Research Article

Weighted Composition Operators and Integral-Type Operators between Weighted Hardy Spaces on the Unit Ball

Stevo Stević¹ and Sei-Ichiro Ueki²

¹ Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia
² Faculty of Engineering, Ibaraki University, Hitachi 316-8511, Japan

Correspondence should be addressed to Stevo Stević, sstevic@ptt.rs

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We study the boundedness and compactness of the weighted composition operators as well as integral-type operators between weighted Hardy spaces on the unit ball.

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1. Introduction

Let ℬ denote the open unit ball of the n-dimensional complex vector space Cⁿ, ∂ℬ its boundary, and let H(ℬ) denote the space of all holomorphic functions on ℬ. For 0 < p < ∞ and α ≥ 0 we define the weighted Hardy space Hᵃᵖ(ℬ) as follows:

\[
Hⁿᵃᵖ(ℬ) = \left\{ f \in H(ℬ) : \sup_{0 < r < 1} (1 - r)^α \int_{∂ℬ} |f(rζ)|^p dσ(ζ) < ∞ \right\},
\]

(1.1)

where dσ is the normalized Lebesgue measure on ∂ℬ (see, also [1], as well as [2], for an equivalent definition of the space). Note that for α = 0 the weighted Hardy space becomes the Hardy space Hᵖ(ℬ). We define the norm \|f\|_{Hⁿᵃᵖ} on this space as follows:

\[
\|f\|_{Hⁿᵃᵖ} = \sup_{0 < r < 1} (1 - r)^α \int_{∂ℬ} |f(rζ)|^p dσ(ζ).
\]

(1.2)
With this norm \( H^p_n(\mathbb{B}) \) is a Banach space when \( 1 \leq p < \infty \). For a related space on the unit polydisk; see [3]. In this paper, we investigate two types of operators acting between weighted Hardy spaces.

Let \( \varphi \) be a holomorphic self-map of \( \mathbb{B} \) and \( u \in H(\mathbb{B}) \). Then \( \varphi \) and \( u \) induce a \textit{weighted composition operator} \( uC_\varphi \) on \( H(\mathbb{B}) \) which is defined by \( uC_\varphi f = u(f \circ \varphi) \). This type of operators has been studied on various spaces of holomorphic functions in \( \mathbb{C}^n \), by many authors; see, for example, [4], recent papers [5–17], and the references therein.

Let \( g \in H(\mathbb{D}) \) and \( \varphi \) be a holomorphic self-map of the open unit disk \( \mathbb{D} \) in the complex plane. Products of integral and composition operators on \( H(\mathbb{D}) \) were introduced by S. Li and S. Stević in a private communication (see [18–21], as well as papers [22] and [23] for closely related operators) as follows:

\[
C_\varphi J_g f(z) = \int_0^{\varphi(z)} f(\zeta) g'(\zeta) d\zeta.
\]
\[
J_g C_\varphi f(z) = \int_0^{\varphi(z)} f(\zeta) g'(\zeta) d\zeta.
\]

In [24] the author of this paper has extended the operator in (1.4) in the unit ball settings as follows (see also [25, 26]). Assume \( g \in H(\mathbb{D}) \), \( g(0) = 0 \), and \( \varphi \) is a holomorphic self-map of \( \mathbb{B} \), then we define an operator on the unit ball as follows:

\[
P_\varphi^g f(z) = \int_0^1 f(\varphi(tz)) g_0(tz) \frac{dt}{t}, \quad f \in H(\mathbb{B}), \quad z \in \mathbb{B}.
\]

If \( n = 1 \), then \( g \in H(\mathbb{D}) \) and \( g(0) = 0 \), so that \( g(z) = zg_0(z) \), for some \( g_0 \in H(\mathbb{D}) \). By the change of variable \( \zeta = tz \), it follows that

\[
P_\varphi^g f(z) = \int_0^1 f(\varphi(tz)) t g_0(tz) \frac{dt}{t} = \int_0^{\varphi(0)} f(\varphi(\zeta)) g_0(\zeta) d\zeta.
\]

Thus the operator (1.5) is a natural extension of operator \( J_g C_\varphi \) in (1.4). For related operators see [27–33] as well as the references therein.

In this paper we study the boundedness and compactness of the weighted composition operators as well as the integral-type operator \( P_\varphi^g \), between different weighted Hardy spaces on the unit ball.

Throughout this paper, constants are denoted by \( C \), they are positive and may differ from one occurrence to the other. The notation \( a \leq b \) means that there is a positive constant \( C \) such that \( a \leq Cb \). Moreover, if both \( a \leq b \) and \( b \leq a \) hold, then one says that \( a \asymp b \).

\section{2. Weighted Composition Operators}

This section is devoted to studying weighted composition operators between weighted Hardy spaces. Weighted composition operators between different Hardy spaces on the unit ball were previously studied in [15, 34], while the composition operators on the unit ball were studied in [35, 36]. For the case of the unit disk see also [37].
Before we formulate the main results in this section we quote several auxiliary results which will be used in the proofs of these ones.

\textbf{Lemma 2.1.} Let $0 < p < \infty$ and $\alpha \geq 0$. Suppose that $u \in H(\mathbb{B})$ and $\varphi$ is a holomorphic self-map of $\mathbb{B}$. Then for each $f \in H(\mathbb{B})$

\[
\|uC_{\varphi}f\|_{H^p_\alpha} \leq \lim \inf_{R \to 1^-} \|uC_{\varphi}f R\|_{H^p_\alpha}
\]

where $uC_{\varphi}f_R(z) = u(z)f(R\varphi(z))$.

\textit{Proof.} Fix $r \in (0, 1)$. Fatou’s lemma shows that

\[
(1 - r)^{\alpha}\int_{\partial \mathbb{B}} |u(r\zeta) f(\varphi(r\zeta))| d\sigma(\zeta) \leq (1 - r)^{\alpha}\lim \inf_{R \to 1^-} \int_{\partial \mathbb{B}} |u(r\zeta) f(\varphi(r\zeta))| |^p d\sigma(\zeta)
\]

\[
= \lim \inf_{R \to 1^-} (1 - r)^{\alpha}\int_{\partial \mathbb{B}} |u(r\zeta) f(\varphi(r\zeta))| |^p d\sigma(\zeta)
\]

\[
\leq \lim \inf_{R \to 1^-} \|uC_{\varphi}f R\|_{H^p_\alpha}.
\]

Hence we have the desired inequality. \hfill \square

Recall that an $f \in H(\mathbb{B})$ has the homogeneous expansion

\[
f(z) = \sum_{k=0}^{\infty} \sum_{|\gamma|=k} c(\gamma) z^\gamma,
\]

where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multi-index, $|\gamma| = \gamma_1 + \cdots + \gamma_n$ and $z^\gamma = z_1^{\gamma_1} \cdots z_n^{\gamma_n}$. For the homogeneous expansion of $f$ and any integer $j \geq 1$, let

\[
R_j f(z) = \sum_{k=0}^{\infty} \sum_{|\gamma|=k} c(\gamma) z^\gamma,
\]

and $K_j = I - R_j$ where $I f = f$ is the identity operator. Note that $K_j$ is compact operator on $H^p_\alpha(\mathbb{B})$ for each $j \in \mathbb{N}$.

\textbf{Lemma 2.2.} If $1 < p < \infty$, then $R_j$ converges to 0 pointwise in the Hardy space $H^p(\mathbb{B})$ as $j \to \infty$.

\textit{Proof.} See [34, Corollary 3.4]. \hfill \square

Lemma 2.2 and the uniform boundedness principle show that \{R_j\} is an uniformly bounded sequence in $H^p(\mathbb{B})$. 

The following lemma is proved similar to [4, Lemma 3.16]. We omit its proof.

**Lemma 2.3.** If $uC_\varphi$ is bounded from $H^p_\alpha(B)$ into $H^q_\beta(B)$, then

$$
\|uC_\varphi\|_{H^p_\alpha(B) \to H^q_\beta(B)} \leq \lim_{j \to \infty} \inf \|uC_\varphi R_j\|_{H^p_\alpha(B) \to H^q_\beta(B)}
$$

(2.5)

where $\| \cdot \|_{H^p_\alpha(B) \to H^q_\beta(B)}$ and $\| \cdot \|_{H^p_\alpha(B) \to H^q_\beta(B)}$ denote the essential norm and the operator norm, respectively.

