Research Article

K-nacci Sequences in Finite Triangle Groups

Erdal Karaduman and Ömür Deveci

Department of Mathematics, Faculty of Science, Atatürk University, 25240 Erzurum, Turkey

Correspondence should be addressed to Erdal Karaduman, eduman@atauni.edu.tr

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A k-nacci sequence in a finite group is a sequence of group elements $x_0, x_1, x_2, \ldots, x_n, \ldots$ for which, given an initial (seed) set $x_0, x_1, x_2, \ldots, x_{j-1}$, each element is defined by $x_n = x_0 x_1 \cdots x_{n-1}$, for $j \leq n < k$, and $x_n = x_{n-k} x_{n-k+1} \cdots x_{n-1}$, for $n \geq k$. We also require that the initial elements of the sequence $x_0, x_1, x_2, \ldots, x_{j-1}$, generate the group, thus forcing the k-nacci sequence to reflect the structure of the group. The K-nacci sequence of a group generated by $x_0, x_1, x_2, \ldots, x_{j-1}$ is denoted by $F_k(G; x_0, x_1, \ldots, x_{j-1})$ and its period is denoted by $P_k(G; x_0, x_1, \ldots, x_{j-1})$. In this paper, we obtain the period of K-nacci sequences in finite polyhedral groups and the extended triangle groups.

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1. Introduction

The Fibonacci sequences and their related higher-order (tribonacci, quatranacci, k-nacci) are generally viewed as sequences of integers. In [1] the Fibonacci length of a 2-generator group is defined, thus extending the idea of forming a sequence of group elements based on a Fibonacci-like recurrence relation first introduced by Wall in [2]. There he considered the Fibonacci length of the cyclic group $C_n$. The concept of Fibonacci length for more than two generators has also been considered, see, for example [3, 4]. Also, the theory has been expanded to the nilpotent groups, see, for example [5–7]. Other works on Fibonacci length are discussed in, for example, [8–12]. Knox proved that the periods of k-nacci (k-step Fibonacci) sequences in dihedral groups are equal to $2k + 2$ [13]. Campbell and Campbel, examined the behaviour of the Fibonacci length of the finite polyhedral, binary polyhedral groups, and related groups in [14].

This paper discusses the period of k-nacci Fibonacci sequences in the polyhedral groups $(2,2,2)$, $(n,2,2)$, $(2,n,2)$, $(2,2,n)$ for any $n$ and in the extended triangle groups $E(2,2,2)$, $E(n,2,2)$, $E(2,n,2)$, $E(2,2,n)$ for any $n > 2$. We consider polyhedral groups both as 2-generator and as 3-generator groups. A 2-step Fibonacci sequence in the integers modulo $m$ can be written as $F_2(Z_m,0,1)$. A 2-step Fibonacci sequence of group elements is called

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**Note:** The document includes a block of mathematical content and references to further reading and additional information. The text is structured to provide a comprehensive overview of the research article, focusing on the introduction of the K-nacci sequences and their properties in finite triangle groups.
a Fibonacci sequence of a finite group. A finite group $G$ is $k$-nacci sequenceable if there exists a $k$-nacci sequence of $G$ such that every element of the group appears in the sequence. A sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called the period of the sequence. For example, the sequence $x_0, x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4, \ldots$ is periodic after the initial element $x_0$ and has period 4. A sequence of group elements is simply periodic with period $k$ if the first $k$ elements in the sequence form a repeating subsequence. For example, the sequence $x_0, x_1, x_2, x_3, x_4, x_0, x_1, x_2, x_3, x_4, \ldots$ is simply periodic with period 5. It is important to note that the Fibonacci length depends on the chosen generating $n$-tuple for a group.

Definition 1.1. For a finitely generated group $G = \langle A \rangle$ where $A = \{ a_1, a_2, \ldots, a_n \}$ the sequence $x_i = a_{i+1}, 0 \leq i \leq n-1, x_{i+n} = \prod_{j=1}^{n} x_{i+j-1}, i \geq 0,$ is called the Fibonacci orbit of $G$ with respect to the generating set $A$, denoted $F_A(G)$.

