Research Article

Boundedness, Attractivity, and Stability of a Rational Difference Equation with Two Periodic Coefficients

G. Papaschinopoulos, G. Stefanidou, and C. J. Schinas

School of Engineering, Democritus University of Thrace, 67100 Xanthi, Greece

Correspondence should be addressed to G. Papaschinopoulos, gpapas@env.duth.gr

Received 24 August 2008; Accepted 11 January 2009

Recommended by Yong Zhou

We study the boundedness, the attractivity, and the stability of the positive solutions of the rational difference equation

\[ x_{n+1} = \frac{p_n x_{n-2} + x_{n-3}}{q_n + x_{n-3}}, \quad n = 0, 1, \ldots, \]

where \( p_n, q_n, n = 0, 1, \ldots \) are positive sequences of period 2.

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1. Introduction

In [1], Camouzis et al. studied the global character of the positive solutions of the difference equation:

\[ x_{n+1} = \frac{\delta x_{n-2} + x_{n-3}}{A + x_{n-3}}, \quad n = 0, 1, \ldots, \]

(1.1)

where \( \delta, A \) are positive parameters and the initial values \( x_{-3}, x_{-2}, x_{-1}, x_0 \) are positive real numbers.

The mathematical modeling of a physical, physiological, or economical problem very often leads to difference equations (for partial review of the theory of difference equations and their applications see [2–12]). Moreover, a lot of difference equations with periodic coefficients have been applied in mathematical models in biology (see [13–15]). In addition, between others in [16–19], we can see some more difference equations with periodic coefficients that have been studied.
In this paper, we investigate the difference equation

\[
x_{n+1} = \frac{p_n x_{n-2} + x_{n-3}}{q_n + x_{n-3}}, \quad n = 0, 1, \ldots,
\]

where \(p_n, q_n\), \(n = 0, 1, \ldots\) are positive sequences of period 2 and the initial values \(x_i, i = -3, -2, -1, 0\) are positive numbers.

Our goal in this paper is to extend some results obtained in [1]. More precisely, we study the existence of a unique positive periodic solution of (1.2) of prime period 2. In the sequel, we investigate the boundedness, the persistence, and the convergence of the positive solutions to the unique periodic solution of (1.2). Finally, we study the stability of the positive periodic solution and the zero solution of (1.2).

If we set \(y_n = x_{2n-1}, z_n = x_{2n}\), it is easy to prove that (1.2) is equivalent to the following system of difference equations:

\[
y_{n+1} = \frac{p_0 z_{n-1} + y_{n-1}}{q_0 + y_{n-1}}, \quad z_{n+1} = \frac{p_1 y_n + z_{n-1}}{q_1 + z_{n-1}}, \quad n = 0, 1, \ldots,
\]

where \(p_i, q_i\), \(i = 0, 1\) are positive constants and the initial values \(y_i, z_i, i = -1, 0\) are positive numbers. So in order to study (1.2) we investigate system (1.3).

2. Existence of the Unique Positive Equilibrium of System (1.3)

In the following proposition, we study the existence of the unique positive equilibrium of system (1.3).

**Proposition 2.1.** Consider system (1.3) where \(p_i, q_i\), \(i = 0, 1\) are positive constants and the initial values \(y_i, z_i, i = -1, 0\) are positive numbers. Suppose that

\[
q_0 - 1 < p_0, \quad q_1 - 1 < p_1
\]

are satisfied. Then system (1.3) possesses a unique positive equilibrium.