**Lemma 2.4.** Let $0 < p \leq q < \infty$. Suppose that $\mu$ is a positive Borel measure on $B$ which satisfies

$$
\mu(B(\zeta, t)) \leq C_1 t^{\alpha/n/p} \quad (\zeta \in \partial B, t > 0),
$$

(2.6)

for some positive constant $C_1$. Then there exists a positive constant $C_2$ which depends only on $p, q,$ and the dimension $n$ such that

$$
\int_B |f|^q d\mu \leq C_1 C_2 \|f\|_{H^p}^q,
$$

(2.7)

for any $f \in H^p(B)$. Here $B(\zeta, t) = \{z \in B : |1 - \langle z, \zeta \rangle| < t\}$.

**Proof.** See [38, page 13, Theorem ] or [34, Lemma 2.1] .

Let $0 < q < \infty$. For each $r \in (0, 1)$, a holomorphic self-map $\varphi$ of $B$ and $u \in H(B)$, we define a positive Borel measure $\mu_{u, \varphi}$ on $B$ by

$$
\mu_{u, \varphi}(E) = \int_{(\varphi)^{-1}(E)} |u(r\zeta)|^q d\sigma, \quad \forall E \in \mathcal{B}.
$$

(2.8)

for all Borel sets $E$ of $B$. By the change of variables formula from measure theory, we can verify

$$
\int_B g d\mu_{u, \varphi} = \int_{\partial B} |u(r\zeta)|^q (g \circ \varphi)(r\zeta) d\sigma(\zeta),
$$

(2.9)

for each nonnegative measurable function $g$ in $B$.

**Theorem 2.5.** Let $0 < p \leq q < \infty$ and $\alpha, \beta \geq 0$. Suppose that $u \in H(B)$ and $\varphi$ is a holomorphic self-map of $B$. Then $uC_\varphi : H^p_\alpha(B) \to H^q_\beta(B)$ is bounded if and only if

$$
\sup_{w \in B} \sup_{0 < r < 1} \int_{\partial B} |u(r\zeta)|^q \left\{ \frac{1 - |w|^2}{1 - \langle \varphi(r\zeta), w \rangle} \right\}^{\alpha(n+\beta)/p} d\sigma(\zeta) < \infty.
$$

(2.10)
Proof. For \( w \in \mathbb{B} \) we put

\[
 f_w(z) = \left\{ \frac{1 - |w|^2}{(1 - \langle z, w \rangle)^2} \right\}^{(a+n)/p}. \tag{2.11}
\]

Then we see that \( f_w \in H_u^q(\mathbb{B}) \) and moreover \( \sup_{w \in \mathbb{B}} \| f_w \|_{H_u^q} \leq C \). By a straightforward calculation, we have

\[
 \| u C_w f_w \|_{H_u^q}^q = \sup_{0 < r < 1} (1 - r)^{\beta} \int_{\partial \mathbb{B}} \left| u(r_{\zeta}) \right|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \varphi(r_{\zeta}), w \rangle|^2} \right\}^{q(a+n)/p} d\sigma(\zeta), \tag{2.12}
\]

for all \( w \in \mathbb{B} \). Hence if \( u C_w : H_u^q(\mathbb{B}) \to H_u^q(\mathbb{B}) \) is bounded, then \( u \) and \( \varphi \) satisfy the condition

\[
 \sup \sup_{w \in \mathbb{B} : 0 < r < 1} (1 - r)^{\beta} \int_{\partial \mathbb{B}} \left| u(r_{\zeta}) \right|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \varphi(r_{\zeta}), w \rangle|^2} \right\}^{q(a+n)/p} d\sigma(\zeta) \leq C \| u C_w \|_{H_u^q(\mathbb{B})}^q < \infty. \tag{2.13}
\]

Next we assume

\[
 M := \sup \sup_{w \in \mathbb{B} : 0 < r < 1} (1 - r)^{\beta} \int_{\partial \mathbb{B}} \left| u(r_{\zeta}) \right|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \varphi(r_{\zeta}), w \rangle|^2} \right\}^{q(a+n)/p} d\sigma(\zeta) < \infty. \tag{2.14}
\]

Fix \( r \in (0, 1) \) and \( R \in (0, 1) \), respectively. For \( \zeta \in \partial \mathbb{B} \) and \( t, 0 < t \leq t_R = 1 - R \), we put \( \omega = (1 - t) \zeta \) and \( \omega_R = (1 - t_R) \zeta \). Since the function \( f_w(z) \), which is defined by (2.11) for this \( \omega \), satisfies

\[
 \left| f_w(z) \right|^q > 4^{-q(a+n)/p} t^{-qn/p} (1 - R)^{-qa/p} \tag{2.15}
\]

for all \( z \in B(\zeta, t) \), we have

\[
 \frac{H^q_{u, \varphi}(B(\zeta, t))}{t^{qn/p}} \leq 4^{q(a+n)/p} (1 - R)^{qa/p} \int_{B(\zeta, t)} \left| f_w(z) \right|^q d\mu^q_{u, \varphi}(z) \leq 4^{q(a+n)/p} (1 - R)^{qa/p} \int_{\partial \mathbb{B}} \left| u(r_{\zeta}) \right|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \varphi(r_{\zeta}), w \rangle|^2} \right\}^{q(a+n)/p} d\sigma(\zeta) \leq 4^{q(a+n)/p} (1 - R)^{qa/p} \left( \frac{1}{1 - r} \right)^{\beta} M. \tag{2.16}
\]

By the same argument, the function \( f_{w_R}(z) \) gives the following estimate:

\[
 \frac{H^q_{u, \varphi}(B(\zeta, 2t_R))}{t_R^{qn/p}} \leq 4^{q(a+n)/p} (1 - R)^{qa/p} \left( \frac{1}{1 - r} \right)^{\beta} M t_R^{qn/p}. \tag{2.17}
\]
Now we need to prove that there exists a positive constant \( C \) such that

\[
\mu^r_{u,p}(B(\xi, t)) \leq C \left( \frac{(1 - R)^{\alpha/p}}{(1 - r)^\beta} \right) M t^q q_n/p,
\] (2.18)

for all \( \zeta \in \partial B \) and \( t > 0 \). By the estimate (2.16), we see that the inequality (2.18) is true for all \( t \in (0, t_R] \). Thus we assume \( t > t_R \). By the same argument as in [36, pages 241-242, proof of Theorem 1.1], we see that the inequality (2.17) shows that there exists a positive constant \( C_R \) which depends only on the dimension \( n \) such that

\[
\mu^r_{u,p}(B(\xi, t)) \leq C^9 (1 - R)^{\alpha/p} \frac{(1 - R)^q/q_n}{(1 - r)^\beta} M t^q q_n/p
\]

(2.19)

Hence for \( C = \max\{4^{q(\alpha+n)/p}, C_R q^{q(\alpha+n)/p}\} \), we have the inequality in (2.18).

For \( f \in H^p_\alpha(\mathbb{B}) \) the dilate function \( f_R \) belongs to the ball algebra, and so \( f_R \) is in the Hardy space \( H^p(\mathbb{B}) \). Hence Lemma 2.4 gives

\[
\int_B |f_R(z)|^q d\mu^r_{u,p}(z) \leq C' C M \left( \frac{(1 - R)^{\alpha/p}}{(1 - r)^\beta} \right) M ||f_R||^q_{H^p},
\] (2.20)

for some positive constant \( C' \) and all \( R \in (0, 1) \). This implies that

\[
(1 - r)\beta \int_{\partial B} |uC_{\varphi}f_R(r \xi)|^q d\sigma(\xi) \leq C' C M \left( (1 - R)^{\alpha/p} \int_{\partial B} |f(R \xi)|^p d\sigma(\xi) \right)^{q/p},
\] (2.21)

and so we have

\[
||uC_{\varphi}f_R||^q_{H^p_{\varphi}} \leq C' C M ||f||^q_{H^p_{\varphi}},
\] (2.22)

for all \( R \in (0, 1) \). By Lemma 2.1 we have

\[
||uC_{\varphi}f||^q_{H^p_{\varphi}} \leq C' C ||f||^q_{H^p_{\varphi}}
\]

\[
\times \sup_{w \in \mathbb{B}} \sup_{0 < \gamma < 1} (1 - r)\beta \int_{\partial B} |u(r \xi)|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \varphi(r \xi), w \rangle|^2} \right\}^{q(\alpha+n)/p} d\sigma(\xi).
\] (2.23)

This completes the proof. \( \square \)
The following proposition is proved in a standard way; see, for example, the proofs of the corresponding results in [4, 32, 33, 39]. Hence we omit its proof.