Notice that the orbit of a $k$-generated group is a $k$-nacci sequence. The orbits of $(n,2,2)$, $(2,n,2)$, $(2,2,n)$ for any $n > 2$ and $E(2,q,2)$ for any $q > 2$ are studied in [14].

2. The Groups $(2,2,2)$, $(n,2,2)$, $(2,n,2)$, and $(2,2,n)$

Definition 2.1. The polyhedral group $(l,m,n)$ for $l, m, n > 1$ is defined by the presentation

$$\langle x, y, z : x^l = y^m = z^n = xyz = e \rangle \quad (2.1)$$

or

$$\langle x, y : x^l = y^m = (xy)^n = e \rangle. \quad (2.2)$$

The polyhedral group $(l,m,n)$ is finite if and only if the number

$$\mu = lmn \left( \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1 \right) = mn + nl + lm - lmn \quad (2.3)$$

is positive, that is, in the case $(2,2,n)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$. Its order is $2lnm/\mu$. Using Tietze transformations, we may show that $(l,m,n) \cong (m,n,l) \cong (n,l,m)$. For more information on these groups see [15] and [16, pages 67-68]. The groups considered in Theorems 2.3 and 2.4 are the same group, namely, $D_n$, the dihedral group of $2n$ elements, except the generators $x$, $y$, and $z$ are different from one theorem to the other.

Theorem 2.2. Let $G_2$ be the group defined by the presentation $G_2 = \langle x, y, z : x^2 = y^2 = z^2 = xyz = e \rangle$. Then $P_k(G_2, x, y, z) = k + 1$.

Proof. Firstly, let us consider the 2-generator case. Notice that $G_2$ is $Z_2 \oplus Z_2$ and $P_k(Z_2;0,1) = k + 1$. Under these identifications, since the period of a Fibonacci sequence in a direct product of groups is the least common multiple of the periods in each the factors we get $P_k(G_2; x, y) = k + 1$. On the other hand, since $z = xy$ the formulas in the “three generator case” with recurrences of period $k + 1$ are the same as the formulas the two generator case as long as $k \geq 4$. \qed
Theorem 2.3. Let $G_n, n > 2$, be the group defined by the presentation $(x, y, z : x^n = y^2 = z^2 = xyz = e)$. Then $P_k(G_n; x, y, z) = 2k + 2$.

Proof. Let us consider the 3-generator case. We first note that the orders of $x, y, z$ are $n, 2, 2$, respectively. If $k = 2$, we have the sequence

$$x, y, z, yz, z, x, y, \ldots,$$

which has period 6. If $k = 3$, we have the sequence

$$x, y, z, xyz = e, yz, zyz, z, x, y, z, \ldots,$$

which has period 8. If $k \geq 4$, the first $k$ elements of sequence are

$$x_0 = x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = xyz, \quad x_4 = (xyz)^2, \ldots, \quad x_{k-1} = (xyz)^{2k-3}.$$

Thus, using the above information the sequence reduces to

$$x_0 = x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = e, \quad x_4 = e, \ldots, e,$$

where $x_j = e$ for $3 \leq j \leq k - 1$. Thus,

$$x_k = \prod_{i=0}^{k-1} x_i = (xy)^{2k-2} = e, \quad x_{k+1} = \prod_{i=1}^{k} x_i = yz = x^{n-1}, \quad x_{k+2} = \prod_{i=2}^{k+1} x_i = zyz = xz,$$

$$x_{k+3} = \prod_{i=3}^{k+2} x_i = z, \quad x_{k+4} = \prod_{i=4}^{k+3} x_i = e, \ldots, e.$$

It follows that $x_{k+j} = e$ for $4 \leq j \leq k$. We also have,

$$x_{k+k+1} = \prod_{i=k+1}^{k+k} x_i = e, \quad x_{k+k+2} = \prod_{i=k+2}^{k+k+1} x_i = x,$$

$$x_{k+k+3} = \prod_{i=k+3}^{k+k+2} x_i = y, \quad x_{k+k+4} = \prod_{i=k+4}^{k+k+3} x_i = z.$$

Since the elements succeeding $x_{2k+2}$, $x_{2k+3}$, $x_{2k+4}$, depend on $x$, $y$, and $z$ for their values, the cycle begins again with the $2k + 2$nd element; that is, $x_0 = x_{2k+2}$, $x_1 = x_{2k+3}$, $x_2 = x_{2k+4}$, $\ldots$. Thus, $P_k(G_n; x, y, z) = 2k + 2$.