**Proof.** Let \((y, z)\) be a positive equilibrium of system (1.3) then

\[
y = \frac{p_0 z + y}{q_0 + y}, \quad z = \frac{p_1 y + z}{q_1 + z}.
\]

Equations (2.2) imply that \(z\) is a solution of the equation

\[
f(x) = x^3 + 2(q_1 - 1)x^2 + \left[(q_1 - 1)^2 + p_1(q_0 - 1)\right]x + (q_1 - 1)(q_0 - 1)p_1 - p_0 p_1^2 = 0.
\]

Suppose that

\[
q_1 \geq 1.
\]
Let $\lambda_1$, $\lambda_2$, and $\lambda_3$ be the solutions of (2.3). Then from (2.1), (2.3), and (2.4) we take

$$\lambda_1 + \lambda_2 + \lambda_3 = 2(1 - q_1) \leq 0,$$

$$\lambda_1 \lambda_2 \lambda_3 = -(q_1 - 1)(q_0 - 1)p_1 + p_0 p_1^2 > 0,$$

and so (2.3) has unique positive solution $z$. Then from (2.2) and (2.4) we have

$$z > 1 - q_1, \quad y = \frac{z^2 + (q_1 - 1)z}{p_1} > 0,$$

and so system (1.3) has a unique positive equilibrium.

Now suppose that

$$q_1 < 1, \quad (q_1 - 1)(q_0 - 1) > p_1 p_0.$$  \hspace{1cm} (2.7)

If $\lambda_1$, $\lambda_2$, and $\lambda_3$ are the solutions of (2.3), then from (2.3) and (2.7) we take

$$\lambda_1 + \lambda_2 + \lambda_3 = 2(1 - q_1) > 0,$$

$$\lambda_1 \lambda_2 \lambda_3 = -(q_1 - 1)(q_0 - 1)p_1 + p_0 p_1^2 < 0,$$

and so (2.3) has a negative solution, but also (2.3) has a solution in the interval $(0, 1 - q_1)$, since

$$f(0) = (q_1 - 1)(q_0 - 1)p_1 - p_0 p_1^2 > 0,$$

$$f(1 - q_1) = -p_0 p_1^2 < 0.$$  \hspace{1cm} (2.9)

Moreover, (2.3) has a solution $z$ in the interval $(1 - q_1, \infty)$, since

$$\lim_{x \to \infty} f(x) = \infty,$$  \hspace{1cm} (2.10)

therefore, we get (2.6) and so system (1.3) has a unique positive equilibrium.

Finally, suppose that

$$q_1 < 1, \quad (q_1 - 1)(q_0 - 1) < p_1 p_0.$$  \hspace{1cm} (2.11)

If $\lambda_1$, $\lambda_2$, and $\lambda_3$ are the solutions of (2.3), then from (2.3) and (2.11), we take

$$\lambda_1 + \lambda_2 + \lambda_3 = 2(1 - q_1) > 0,$$

$$\lambda_1 \lambda_2 \lambda_3 = -(q_1 - 1)(q_0 - 1)p_1 + p_0 p_1^2 > 0.$$  \hspace{1cm} (2.12)
We have \( \lim_{x \to \infty} f(x) = \infty \), and since \( f(1 - q_1) < 0 \), it is obvious that (2.3) has a solution \( z \) in the interval \((1 - q_1, \infty)\). From (2.3), we get

\[
f'(x) = 3x^2 + 4x(q_1 - 1) + (q_1 - 1)^2 + p_1(q_0 - 1). \tag{2.13}
\]

If equation \( f'(x) = 0 \) has complex roots, then it is obvious that \( z \) is the unique solution of (2.3). Therefore, we get (2.6), and so system (1.3) has a unique positive equilibrium.

Now, suppose that the roots of \( f'(x) = 0 \)

\[
\mu_1 = \frac{2(1 - q_1) - \sqrt{D}}{3}, \quad \mu_2 = \frac{2(1 - q_1) + \sqrt{D}}{3}, \quad D = (1 - q_1)^2 + 3p_1(1 - q_0), \tag{2.14}
\]

are real numbers.

Suppose that \( q_0 < 1 \), then it is obvious that

\[
\mu_1 < 1 - q_1 < \mu_2, \tag{2.15}
\]

and so we have that (2.3) has a unique solution \( z \in (1 - q_1, \infty) \).

If \( q_0 \geq 1 \), then it holds that

\[
0 < \mu_1 \leq \mu_2 \leq 1 - q_1, \tag{2.16}
\]

which implies that (2.3) has a unique solution \( z \in (1 - q_1, \infty) \).