**Proposition 2.6.** Let $0 < p, q < \infty$ and $\alpha, \beta \geq 0$. Suppose that $u \in H(\mathbb{B})$ and $\varphi$ is a holomorphic self-map of $\mathbb{B}$ which induce the bounded operator $uC_{\varphi} : H^p(\mathbb{B}) \rightarrow H^q(\mathbb{B})$. Then $uC_{\varphi} : H^p(\mathbb{B}) \rightarrow H^q(\mathbb{B})$ is compact if and only if for every bounded sequence $\{f_j\}_{j \in \mathbb{N}}$ in $H^p(\mathbb{B})$ which converges to 0 uniformly on compact subsets of $\mathbb{B}$, $\{uC_{\varphi}f_j\}_{j \in \mathbb{N}}$ converges to 0 in $H^q(\mathbb{B})$.

In the proof of Theorem 2.8, we need the following lemma.

**Lemma 2.7.** Let $1 < p < \infty$, $\alpha \geq 0$, and $f_w$ be the family of test functions defined in (2.11). Then $f_w \rightarrow 0$ weakly in $H^p(\mathbb{B})$ as $|w| \rightarrow 1$.

**Proof.** The family $\{f_w\}_{w \in \mathbb{B}}$ is bounded in $H^p(\mathbb{B})$ and $f_w \rightarrow 0$ uniformly on compact subsets of $\mathbb{B}$ as $|w| \rightarrow 1$. By the definitions of the space $H^p(\mathbb{B})$ and the norm $\| \cdot \|_{H^p}$, we see that $H^p(\mathbb{B})$ is a subspace of the weighted Bergman space $A^p_\alpha(\mathbb{B})$ and

$$\|f\|_{A^p_\alpha} \leq C(\alpha, p, n)\|f\|_{H^p} \quad (f \in H^p(\mathbb{B})), \quad (2.24)$$

for some positive constant $C(\alpha, p, n)$ which depends on $\alpha, p$, and $n$. This inequality implies that the family $\{f_w\}_{w \in \mathbb{B}}$ is also bounded in $A^p_\alpha(\mathbb{B})$. Note also that the family converges to 0 uniformly on compact subsets of $\mathbb{B}$ as $|w| \rightarrow 1$. Hence $f_w \rightarrow 0$ weakly in $A^p_\alpha(\mathbb{B})$ as $|w| \rightarrow 1$.

In order to prove that $f_w \rightarrow 0$ weakly in $H^p(\mathbb{B})$ as $|w| \rightarrow 1$, we take an arbitrary bounded linear functional $\Lambda$ on $H^p_\alpha(\mathbb{B})$. By the Hahn-Banach theorem, $\Lambda$ can be extended to a bounded linear functional $\tilde{\Lambda}$ on $A^p_\alpha(\mathbb{B})$ so that $\tilde{\Lambda}(f_w) = \Lambda(f_w)$ for all $w \in \mathbb{B}$. Since $f_w \rightarrow 0$ weakly in $A^p_\alpha(\mathbb{B})$ as $|w| \rightarrow 1$, we have $\Lambda(f_w) = \tilde{\Lambda}(f_w) \rightarrow 0$ as $|w| \rightarrow 1$, and so $f_w \rightarrow 0$ weakly in $H^p(\mathbb{B})$ as $|w| \rightarrow 1$.

**Theorem 2.8.** Let $1 < p \leq q < \infty$ and $\alpha, \beta \geq 0$. Suppose that $u \in H(\mathbb{B})$ and $\varphi$ is a holomorphic self-map of $\mathbb{B}$ such that $uC_{\varphi} : H^p(\mathbb{B}) \rightarrow H^q(\mathbb{B})$ is bounded. Then the $q$th power of the essential norm $\|uC_{\varphi}\|_{\epsilon H^p(\mathbb{B}) \rightarrow H^q(\mathbb{B})}$ is comparable to

$$\lim_{|w| \rightarrow 1^-} \sup_{0 < r < 1} (1 - r)^\theta \int_{\partial \mathbb{B}} |u(r\zeta)|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \varphi(r\zeta), w \rangle|^2} \right\}^{(\alpha + n)/p} d\sigma(\zeta). \quad (2.25)$$

Hence $uC_{\varphi} : H^p_\alpha(\mathbb{B}) \rightarrow H^q_\beta(\mathbb{B})$ is compact if and only if

$$\lim_{|w| \rightarrow 1^-} \sup_{0 < r < 1} (1 - r)^\theta \int_{\partial \mathbb{B}} |u(r\zeta)|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \varphi(r\zeta), w \rangle|^2} \right\}^{(\alpha + n)/p} d\sigma(\zeta) = 0. \quad (2.26)$$
Proof. To prove a lower estimate

\[ \|uC_q\|_{H^p_\alpha(B) \to H^q_\beta(B)} \geq \lim \sup_{|w| \to 1^-} (1 - r)^\beta \int_{\partial B} |u(r\zeta)|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \varphi(r\zeta), w \rangle|^2} \right\}^{q(\alpha - n)/p} \, d\sigma(\zeta), \]

(2.27)

we consider the test functions \( f_w \) defined in (2.11). The family \( \{f_w\}_{w \in B} \) is bounded in \( H^p_\alpha(B) \), say by \( L \), and \( f_w \to 0 \) uniformly on compact subsets of \( B \) as \( |w| \to 1^- \). Thus by Lemma 2.2 we have that \( f_w \to 0 \) weakly in \( H^p_\alpha(B) \) as \( |w| \to 1^- \), so that \( \|Kf_w\|_{H^q_\beta} \to 0 \) as \( |w| \to 1^- \) for every compact operator \( K : H^p_\alpha(B) \to H^q_\beta(B) \). Hence

\[ \|uC_q - K\|_{H^p_\alpha(B) \to H^q_\beta(B)} \geq \lim \sup_{|w| \to 1^-} \|uC_q - K\|_{H^q_\beta} \]

(2.28)

This inequality and (2.12) give the lower estimate for \( \|uC_q\|_{H^p_\alpha(B) \to H^q_\beta(B)} \).

Next we prove an upper estimate. Take \( f \in H^p_\alpha(B) \) with \( \|f\|_{H^p_\alpha} \leq 1 \). Fix \( \epsilon > 0 \) and put

\[ M_1 := \lim \sup_{|w| \to 1^-} (1 - r)^\beta \int_{\partial B} |u(r\zeta)|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \varphi(r\zeta), w \rangle|^2} \right\}^{q(\alpha - n)/p} \, d\sigma(\zeta). \]

(2.29)

Then we can choose \( R_0 \in (0,1) \) such that

\[ \sup_{0 < r < 1} (1 - r)^\beta \int_{\partial B} |u(r\zeta)|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \varphi(r\zeta), w \rangle|^2} \right\}^{q(\alpha - n)/p} \, d\sigma(\zeta) < M_1 + \epsilon, \]

(2.30)

for \( w \in B \) with \( |w| > R_0 \). Fix \( r \in (0,1) \) and \( R \in (R_0,1) \). By the same argument as in the proof of inequality (2.20) in Theorem 2.5, we obtain that

\[ \int_B \left| (R_jf) (z) \right|^q \, d\mu_{H^p}(z) \leq C \frac{(1 - R)^{q\alpha/p}}{(1 - r)^\beta} \| (R_jf) \|_{H^q} (M_1 + \epsilon) \frac{\| (R_jf) \|_{H^q}}{R}, \]

(2.31)

where the positive constant \( C \) is independent of \( r, R \) and a positive integer \( j \). Since \( f_R \) is in the ball algebra, Lemma 2.2 gives

\[ \| (R_jf) \|_{H^q} = \| R_j (f_R) \|_{H^q} \leq \sup_{j \geq 1} \| R_j \|_{H^p(B) \to H^q(B)} \| f_R \|_{H^q}. \]