Similarly, it is easy to show that for 2-generator, $P_k(G_n; x, y, z) = 2k + 2$ in $(n, 2, 2)$, and it can be shown that $P_k(G_n; x, y, z) = 2k + 2$ for $(2, n, 2)$.

Because of $(n, 2, 2) \cong (2, n, 2) \cong (2, 2, n) \cong D_n$ for any $n > 2$ and using Tietze transformations we can obtain the same presentation for this groups, it is easy to show that for 2-generator $P_k(G_n; x, y) = 2k + 2$ in the groups $(n, 2, 2)$, $(2, n, 2)$, and $(2, 2, n)$. □
Theorem 2.4. Let \( G_n, n > 2, \) be the group defined by the presentation \( \langle x, y, z : x^2 = y^2 = z^n = xyz = e \rangle \)

(i) \( P_2(G_n; y, x, z) = 6: \)

\[
P_4(G_n; x, y, z) = \begin{cases} 
  n \left( \frac{5}{2} \right), & n \equiv 0 \mod 4, \\
  5n, & n \equiv 2 \mod 4, \\
  10n, & \text{otherwise},
\end{cases}
\] (2.10)

(ii) \( k \geq 5. \)

(1) If there is no \( t \in [3, k - 2] \) such that \( t \) is a odd factor of \( n \), then

\[
P_k(G_n; x, y, z) = \begin{cases} 
  n \left( \frac{k + 1}{2} \right), & n \equiv 0 \mod 4, \\
  n(k + 1), & n \equiv 2 \mod 4, \\
  2n(k + 1), & \text{otherwise}.
\end{cases}
\] (2.11)

(2) Let \( \alpha \) be the biggest odd factor of \( n \) in \([3, k - 2]\). Then two cases occur:

(i') if \( \alpha 3^j \not\in [3, k - 2] \) for \( j \in \mathbb{N} \), then

\[
P_k(G_n; x, y, z) = \begin{cases} 
  \alpha \left( n \left( \frac{k + 1}{2} \right) \right), & n \equiv 0 \mod 4, \\
  \alpha(n(k + 1)), & n \equiv 2 \mod 4, \\
  \alpha(2n(k + 1)), & \text{otherwise};
\end{cases}
\] (2.12)

(ii') if \( \beta \) is the biggest odd number which is in \([3, k - 2]\) and \( \beta = \alpha 3^j \) for \( j \in \mathbb{N} \), then

\[
P_k(G_n; x, y, z) = \begin{cases} 
  \beta \left( n \left( \frac{k + 1}{2} \right) \right), & n \equiv 0 \mod 4, \\
  \beta(n(k + 1)), & n \equiv 2 \mod 4, \\
  \beta(2n(k + 1)), & \text{otherwise}.
\end{cases}
\] (2.13)

Proof. We consider \( G_n \) as \( D_n \), the dihedral group of 2\( n \) elements. Now \( D_n \) being the group of symmetries of the regular polygon with \( n \) elements admits a presentation as the group generated by the two matrices:

\[
a := \begin{pmatrix}
  \cos \left( \frac{2\pi}{n} \right) & -\sin \left( \frac{2\pi}{n} \right) \\
  \sin \left( \frac{2\pi}{n} \right) & \cos \left( \frac{2\pi}{n} \right)
\end{pmatrix}, \quad b := \begin{pmatrix}
  1 & 0 \\
  0 & -1
\end{pmatrix}.
\] (2.14)
Under these identifications, we can take $z = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ \sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$, $y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $x = \begin{pmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ -\sin(2\pi/n) & \cos(2\pi/n) \end{pmatrix}$.

