + \frac{\sum_{i=1}^{k} \alpha_i x_{n-p_i/2^s}}{\sum_{j=1}^{m} \beta_j x_{n-q_j/2^s}}, \quad n \in \mathbb{N}_0,
\]
(2.28)
where $t \in \{0, 1, \ldots, 2^s - 1\}$. Note that by the definition of $2^s$, it follows that at least one of the numbers $p_i/2^s, i \in \{1, \ldots, k\},$ and $q_j/2^s, j \in \{1, \ldots, m\},$ is odd. Hence, Theorem 2.1 can be applied to the equations in (2.28).

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