(2.32)
Combining this with (2.31), we have

$$
(1 - r)^\beta \int_{\partial B} |uC_\psi(R_jf_R)(r\zeta)|^q d\sigma(\zeta) \leq C'(M_1 + \epsilon) \left[ (1 - R)^a \int_{\partial B} |f_R(\zeta)|^p d\sigma(\zeta) \right]^{q/p} 
\leq C'(M_1 + \epsilon) \|f\|^q_{H^p_R}
$$

(2.33)

and so we have

$$
\|uC_\psi(R_jf_R)\|^q_{H^p_R} \leq C'(M_1 + \epsilon)\|f\|^q_{H^p_R}.
$$

(2.34)

Letting $R \to 1^-$, by Lemma 2.1, we obtain

$$
\|uC_\psi f\|^q_{H^p_R} \leq C'(M_1 + \epsilon)\|f\|^q_{H^p_R}.
$$

(2.35)

Since $\epsilon > 0$ is arbitrary, this estimate and Lemma 2.3 imply

$$
\|uC_\psi\|^q_{e,H^p_R(\mathbb{B}) \to H^q_R(\mathbb{B})} \leq \lim_{j \to \infty} \inf \|uC_\psi R_j\|^q_{H^p_R(\mathbb{B}) \to H^q_R(\mathbb{B})} 
\leq C'\lim_{\|\cdot\|_{H^p_R(\mathbb{B})}} \sup (1 - r)^\beta \int_{\partial B} |u(r\zeta)|^q \left\{ \frac{1 - |w|^2}{1 - \langle \varphi(r\zeta), w \rangle^2} \right\}^{q(a+n)/p} d\sigma(\zeta),
$$

(2.36)

which completes the proof.

\[\square\]

Remark 2.9. In the above proof, we used Lemma 2.2. This lemma required the assumption $1 < p < \infty$. Hence we cannot have an upper estimate for $\|uC_\psi\|_{e,H^p_R(\mathbb{B}) \to H^q_R(\mathbb{B})}$ in the case $0 < p \leq 1$. However, Proposition 2.6 shows that the compactness of $uC_\psi : H^p_R(\mathbb{B}) \to H^q_R(\mathbb{B})$ ($0 < p \leq q < \infty$) is equivalent to

$$
\lim_{\|\cdot\|_{H^p_R(\mathbb{B})}} \sup (1 - r)^\beta \int_{\partial B} |u(r\zeta)|^q \left\{ \frac{1 - |w|^2}{1 - \langle \varphi(r\zeta), w \rangle^2} \right\}^{q(a+n)/p} d\sigma(\zeta) = 0.
$$

(2.37)

3. Integral-Type Operators

Here we study the boundedness and compactness of the integral-type operators $P_\psi^S$ between weighted Hardy spaces on the unit ball.

For $f \in H(B)$ with the Taylor expansion $f(z) = \sum_{|j| \geq 0} a_j z^j$, let $\Re f(z) = \sum_{|j| \geq 0} |a_j| z^j$ be the radial derivative of $f$. 
The following lemma was proved in [24] (see also [25]).

**Lemma 3.1.** Assume that \( \varphi \) is a holomorphic self-map of \( \mathbb{B} \), \( g \in H(\mathbb{B}) \) and \( g(0) = 0 \). Then for every \( f \in H(\mathbb{B}) \) one holds

\[
\Re \left[ P^\varphi_0(f) \right](z) = f(\varphi(z))g(z). \tag{3.1}
\]

A positive continuous function \( \omega \) on the interval \([0,1]\) is called normal [40] if there is a \( \delta \in [0,1) \) and \( a \) and \( b, 0 < a < b \) such that

\[
\frac{\omega(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r \to 1^-} \frac{\omega(r)}{(1-r)^a} = 0,
\]

\[
\frac{\omega(r)}{(1-r)^b} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r \to 1^-} \frac{\omega(r)}{(1-r)^b} = \infty. \tag{3.2}
\]

If it is said that a function \( \omega : \mathbb{B} \to [0,\infty) \) is normal, it is also assume that it is radial.

**Lemma 3.2.** Assume that \( 0 < q \leq \infty \), \( m \) is a positive integer and \( \omega \) is normal. Then for every \( f \in H(\mathbb{B}) \)

\[
\sup_{0 < r < 1} \omega(r) M_q(f, r) \leq |f(0)| + \sup_{0 < r < 1} (1-r)^m \omega(r) M_q(\Re f, r), \tag{3.3}
\]

where

\[
M_q(f, r) = \left( \int_{\partial \mathbb{B}} |f(rζ)|^q d\sigma(ζ) \right)^{1/q}, \quad M_\infty(f, r) = \sup_{ζ \in \partial \mathbb{B}} |f(rζ)|. \tag{3.4}
\]

**Proof.** The proof of the lemma in the case \( 1 \leq q \leq \infty \) can be found in [27, Theorem 2]. However, due to an overlook, the proof for the case \( q \in (0,1) \) has a gap. Hence we will give a correct proof here in the case.

We may assume that \( f(0) = 0 \), otherwise we can consider the functions \( h(z) = f(z) - f(0) \). Also we may assume that \( \delta = 0 \), to avoid some minor technical difficulties.

By [27, Lemma 1], for each fixed \( q \in (0,1] \), there is a positive constant \( C \) depending only on \( q \) and the dimension \( n \) such that

\[
M_q(f, r) \leq C \left( \int_0^r (r-t)^{n-1} M_q^2(\Re f, t) dt \right)^{1/q}, \tag{3.5}
\]

for every \( r \in (0,1) \) and \( f \in H(\mathbb{B}) \) such that \( f(0) = 0 \).
From (3.5) and the fact that \( \omega \) is normal, we have

\[
\sup_{0 \leq r < 1} \omega(r) M_q(f, r) \leq C \sup_{0 \leq r < 1} \omega(r) \left( \frac{1}{r} \left( \int_0^r (r-t)^{q-1} M_q^d(\Re f, t) \, dt \right)^{1/q} \right)
\]

\[
\leq C \sup_{0 \leq r < 1} (1-r)^a \left( \frac{1}{r} \left( \int_0^r (r-t)^{q-1} \frac{\omega(t)}{(1-t)^aq} M_q^d(\Re f, t) \, dt \right)^{1/q} \right)
\]

\[
\leq C \sup_{0 \leq r < 1} (1-r)^a \left( \frac{1}{r} \left( \int_0^r (r-t)^{q-1} \frac{\omega(t)}{(1-t)^aq} \, dt \right)^{1/q} \right) \sup_{0 \leq t < 1} (1-t)\omega(t) M_q(\Re f, t)
\]

\[
= C \sup_{0 \leq r < 1} (1-r)^a \left( \int_0^1 \frac{(1-u)^{q-1}}{(1-ur)^aq} \, du \right)^{1/q} \sup_{0 \leq t < 1} (1-t)\omega(t) M_q(\Re f, t).
\]

(3.6)

By [40, page 291, Lemma 6] there exists a positive constant \( C \) such that

\[
\int_0^1 \frac{(1-u)^{q-1}}{(1-ur)^aq} \, du \leq \frac{C}{(1-r)^aq},
\]

for every \( r \in (0, 1) \). Combining this with (3.6), we have

\[
\sup_{0 \leq r < 1} \omega(r) M_q(f, r) \leq C \sup_{0 \leq r < 1} (1-r)^a \left( \frac{1}{(1-r)^aq} \right)^{1/q} \sup_{0 \leq t < 1} (1-t)\omega(t) M_q(\Re f, t)
\]

\[
= C \sup_{0 \leq r < 1} (1-r)^a \omega(t) M_q(\Re f, t).
\]

(3.8)

The reverse inequality is proved by the following inequality:

\[
(1-r)^a \omega(\Re f, r) \leq CM_q \left( f, \frac{1+r}{2} \right)
\]

(3.9)

and the fact that \( \omega(r) = \omega(1+r/2) \) for \( \omega \) normal (see [27]). Hence, we obtain the result for the case \( m = 1 \).