(i) If $k = 2$, we have the sequence

$$x_0 = y, \quad x_1 = x, \quad x_2 = z, \quad x_3 = \begin{pmatrix} \cos(4\pi/n) & -\sin(4\pi/n) \\ -\sin(4\pi/n) & \cos(4\pi/n) \end{pmatrix} = xz, \quad x_4 = x,$$

$$x_5 = \begin{pmatrix} \cos(\frac{2\pi}{n}) & \sin(\frac{2\pi}{n}) \\ -\sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{pmatrix} = xy, \quad x_6 = y, \quad x_7 = x, \quad x_8 = z, \ldots.$$

Thus we get $P_2(G_n; y, x, z) = 6$.

(ii) If $k = 4$, we have the sequence

$$x, y, z, xyz = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e, \quad (xyz)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e, \quad \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ -\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix} = x,$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = y, \quad \begin{pmatrix} \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ -\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix} = xy, \quad \begin{pmatrix} \cos\frac{4\pi}{n} & \sin\frac{4\pi}{n} \\ -\sin\frac{4\pi}{n} & \cos\frac{4\pi}{n} \end{pmatrix} = z^{-2},$$

$$\begin{pmatrix} \cos\frac{6\pi}{n} & \sin\frac{6\pi}{n} \\ \sin\frac{6\pi}{n} & -\cos\frac{6\pi}{n} \end{pmatrix} = z^4x, \quad \begin{pmatrix} \cos\frac{14\pi}{n} & \sin\frac{14\pi}{n} \\ \sin\frac{14\pi}{n} & -\cos\frac{14\pi}{n} \end{pmatrix} = z^8x,$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = y, \quad \begin{pmatrix} \cos\frac{2\pi}{n} & -\sin\frac{2\pi}{n} \\ \sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix} = z, \quad \begin{pmatrix} \cos\frac{8\pi}{n} & -\sin\frac{8\pi}{n} \\ \sin\frac{8\pi}{n} & \cos\frac{8\pi}{n} \end{pmatrix} = z^4,$$

$$\begin{pmatrix} \cos\frac{24\pi}{n} & -\sin\frac{24\pi}{n} \\ \sin\frac{24\pi}{n} & \cos\frac{24\pi}{n} \end{pmatrix} = z^{12}, \quad \begin{pmatrix} \cos\frac{34\pi}{n} & -\sin\frac{34\pi}{n} \\ -\sin\frac{34\pi}{n} & \cos\frac{34\pi}{n} \end{pmatrix} = xz^{16}, \ldots.$$

(2.16)
Now we consider what happens to the 4-nacci sequence when we have a section of the form ...
\( z^\tau x, zx, z, \ldots \):
\[
z^\tau x, zx = y, z, z^\tau, z^{2\tau}, z^{2\tau+1}, y, x z x = xy, x z^2 x = z^{-(\tau+2)},
\]
\[
x z^{3\tau+4} x = z^{-(3\tau+4)}, z^{4\tau+8} x, z x = y, z, \ldots.
\]
(2.17)

The 4-nacci sequence can be said to form layers of length 10. Using the above, the 4-nacci sequence becomes
\[
\begin{align*}
x_0 &= x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = \epsilon, \ldots, \\
x_{10} &= z^8 x, \quad x_{11} = z^6 x = y, \quad x_{12} = z = 4, \ldots, \\
x_{20} &= z^{32} x, \quad x_{21} = z^8 x = y, \quad x_{22} = z = 8, \ldots, \\
x_{10i} &= z^{8i} x, \quad x_{10i+1} = z^i x = y, \quad x_{10i+2} = z = i, \quad x_{10i+3} = z^{4i}, \ldots,
\end{align*}
\]
(2.18)

where \( z^{8i} = \left( \cos \frac{8i}{n} \pi - \sin \frac{8i}{n} \pi \right) \) and \( z^{4i} = \left( \cos \frac{4i}{n} \pi - \sin \frac{4i}{n} \pi \right) \).

So, we need the smallest \( i \in N \) such that \( 8i^2 = nv1 \) and \( 4i = nv2 \) for \( v1, v2 \in N \).

If \( n \equiv 0 \mod 4 \), \( z^{8i} = \left( \frac{1}{0}, \frac{1}{1} \right) \) and \( z^{4i} = \left( \frac{1}{0}, \frac{1}{1} \right) \) for \( i = n/4 \).

Thus, \( 10i = (5/2)n \) and \( P_4 = (n, x, y, z) = n((k+1)/2) = (5/2)n \).