For \( m \geq 2 \) it should be only noticed that \( (1-r)^m \omega(r) \) is still normal, that \( \Re^m f(0) = 0 \) for every integer \( m \geq 1 \), and use the method of induction.

\[\square\]

**Theorem 3.3.** Let \( 0 < p \leq q < \infty \) and \( \alpha, \beta > 0 \). Suppose that \( g \in H(\mathbb{B}) \) with \( g(0) = 0 \) and \( \varphi \) is a holomorphic self-map of \( \mathbb{B} \). Then \( P^\infty_\varphi : H^p_{\alpha}(\mathbb{B}) \rightarrow H^p_{\beta}(\mathbb{B}) \) is bounded if and only if

\[
\sup_{w \in \mathbb{B}} \sup_{0 \leq r < 1} (1-r)^{\beta/p} \int_{\mathbb{B}} |g(r\xi)|^q \left\{ \frac{1-|w|^2}{|1-\langle \varphi(r\xi), w \rangle|^2} \right\}^{q(\alpha+n)/p} \, d\sigma(\xi) < \infty.
\]

(3.10)
Proposition 3.4. Let $\|f\|_{H^p_a} \leq 1$. Since the function $(1 - r)^{\beta/q}$ for $\beta > 0$ and $0 < q < \infty$ is normal, Lemma 3.2 gives

$$\sup_{0 \leq r < 1} (1 - r)^{\beta/q} M_q \left( \mathcal{P}_\alpha^p f, r \right) = \left| \mathcal{P}_\alpha^p f(0) \right| + \sup_{0 \leq r < 1} (1 - r)^{(\beta/q) + 1} M_q \left( \Re \mathcal{P}_\alpha^p f, r \right).$$  \hspace{1cm} (3.11)

The assumption $g(0) = 0$ implies $\mathcal{P}_\alpha^p f(0) = 0$, and Lemma 3.1 shows $\Re [\mathcal{P}_\alpha^p f] = g C_{\alpha} f$. Hence we obtain

$$\sup_{0 \leq r < 1} (1 - r)^{\beta/q} M_q \left( \mathcal{P}_\alpha^p f, r \right) \asymp \sup_{0 \leq r < 1} (1 - r)^{\beta/q} M_q \left( g C_{\alpha} f, r \right),$$  \hspace{1cm} (3.12)

and so we obtain $\|\mathcal{P}_\alpha^p f\|_{H^q_{\beta}} \asymp \|g C_{\alpha} f\|_{H^q_{\beta}}$. This implies that the boundedness of $\mathcal{P}_\alpha^p : H^p_a(\mathbb{B}) \rightarrow H^q_{\beta}(\mathbb{B})$ is equivalent to the boundedness of $g C_{\alpha} : H^p_a(\mathbb{B}) \rightarrow H^q_{\beta}(\mathbb{B})$. So Theorem 2.5 shows that the condition

$$\sup_{w \in \overline{B}} \sup_{0 \leq r < 1} (1 - r)^{\beta/q} \int_{\partial \overline{B}} \left| g(r \zeta) \right|^q \frac{1 - |w|^2}{\left| 1 - \langle \varphi(r \zeta), w \rangle \right|^2} \frac{q(\alpha + n)/p}{d \sigma(\zeta)} < \infty$$  \hspace{1cm} (3.13)

is a necessary and sufficient condition for the boundedness of $\mathcal{P}_\alpha^p : H^p_a(\mathbb{B}) \rightarrow H^q_{\beta}(\mathbb{B})$. This completes the proof. \hfill \Box

The next proposition is proved similar to Proposition 2.6.

**Proposition 3.4.** Let $0 < p, q < \infty$, and $\alpha, \beta > 0$. Suppose that $g \in H(\mathbb{B})$ with $g(0) = 0$ and $\varphi$ is a holomorphic self-map of $\mathbb{B}$ which induce the bounded operator $P_{\alpha}^p : H^p_a(\mathbb{B}) \rightarrow H^q_{\beta}(\mathbb{B})$. Then $P_{\alpha}^p : H^p_a(\mathbb{B}) \rightarrow H^q_{\beta}(\mathbb{B})$ is compact if and only if for every bounded sequence $\{f_j \}_{j \in \mathbb{N}}$ in $H^p_a(\mathbb{B})$ which converges to 0 uniformly on compact subsets of $\mathbb{B}$, $\{P_{\alpha}^p f_j \}_{j \in \mathbb{N}}$ converges to 0 in $H^q_{\beta}(\mathbb{B})$.

**Theorem 3.5.** Let $0 < p \leq q < \infty$ and $\alpha, \beta > 0$. Suppose that $g \in H(\mathbb{B})$ with $g(0) = 0$ and $\varphi$ is a holomorphic self-map of $\mathbb{B}$ which induce the bounded operator $P_{\alpha}^p : H^p_a(\mathbb{B}) \rightarrow H^q_{\beta}(\mathbb{B})$. Then $P_{\alpha}^p : H^p_a(\mathbb{B}) \rightarrow H^q_{\beta}(\mathbb{B})$ is compact if and only if

$$\lim_{|w| \rightarrow 1} \sup_{0 \leq r < 1} (1 - r)^{\beta/q} \int_{\partial \overline{B}} \left| g(r \zeta) \right|^q \frac{1 - |w|^2}{\left| 1 - \langle \varphi(r \zeta), w \rangle \right|^2} \frac{q(\alpha + n)/p}{d \sigma(\zeta)} = 0.$$  \hspace{1cm} (3.14)

**Proof.** First we assume that condition (3.14) holds. Take a bounded sequence $\{f_j \}_{j \in \mathbb{N}} \subset H^p_a(\mathbb{B})$ which converges to 0 uniformly on compact subsets of $\mathbb{B}$. Then Lemma 2.8 implies that $g C_{\alpha} : H^p_a(\mathbb{B}) \rightarrow H^q_{\beta}(\mathbb{B})$ is compact. This is Proposition 2.6 implies that

$$\lim_{j \rightarrow \infty} \|g C_{\alpha} f_j\|_{H^q_{\beta}} = 0.$$  \hspace{1cm} (3.15)
From (3.15) and since $\|P^j_{\psi}f\|_{H^p_{\psi}} \leq \|gC_{\psi}f\|_{H^p_{\psi}}^q$, we have that $\|P^j_{\psi}f\|_{H^p_{\psi}}^q \to 0$ as $j \to \infty$. By Proposition 3.4, we see that $P^j_{\psi} : H^p_a(\mathbb{B}) \to H^p_{\psi}(\mathbb{B})$ is compact.

To prove the necessity of the condition in (3.14), we consider the family of test functions $f_w$ which is defined by (2.11). Hence we have

$$\|P^j_{\psi}f_w\|_{H^p_{\psi}}^q \leq \|gC_{\psi}f_w\|_{H^p_{\psi}}^q = \sup_{0 < r < 1} \int_{\partial \mathbb{B}} |g(r\zeta)|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \varphi(r\zeta), w \rangle|^2} \right\}^{q(\alpha+n)/p} d\zeta,$$

for all $w \in \mathbb{B}$. Since $\{f_w\}_{w \in \mathbb{B}}$ is a bounded sequence in $H^p_a(\mathbb{B})$ and $f_w \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $|w| \to 1^-$, the compactness of $P^j_{\psi}$ and Proposition 3.4 show that $\|P^j_{\psi}f_w\|_{H^p_{\psi}}^q \to 0$ as $|w| \to 1^-$. This fact along with (3.16) implies the condition in (3.14), finishing the proof of the theorem.

Theorem 3.6. Let $1 < p \leq q < \infty$ and $\alpha, \beta > 0$. Suppose that $g \in H(\mathbb{B})$ with $g(0) = 0$ and $\varphi$ is a holomorphic self-map of $\mathbb{B}$ which induce the bounded operator $P^\alpha_{\psi} : H^p_a(\mathbb{B}) \to H^p_{\psi}(\mathbb{B})$. Then the $q$th power of the essential norm of $P^\alpha_{\psi}$ is comparable to

$$\limsup_{|w| \to 1^-} \sup_{0 < r < 1} (1 - r)^{\beta+q} \int_{\partial \mathbb{B}} |g(r\zeta)|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \varphi(r\zeta), w \rangle|^2} \right\}^{q(\alpha+n)/p} d\zeta.$$

Proof. To prove a lower estimate, we take an arbitrary compact operator $\mathcal{K} : H^p_a(\mathbb{B}) \to H^p_{\psi}(\mathbb{B})$. Since Lemma 2.7 implies that the family of functions $f_w$ defined by (2.11) converges to 0 weakly in $H^p_{\psi}(\mathbb{B})$ as $|w| \to 1^-$, we obtain

$$C\|P^\alpha_{\psi} - \mathcal{K}\|_{H^p_a(\mathbb{B}) \to H^p_{\psi}(\mathbb{B})} \geq \limsup_{|w| \to 1^-} \left( \|P^\alpha_{\psi}f_w\|_{H^p_{\psi}} - \|\mathcal{K}f_w\|_{H^p_{\psi}} \right) \geq \limsup_{|w| \to 1^-} \|P^\alpha_{\psi}f_w\|_{H^p_{\psi}}.$$

Combining this with (3.16), we have

$$C\|P^\alpha_{\psi}\|_{H^p_a(\mathbb{B}) \to H^p_{\psi}(\mathbb{B})} \geq \limsup_{|w| \to 1^-} \sup_{0 < r < 1} (1 - r)^{\beta+q} \int_{\partial \mathbb{B}} |g(r\zeta)|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \varphi(r\zeta), w \rangle|^2} \right\}^{q(\alpha+n)/p} d\zeta,$$

which is a lower estimate.