If \( n \equiv 2 \mod 4 \), \( z^{8i} = \left( \frac{1}{0}, \frac{1}{1} \right) \) and \( z^{4i} = \left( \frac{1}{0}, \frac{1}{1} \right) \) for \( i = n/2 \).

Thus, \( 10i = 5n \) and \( P_4 = (n, x, y, z) = n(k+1) = 5n \).

If \( n \equiv 1 \mod 4 \) or \( n \equiv 3 \mod 4 \), \( z^{8i} = \left( \frac{1}{0}, \frac{1}{1} \right) \) and \( z^{4i} = \left( \frac{1}{0}, \frac{1}{1} \right) \) for \( i = n \).

Thus, \( 10i = 10n \) and \( P_4 = (n, x, y, z) = 2n(k+1) = 10n \).

(iii) If \( k \geq 5 \), the first \( k+1 \) elements of the sequence are
\[
\begin{align*}
x_0 &= x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = z^n, \quad x_4 = z^{2n}, \ldots, \quad x_k = z^{kn}.\n\end{align*}
\]
(2.19)

Thus, using the above information, the sequence reduces
\[
\begin{align*}
x_0 &= x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = \epsilon, \quad x_4 = e, \ldots, e, \nonumber
\end{align*}
\]
(2.20)

where \( x_j = e \) for \( 3 \leq j \leq k \).

Now we consider what happens to the \( k \)-nacci sequence when we have a section of the form ...
\( z^\tau x, zx, z, \ldots \):
\[
\begin{align*}
x_{2k+2} &= \prod_{i=k+2}^{2k+1} x_i = z^\tau x, \quad x_{2k+2+1} = \prod_{i=k+3}^{2k+2} x_i = z^x, \quad x_{2k+2+2} = \prod_{i=k+4}^{2k+3} x_i = z, \\
x_{2k+2+3} &= \prod_{i=k+5}^{2k+4} x_i = z^e, \quad x_{2k+2+4} = \prod_{i=k+6}^{2k+5} x_i = z^e, \quad x_{2k+2+5} = \prod_{i=k+7}^{2k+6} x_i = z^{\epsilon}, \ldots, \\
x_{2k+2+k} &= \prod_{i=2k+2}^{3k+1} x_i = z^{\epsilon}, \ldots.
\end{align*}
\]
(2.21)
The *k-nacci* sequence can be said to form layers of length $(2k + 2)$. Using the above, the *k-nacci* sequence becomes

\[
\begin{align*}
x_0 &= x, \quad x_1 = y, \quad x_2 = z, \quad x_3 = e, \ldots, \quad x_k = z^{n-3}n = e, \ldots, \\
x_{i(2k+2)} &= z^i x, \quad x_{i(2k+2)+1} = z x, \quad x_{i(2k+2)+2} = z, \quad x_{i(2k+2)+3} = z^j, \\
x_{i(2k+2)+4} &= z^{8i+4j}, \quad x_{i(2k+2)+5} = z^{4j}, \ldots, \quad x_{i(2k+2)+k} = z^{8i-4}, \ldots.
\end{align*}
\]  

(2.22)

So, we need the smallest $i \in N$ such that $4i = nv_1$ and $8i^2 + 4i = nv_2$ for $v_1, v_2 \in N$.

(1) If there is no $t \in [3, k - 2]$ such that $t$ is an odd factor of $n$, there are 3 subcases.

Case 1. If $n \equiv 0 \pmod{4}$ and $n \mid \tau, n \mid u_1, \ldots, u_{k-4}, z^{4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$, and $z^{8i^2+4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$ for $i = n/4$. So, we get $P_k = (G_n; x, y, z) = n((k + 1)/2)$ since $i(2k + 2) = n((k + 1)/2)$ (where by $n \mid \tau$ we mean that $n$ divides $\tau$).

Case 2. If $n \equiv 2 \pmod{4}$ and $n \mid \tau, n \mid u_1, \ldots, u_{k-4}, z^{4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$, and $z^{8i^2+4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$ for $i = n/2$. So, we get $P_k = (G_n; x, y, z) = n(k + 1)$ since $i(2k + 2) = n(k + 1)$.