By some modification of Lemma 2.3 and the application of Lemmas 3.1 and 3.2, we get

$$\|P^\alpha_{\psi}\|_{H^p_a(\mathbb{B}) \to H^p_{\psi}(\mathbb{B})} \leq \liminf_{j \to \infty} \sup_{\|f\|_{H^p_{\psi}} \leq 1} \|\mathcal{K}_j f\|_{H^p_{\psi}} \leq C \liminf_{j \to \infty} \sup_{\|f\|_{H^p_{\psi}} \leq 1} \|gR_j f\|_{H^p_{\psi}}^q.$$

(3.20)
As in the proof of Theorem 2.8, we obtain

\[
\liminf_{j \to \infty} \sup_{\|f\|_{p,j} \leq 1} \left\| g C_{p,j} f \right\|_{H_\beta}^q \leq C \limsup_{|\omega| \to 1^-} \sup_{0 < r < 1} (1 - r)^{\beta + q} \times \int_{\partial \mathbb{B}} |g(r\zeta)|^q \left\{ 1 - \frac{|\omega|^2}{|1 - \langle \phi(r\zeta), \omega \rangle|^2} \right\} q(\alpha + n)/p \, d\sigma(\zeta),
\]

and so we have an upper estimate for \( \|P^q_{\psi}\|_{\ell, H^\infty_\beta} \to H^\infty_\beta \).

4. The Case \( P^q_{\psi} : H^\infty_\alpha(\mathbb{B}) \to H^\infty_\beta(\mathbb{B}) \)

When \( p = \infty \) and \( \alpha > 0 \), we define the weighted-type space \( H^\infty_\alpha(\mathbb{B}) \) as follows:

\[
H^\infty_\alpha(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) : \sup_{0 < r < 1} (1 - r)^{\alpha} M_\infty (f, r) < \infty \right\}.
\]

(4.1)

It is easy to see that \( f \in H^\infty_\alpha(\mathbb{B}) \) if and only if \( \sup_{z \in \mathbb{B}} (1 - |z|)^{\alpha} |f(z)| < \infty \), so we define the norm \( \|f\|_{H^\infty_\alpha} \) on \( H^\infty_\alpha(\mathbb{B}) \) by this supremum.

Furthermore we consider the subspace \( H^\infty_{\alpha,0}(\mathbb{B}) \) defined by

\[
H^\infty_{\alpha,0}(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) : \lim_{r \to 1^-} (1 - r)^{\alpha} M_\infty (f, r) = 0 \right\}.
\]

(4.2)

**Theorem 4.1.** Let \( \alpha, \beta > 0 \). Suppose that \( g \in H(\mathbb{B}) \) with \( g(0) = 0 \) and \( \phi \) is a holomorphic self-map of \( \mathbb{B} \). Then \( P^q_{\psi} : H^\infty_\alpha(\mathbb{B}) \to H^\infty_\beta(\mathbb{B}) \) is bounded if and only if

\[
\sup_{z \in \mathbb{B}} \left( \frac{(1 - |z|)^{\beta + 1} \left| g(z) \right|}{(1 - |\phi(z)|)^{\alpha}} \right) < \infty.
\]

(4.3)

In this case, the operator norm \( \|P^q_{\psi}\|_{H^\infty_\alpha(\mathbb{B})} \to H^\infty_\beta(\mathbb{B}) \) is comparable to the above supremum.

**Proof.** By the definition of the space \( H^\infty_\alpha(\mathbb{B}) \), \( f \in H^\infty_\alpha(\mathbb{B}) \) satisfies the growth condition

\[
|f(\omega)| \leq (1 - |\omega|)^{-\alpha} \|f\|_{H^\infty_\alpha} \quad (\omega \in \mathbb{B}),
\]

(4.4)

so it follows from Lemma 3.1 and Lemma 3.2 that

\[
\|P^q_{\psi} f\|_{H^\infty_\beta} \leq \sup_{z \in \mathbb{B}} (1 - |z|)^{\beta + 1} \left| g C_{\psi} f(z) \right| \leq \|f\|_{H^\infty_\alpha} \sup_{z \in \mathbb{B}} \left( \frac{(1 - |z|)^{\beta + 1} \left| g(z) \right|}{(1 - |\phi(z)|)^{\alpha}} \right),
\]

(4.5)

for every \( f \in H^\infty_\alpha(\mathbb{B}) \).
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Hence we obtain

\[ \| P^S_\varphi \|_{H^\varphi(B)} (or H_{\alpha,0}(B)) \rightarrow H^\varphi(B) \leq C \sup_{z \in B} \frac{(1 - |z|)^{\beta + 1}|g(z)|}{(1 - |\varphi(z)|)^{\alpha}}. \] (4.6)

Now we prove the reverse inequality. For \( w \in B \), we put

\[ f_w(z) = \frac{1}{(1 - \langle z, w \rangle)^{\alpha}}. \] (4.7)

Note that \( f_w \in H_{\alpha,0}^\infty(B) \) for each \( w \in B \) and moreover \( \sup_{w \in B} \| f_w \|_{H^\varphi(B)} \leq 1 \).

When \( \varphi(z) \neq 0 \), we have

\[ \| P^S_\varphi \|_{H^\varphi_{\alpha,0}(B) \rightarrow H^\varphi_{\beta}(B)} \geq \| P^S_\varphi f_{t(\varphi(z)/|\varphi(z)|)} \|_{H^\varphi_{\beta}} \times \sup_{w \in B} (1 - |tw|)^{\beta + 1} |gC_{\varphi}f_{t(\varphi(z)/|\varphi(z)|)}(w)| \]

\[ \geq (1 - |z|)^{\beta + 1}|g(z)| \left| f_{t(\varphi(z)/|\varphi(z)|)}(\varphi(z)) \right| \]

\[ = \frac{(1 - |z|)^{\beta + 1}|g(z)|}{(1 - t|\varphi(z)|)^{\alpha}}, \] (4.8)

for all \( t \in (0, 1) \). Letting \( t \rightarrow 1^- \) in (4.8), we have

\[ \| P^S_\varphi \|_{H^\varphi_{\alpha,0}(B) \rightarrow H^\varphi_{\beta}(B)} \geq C \frac{(1 - |z|)^{\beta + 1}|g(z)|}{(1 - |\varphi(z)|)^{\alpha}}. \] (4.9)

For the constant function \( 1 \in H^\varphi_{\alpha,0}(B) \) we obtain

\[ \| P^S_\varphi 1 \|_{H^\varphi_{\beta}} = \sup_{w \in B} (1 - |tw|)^{\beta + 1} |gC_{\varphi}1(w)| \geq (1 - |z|)^{\beta + 1} |g(z)|. \] (4.10)

Inequality (4.10) shows that the estimate in (4.9) also holds when \( \varphi(z) = 0 \).

Hence, from (4.9) we obtain

\[ \| P^S_\varphi \|_{H^\varphi_{\alpha,0}(B) \rightarrow H^\varphi_{\beta}(B)} \geq C \sup_{z \in B} \frac{(1 - |z|)^{\beta + 1}|g(z)|}{(1 - |\varphi(z)|)^{\alpha}}, \] (4.11)

which along with the obvious inequality

\[ \| P^S_\varphi \|_{H^\varphi_{\alpha,0}(B) \rightarrow H^\varphi_{\beta}(B)} \geq \| P^S_\varphi \|_{H^\varphi_{\alpha,0}(B) \rightarrow H^\varphi_{\beta}(B)} \] (4.12)

completes the proof of the theorem.

For the compactness of \( P^S_\varphi : H^\varphi_{\alpha}(B) \) (or \( H^\varphi_{\alpha,0}(B) \)) \( \rightarrow H^\varphi_{\beta}(B) \), we can also prove the following proposition which is similar to Proposition 2.6.
Proposition 4.2. Let \( \alpha, \beta > 0 \). Suppose that \( g \in H(\mathbb{B}) \) with \( g(0) = 0 \) and \( \varphi \) is a holomorphic self-map of \( \mathbb{B} \) which induce the bounded operator \( P^S_\varphi : H^\infty_\alpha(\mathbb{B}) \) (or \( H^\infty_{\alpha,0}(\mathbb{B}) \)) \( \rightarrow H^\infty_\beta(\mathbb{B}) \). Then \( P^S_\varphi : H^\infty_\alpha(\mathbb{B}) \) (or \( H^\infty_{\alpha,0}(\mathbb{B}) \)) \( \rightarrow H^\infty_\beta(\mathbb{B}) \) is compact if and only if every bounded sequence \( \{f_j\}_{j \in \mathbb{N}} \) in \( H^\infty_\alpha(\mathbb{B}) \) (or \( H^\infty_{\alpha,0}(\mathbb{B}) \)) which converges to 0 uniformly on compact subsets of \( \mathbb{B} \), \( \{P^S_\varphi f_j\}_{j \in \mathbb{N}} \) converges to 0 in \( H^\infty_\beta(\mathbb{B}) \).

Theorem 4.3. Let \( \alpha, \beta > 0 \). Suppose that \( g \in H(\mathbb{B}) \) with \( g(0) = 0 \) and \( \varphi \) is a holomorphic self-map of \( \mathbb{B} \) such that \( P^S_\varphi : H^\infty_\alpha(\mathbb{B}) \) (or \( H^\infty_{\alpha,0}(\mathbb{B}) \)) \( \rightarrow H^\infty_\beta(\mathbb{B}) \) is bounded. Then the essential norm \( \|P^S_\varphi\|_{e,H^\infty_\alpha(\mathbb{B})} \) is comparable to

\[
\limsup_{|\varphi(z)| \to 1^-} \frac{(1 - |z|)^{\beta+1} |g(z)|}{(1 - |\varphi(z)|)^{\alpha}}.
\]

In particular, \( P^S_\varphi : H^\infty_\alpha(\mathbb{B}) \) (or \( H^\infty_{\alpha,0}(\mathbb{B}) \)) \( \rightarrow H^\infty_\beta(\mathbb{B}) \) is compact if and only if

\[
\lim_{|\varphi(z)| \to 1^-} \frac{(1 - |z|)^{\beta+1} |g(z)|}{(1 - |\varphi(z)|)^{\alpha}} = 0.
\]

Proof. First we consider the family \( \{f_w\}_{w \in \mathbb{B}} \) where

\[
f_w(z) = \frac{1 - |w|}{(1 - \langle z, w \rangle)^{\alpha+1}}.
\]

We can easily check that \( f_w \in H^\infty_{\alpha,0}(\mathbb{B}) \), \( \|f_w\|_{H^\infty} \leq 1 \) for all \( w \in \mathbb{B} \) and \( f_w \to 0 \) uniformly on compact subsets of \( \mathbb{B} \) as \( |w| \to 1^- \). Hence \cite[page 296, Theorem 2]{40} implies that \( f_w \to 0 \) weakly in \( H^\infty_{\alpha,0}(\mathbb{B}) \) as \( |w| \to 1^- \).

If \( \|\varphi\|_\infty < 1 \), then as in the proof of \cite[Theorem 3]{26} it can be seen that the operator \( P^S_\varphi : H^\infty_\alpha(\mathbb{B}) \) (or \( H^\infty_{\alpha,0}(\mathbb{B}) \)) \( \rightarrow H^\infty_\beta(\mathbb{B}) \) is compact, so that

\[
\|P^S_\varphi\|_{e,H^\infty_\alpha(\mathbb{B})} = 0.
\]

On the other hand, the limit in (4.13) is vacuously equal to zero, from which the result follows in this case. If \( \|\varphi\|_\infty = 1 \), then take a sequence \( \{\varphi(z_j)\}_{j \in \mathbb{N}} \) in \( \mathbb{B} \) with \( |\varphi(z_j)| \to 1 \) as \( j \to \infty \) and put \( F_j(z) = f_{\varphi(z_j)}(z) \) for each \( j \in \mathbb{N} \). Then \( \{F_j\}_{j \in \mathbb{N}} \) is a bounded sequence in \( H^\infty_{\alpha,0}(\mathbb{B}) \).
and \( \{F_j\}_{j \in \mathbb{N}} \) converges to 0 weakly in \( H^\infty_{\alpha,0}(\mathcal{B}) \), as \( j \to \infty \). Hence for every compact operator \( \mathcal{K} : H^\infty_{\alpha,0}(\mathcal{B}) \to H^\infty_{\beta}(\mathcal{B}) \) we have \( \|\mathcal{K}F_j\|_{H^\infty_{\beta}} \to 0 \) as \( j \to \infty \). So we have

\[
\|P^k_{\psi} - \mathcal{K}\|_{H^\infty_{\alpha,0}(\mathcal{B}) \to H^\infty_{\beta}(\mathcal{B})} \geq \limsup_{j \to \infty} \|P^k_{\psi}F_j - \mathcal{K}F_j\|_{H^\infty_{\beta}} \\
\geq \limsup_{j \to \infty} \|P^k_{\psi}F_j\|_{H^\infty_{\beta}} \\
\geq \limsup_{j \to \infty} \sup_{w \in \mathcal{B}} (1 - |w|)^{\beta+1} \|g(w)\| \left| F_j(\psi(w)) \right| \\
\geq \limsup_{j \to \infty} (1 - |z_j|)^{\beta+1} \|g(z_j)\| \left| F_j(\psi(z_j)) \right| \\
\geq \frac{1}{2^{\alpha+1}} \limsup_{j \to \infty} \frac{(1 - |z_j|)^{\beta+1} \|g(z_j)\|}{(1 - \|\psi(z_j)\|)^{\beta}} \\
\tag{4.17}
\]

for all compact operators \( \mathcal{K} : H^\infty_{\alpha,0}(\mathcal{B}) \to H^\infty_{\beta}(\mathcal{B}) \). Taking the infimum over the set of all compact operators \( \mathcal{K} : H^\infty_{\alpha,0}(\mathcal{B}) \to H^\infty_{\beta}(\mathcal{B}) \), we obtain

\[
\|P^k_{\psi}\|_{C,H^\infty_{\alpha,0}(\mathcal{B}) \to H^\infty_{\beta}(\mathcal{B})} \geq C \limsup_{j \to \infty} \frac{(1 - |z_j|)^{\beta+1} \|g(z_j)\|}{(1 - \|\psi(z_j)\|)^{\beta}}. \\
\tag{4.18}
\]

Combining this with the estimate \( \|P^k_{\psi}\|_{C,H^\infty_{\alpha,0}(\mathcal{B}) \to H^\infty_{\beta}(\mathcal{B})} \geq \|P^k_{\psi}\|_{C,H^\infty_{\alpha,0}(\mathcal{B}) \to H^\infty_{\beta}(\mathcal{B})} \) we have

\[
\|P^k_{\psi}\|_{C,H^\infty_{\alpha,0}(\mathcal{B}) \to H^\infty_{\beta}(\mathcal{B})} \geq C \limsup_{|\psi(z)| \to 1^{-}} \frac{(1 - |z|)^{\beta+1} \|g(z)\|}{(1 - \|\psi(z)\|)^{\beta}}. \\
\tag{4.19}
\]

Next we prove an upper estimate. Assume that \( \{r_l\}_{l \in \mathbb{N}} \subset (0, 1) \) is a sequence which increasingly converges to 1. For this \( \{r_l\}_{l \in \mathbb{N}} \), we define the operators defined by

\[
P^k_{\psi, f}(z) = \int_{0}^{1} g(tz) f(r_t \psi(tz)) \frac{dt}{t}. \\
\tag{4.20}
\]

As in the proof of [26, Theorem 3], Proposition 4.2 shows that \( P^k_{\psi, f} : H^\infty_{\alpha}(\mathcal{B}) \to H^\infty_{\beta}(\mathcal{B}) \) is compact for each \( l \in \mathbb{N} \).