Case 3. If $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$ and $n \mid \tau, n \mid u_1, \ldots, u_{k-4}$, $z^{4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$, and $z^{8i^2+4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$ for $i = n$. So, we get $P_k = (G_n; x, y, z) = 2n(k + 1)$ since $i(2k + 2) = 2n(k + 1)$.

(2) Let $a$ be the biggest factor of $n$ in $[3, k - 2]$. Then two cases occur:

(i) If $a3^j \not\in [3, k - 2]$ for $j \in N$, then there are 3 subcases.

Case 1. If $n \equiv 0 \pmod{4}$ and $n \mid \tau, n \mid u_1, \ldots, u_{k-4}$, $z^{4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$, and $z^{8i^2+4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$ for $i = a(n/4)$. So, we get $P_k = (G_n; x, y, z) = a(n((k + 1)/2))$ since $i(2k + 2) = a(n((k + 1)/2))$.

Case 2. If $n \equiv 2 \pmod{4}$ and $n \mid \tau, n \mid u_1, \ldots, u_{k-4}$, $z^{4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$, and $z^{8i^2+4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$ for $i = a(n/2)$. So, we get $P_k = (G_n; x, y, z) = a(n(k + 1))$ since $i(2k + 2) = a(n(k + 1))$.

Case 3. If $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$ and $n \mid \tau, n \mid u_1, \ldots, u_{k-4}$, $z^{4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$ and $z^{8i^2+4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$ for $i = an$. So, we get $P_k = (G_n; x, y, z) = a(2n(k + 1))$ since $i(2k + 2) = a(2n(k + 1))$.

(ii') If $\beta$ is the biggest odd number which is in $[3, k - 2]$ and $\beta = a3^j$ for $j \in N$, then there are 3 subcases.

Case 1. If $n \equiv 0 \pmod{4}$ and $n \mid \tau, n \mid u_1, \ldots, u_{k-4}$, $z^{4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$, and $z^{8i^2+4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$ for $i = \beta(n/4)$. So, we get $P_k = (G_n; x, y, z) = \beta(n((k + 1)/2))$ since $i(2k + 2) = \beta(n((k + 1)/2))$.

Case 2. If $n \equiv 2 \pmod{4}$ and $n \mid \tau, n \mid u_1, \ldots, u_{k-4}$, $z^{4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$, and $z^{8i^2+4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$ for $i = \beta(n/2)$. So, we get $P_k = (G_n; x, y, z) = \beta(n(k + 1))$ since $i(2k + 2) = \beta(n(k + 1))$.

Case 3. If $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$ and $n \mid \tau, n \mid u_1, \ldots, u_{k-4}$, $z^{4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$, and $z^{8i^2+4i} = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right)$ for $i = \beta n$. So, we get $P_k = (G_n; x, y, z) = \beta(2n(k+1))$ since $i(2k+2) = \beta(2n(k+1))$.

This completes the proof. □
In the case of 2-generator the group has the presentation \( \langle x, y : x^2 = y^2 = (xy)^n = e \rangle \) and the period is the same as in the 3-generator case and proof is similar.

### 3. The Groups \( E(2, 2, 2), E(n, 2, 2), E(2, n, 2), \) and \( E(2, 2, n) \)

**Definition 3.1.** The extended triangle group \( E(p, q, r) \), for \( p, q, r > 1 \), is defined by the presentation

\[
\langle x, y, z : x^2 = y^2 = z^2 = (xy)^p = (yz)^q = (zx)^r = e \rangle.
\] (3.1)

The extended triangle groups are a very important class of groups closely linked to automorphism groups of regular maps, see [17]. The triangle groups (polyhedral groups), \( (p, q, r) \) are index two subgroups of extended triangle groups. To see this, let \( X = xy, Y = yz \) and \( Z = zx \) in \( E(p, q, r) \) and then use the obvious epimorphism. We get the following three cases for \( E(p, q, r) \):

1. the Euclidean case if \( 1/p + 1/q + 1/r = 1 \),
2. the elliptic case if \( 1/p + 1/q + 1/r > 1 \),
3. the hyperbolic case if \( 1/p + 1/q + 1/r < 1 \).

The group \( E(p, q, r) \) is finite if and only if \( 1/p + 1/q + 1/r > 1 \).