Put

\[
M_2 := \limsup_{|\psi(z)| \to 1^{-}} \frac{(1 - |z|)^{\beta+1} \|g(z)\|}{(1 - \|\psi(z)\|)^{\beta}}. \\
\tag{4.21}
\]
and fix $\varepsilon > 0$. Then we can choose $R \in (0, 1)$ such that
\[
(1 - |z|)^{\beta + 1} |g(z)| \left( 1 - |\varphi(z)| \right)^{\alpha} < M_2 + \varepsilon, \tag{4.22}
\]

if $R < |\varphi(z)| < 1$. Take $f \in H_\alpha^\varphi(\mathbb{B})$ with $\|f\|_{H_\alpha^\varphi} \leq 1$ and an integer $l \in \mathbb{N}$. By Lemma 3.1 and Lemma 3.2, we have
\[
\|P^f_\varphi f - P^f_{r|\varphi|} f\|_{H_\beta^w} = \sup_{z \in \mathbb{B}} (1 - |z|)^{\beta + 1} |\Re \left[ P^f_\varphi f \right] (z) - \Re \left[ P^f_{r|\varphi|} f \right] (z) | \]
\[
= \sup_{z \in \mathbb{B}} (1 - |z|)^{\beta + 1} |g(z)| |f(\varphi(z)) - f(r|\varphi|) | \]
\[
= \sup_{|\varphi(z)| \leq R} (1 - |z|)^{\beta + 1} |g(z)| |f(\varphi(z)) - f(r|\varphi|) | \]
\[
+ \sup_{R < |\varphi(z)| < 1} (1 - |z|)^{\beta + 1} |g(z)| |f(\varphi(z)) - f(r|\varphi|) |. \tag{4.23}
\]

By using the mean value theorem and the asymptotic relation
\[
\sup_{z \in \mathbb{B}} (1 - |z|)^{\beta + 1} |\nabla f(z) | = \sup_{z \in \mathbb{B}} (1 - |z|)^{\beta + 1} |\Re f(z) |, \tag{4.24}
\]
we obtain
\[
\sup_{|\varphi(z)| \leq R} |f(\varphi(z)) - f(r|\varphi|) | \leq \sup_{|\varphi(z)| \leq R} (1 - r) |\varphi(z) | \sup_{|w| \leq R} |\nabla f(w) |
\]
\[
\leq \frac{(1 - r) R}{(1 - R)^{\alpha + 1}} \sup_{w \in \mathbb{B}} (1 - |w|)^{\alpha + 1} |\nabla f(w) |
\]
\[
\leq \frac{(1 - r) R}{(1 - R)^{\alpha + 1}} \sup_{w \in \mathbb{B}} (1 - |w|)^{\alpha + 1} |\Re f(w) |
\]
\[
\leq \frac{(1 - r) R}{(1 - R)^{\alpha + 1}} \|f\|_{H_\alpha^w}. \tag{4.25}
\]

Since the boundedness of $P^f_\varphi : H_\alpha^\varphi(\mathbb{B}) \to H_\beta^\varphi(\mathbb{B})$ implies that $P^f_\varphi 1 \in H_\beta^\varphi(\mathbb{B})$, we see
\[
\sup_{z \in \mathbb{B}} (1 - |z|)^{\beta + 1} |g(z) | < \infty, \tag{4.26}
\]
and so we have
\[
\sup_{\|f\|_{H_\alpha^\varphi} \leq 1} \sup_{|\varphi(z)| \leq R} (1 - |z|)^{\beta + 1} |g(z)| |f(\varphi(z)) - f(r|\varphi|) | \leq C \frac{(1 - r) R}{(1 - R)^{\alpha + 1}} \sup_{z \in \mathbb{B}} (1 - |z|)^{\beta + 1} |g(z) |. \tag{4.27}
\]
On the other hand, the monotonicity of $M_\infty(f, r)$ shows

$$|f(r_\varphi(z))| \leq (1 - |\varphi(z)|)^{-\alpha} \|f_n\|_{H^\varphi} \leq (1 - |\varphi(z)|)^{-\alpha} \|f\|_{H^\varphi}. \quad (4.28)$$

Thus we have

$$\sup_{\|f\|_{H^\varphi} \leq 1} \sup_{R < |\varphi(z)| < 1} (1 - |z|)^{\beta+1} |g(z)||f(\varphi(z)) - f(r_\varphi(z))| \leq 2 \sup_{R < |\varphi(z)| < 1} \frac{(1 - |z|)^{\beta+1}|g(z)|}{(1 - |\varphi(z)|)^{\alpha}}$$

$$\leq 2(M_2 + \varepsilon). \quad (4.29)$$

From (4.23), (4.27), (4.29), and the compactness of $P^\varphi_{r_\varphi}$, we obtain

$$\|P^\varphi_{r_\varphi}\|_{e, H^\varphi_{r_\varphi}(B) \to H^\varphi_{r_\varphi}(B)} \leq \|P^\varphi_{r_\varphi} - P^\varphi_{r_\varphi}\|_{H^\varphi_{r_\varphi}(B) \to H^\varphi_{r_\varphi}(B)}$$

$$\leq C \frac{(1 - r_\varphi)R}{(1 - R)^{\alpha+1}} \sup_{z \in B} (1 - |z|)^{\beta+1}|g(z)| + 2(M_2 + \varepsilon). \quad (4.30)$$

Letting $l \to \infty$ and $\varepsilon \to 0$, we have

$$\|P^\varphi_{r_\varphi}\|_{e, H^\varphi_{r_\varphi}(B) \to H^\varphi_{r_\varphi}(B)} \leq 2 \lim_{|\varphi(z)| \to 1^-} \sup_{|\varphi(z)| \to 1^-} \frac{(1 - |z|)^{\beta+1}|g(z)|}{(1 - |\varphi(z)|)^{\alpha}}. \quad (4.31)$$

This completes the proof. \qed

When $P^\varphi_{r_\varphi} : H^\varphi_{\alpha, \beta}(B)$ (or $H^\varphi_{\alpha, 0}(B)$) $\to H^\varphi_{\beta, 0}(B)$ is bounded, we see that $g \in H^\varphi_{\beta+1, 0}(B)$. By a standard argument as in the proof of [26, Corollary 3], we have

$$\lim_{|z| \to 1^-} \frac{(1 - |z|)^{\beta+1}|g(z)|}{(1 - |\varphi(z)|)^{\alpha}} = \lim_{|\varphi(z)| \to 1^-} \frac{(1 - |z|)^{\beta+1}|g(z)|}{(1 - |\varphi(z)|)^{\alpha}}, \quad (4.32)$$

and so

$$\|P^\varphi_{r_\varphi}\|_{e, H^\varphi_{\alpha, \beta}(B) \to H^\varphi_{\beta, 0}(B)} \leq \lim_{|z| \to 1^-} \sup_{|z| \to 1^-} \frac{(1 - |z|)^{\beta+1}|g(z)|}{(1 - |\varphi(z)|)^{\alpha}}. \quad (4.33)$$

Hence we obtain the following characterization for the compactness of the operator $P^\varphi_{r_\varphi} : H^\varphi_{\alpha, \beta}(B)$ (or $H^\varphi_{\alpha, 0}(B)$) $\to H^\varphi_{\beta, 0}(B)$. 
Corollary 4.4. Let $\alpha, \beta > 0$. Suppose that $g \in H(\mathbb{B})$ with $g(0) = 0$ and $\varphi$ is a holomorphic self-map of $\mathbb{B}$ such that $P^g_\varphi : H^\infty_{\alpha,0}(\mathbb{B}) \rightarrow H^\infty_{\beta,0}(\mathbb{B})$ is bounded. Then $P^g_\varphi : H^\infty_{\alpha,0}(\mathbb{B}) \rightarrow H^\infty_{\beta,0}(\mathbb{B})$ is compact if and only if

$$
\lim_{|z| \to 1^-} \frac{(1 - |z|)^{\beta+1} |\varphi(z)|}{(1 - |g(z)|)^{\alpha}} = 0. \quad (4.34)
$$

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References