For more information on these groups, see [14, 18].

**Theorem 3.2.** Let \( E_2 \) be the group defined by the presentation \( \langle x, y, z : x^2 = y^2 = z^2 = (xy)^2 = (yz)^2 = (zx)^2 = e \rangle \). Then \( P_k(E_2; x, y, z) = k + 1 \) for \( k > 2 \).

**Proof.** Since \( E_2 \) can be identified with \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) and \( x, y, z \) with \( (1,0,0), (0,1,0), \) and \( (0,0,1) \), respectively, from a similar argument applied to Theorem 2.2, we get \( P_k(E_2; x, y, z) = k + 1 \). \( \Box \)

**Theorem 3.3.** Let \( E_n, n > 2 \), be the group defined by the presentation \( \langle x, y, z : x^2 = y^2 = z^2 = (xy)^2 = (yz)^2 = (zx)^n = e \rangle \)

(i)

\[
P_{4,5}(E_n; x, y, z) = \begin{cases} 
  n \left( \frac{k + 1}{2} \right), & n \equiv 0 \mod 4, \\
  n(k + 1), & n \equiv 2 \mod 4, \\
  2n(k + 1), & \text{otherwise};
\end{cases}
\] (3.2)

(ii) let \( k \geq 6 \).
(1) If there is no \( t \in [3, k - 3] \) such that \( t \) is an odd factor of \( n \), then

\[
P_k(E_n; x, y, z) = \begin{cases} 
    n \left( \frac{k + 1}{2} \right), & n \equiv 0 \mod 4, \\
    n(k + 1), & n \equiv 2 \mod 4, \\
    2n(k + 1), & \text{otherwise}.
\end{cases}
\]  

(3.3)

(2) Let \( \alpha \) be the biggest odd factor of \( n \) in \([3, k - 3]\). Then two cases occur:

(i') if \( \alpha 3^j \not\in [3, k - 3] \) for \( j \in \mathbb{N} \), then

\[
P_k(E_n; x, y, z) = \begin{cases} 
    \alpha \left( n \left( \frac{k + 1}{2} \right) \right), & n \equiv 0 \mod 4, \\
    \alpha(n(k + 1)), & n \equiv 2 \mod 4, \\
    \alpha(2n(k + 1)), & \text{otherwise};
\end{cases}
\]  

(3.4)

(ii') if \( \beta \) is be the biggest odd number which is in \([3, k - 3]\) and \( \beta = \alpha 3^j \) for \( j \in \mathbb{N} \), then

\[
P_k(E_r; x, y, z) = \begin{cases} 
    \beta \left( n \left( \frac{k + 1}{2} \right) \right), & n \equiv 0 \mod 4, \\
    \beta(n(k + 1)), & n \equiv 2 \mod 4, \\
    \beta(2n(k + 1)), & \text{otherwise}.
\end{cases}
\]  

(3.5)

Proof. Since \( y \) has order 2 and commutes with \( x \) and \( z \) it follows that \( E_n = \mathbb{Z}_2 \oplus D_n \). As a group of matrices, the can be identified with a group of \( 3 \times 3 \) matrices of form

\[
\begin{pmatrix} 
    \pm 1 & 0 \\
    0 & a
\end{pmatrix},
\]

(3.6)

where \( a \) is a \( 2 \times 2 \) matrix in dihedral group generated by \( a \) and \( b \) shown at (2.14). Here,

\[
x = \begin{pmatrix} 
    1 & 0 & 0 \\
    0 & \cos \left( \frac{2\pi}{n} \right) & -\sin \left( \frac{2\pi}{n} \right) \\
    0 & -\sin \left( \frac{2\pi}{n} \right) & -\cos \left( \frac{2\pi}{n} \right)
\end{pmatrix}, \quad y = \begin{pmatrix} 
    -1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1
\end{pmatrix}, \quad z = \begin{pmatrix} 
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    0 & 0 & -1
\end{pmatrix}.
\]  

(3.7)

Now, since the period of a Fibonacci sequence in a direct product of groups is the least common multiple of the periods in each the factors and from a similar argument applied to Theorem 2.4 the proof is done. \( \square \)
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