Therefore, we can take (2.6) and so system (1.3) has a unique positive equilibrium. This completes the proof of the proposition.

\section{3. Boundedness and Persistence of the Solutions of System (1.3)}

In the following propositions we study the boundedness and the persistence of the positive solutions of system (1.3). In the sequel we will use the following result which has proved in [20].

\begin{theorem}
Assume that all roots of the polynomial

\[
P(t) = t^N - s_1t^{N-1} - \cdots - s_N, \tag{3.1}
\]

where \( s_1, s_2, \ldots, s_N \geq 0 \) have absolute value less than 1, and let \( y_n \) be a nonnegative solution of the inequality

\[
y_{n+N} \leq s_1y_{n+N-1} + \cdots + s_Ny_n + z_n. \tag{3.2}
\]

Then, the following statements are true.

(i) If \( z_n \) is a nonnegative bounded sequence, then \( y_n \) is also bounded.

(ii) If \( \lim_{n \to \infty} z_n = 0 \), then \( \lim_{n \to \infty} y_n = 0 \).
\end{theorem}
Proposition 3.2. One considers the system of difference equations (1.3) where $p_i, q_i$, $i = 0, 1$ are positive constants and the initial values $y_i, z_i$, $i = -1, 0$ are positive numbers. Then the following statements are true.

(i) If

$$
\frac{q_0 q_1}{p_0 p_1} \geq 1,
$$

then every solution of (1.3) is bounded.

(ii) If

$$
q_0 - 1 < p_0 \leq q_0, \quad q_1 - 1 < p_1 \leq q_1,
$$

then every solution of (1.3) is bounded and persists.

Proof. Let $(y_n, z_n)$ be an arbitrary solution of (1.3).

(i) From (3.3), we get that one of the three following conditions holds:

$$
\frac{q_0}{p_0} > 1, \quad (3.5)
$$

$$
\frac{q_1}{p_1} > 1, \quad (3.6)
$$

$$
p_0 = q_0 = p, \quad p_1 = q_1 = q. \quad (3.7)
$$

We assume that (3.5) is satisfied. We prove that there exists a positive integer $N$ such that

$$
y_n < 1, \quad z_n < \frac{q_0}{p_0}, \quad n \geq N. \quad (3.8)
$$

First, we show that if there exists a positive integer $n_0$ such that

$$
z_{n_0} < \frac{q_0}{p_0}, \quad (3.9)
$$

then

$$
z_{n_0 + 3p} < \frac{q_0}{p_0}, \quad p = 0, 1, \ldots. \quad (3.10)
$$

In contradiction, we assume that

$$
z_{n_0 + 3} = \frac{p_1 y_{n_0 + 2} + z_{n_0 + 1}}{q_1 + z_{n_0 + 1}} \geq \frac{q_0}{p_0}. \quad (3.11)
$$
Using relations (1.3), (3.5), and (3.11), we get that

\[ y_{n+2} = \frac{p_0 z_n + y_n}{q_0 + y_n} > \frac{q_0 q_1}{p_0 p_1}, \quad (3.12) \]

and so relations (1.3) and (3.3) imply that

\[ z_n > \frac{q_0^2 q_1}{p_0^2 p_1} > \frac{q_0}{p_0}, \quad (3.13) \]

which contradicts (3.9). So \( z_{n+3} < q_0/p_0 \) and working inductively, we get (3.10).

If \( z_{-1} < q_0/p_0 \), then from the analogous relations (3.9) and (3.10), we get

\[ z_{-1+3p} < \frac{q_0}{p_0}, \quad p = 0, 1, \ldots \quad (3.14) \]

Now, suppose that

\[ z_{-1} \geq \frac{q_0}{p_0}, \quad (3.15) \]

we prove that there exists a positive integer \( q \) such that

\[ z_{-1+3q} < \frac{q_0}{p_0}. \quad (3.16) \]

From (3.3), there exists a positive integer \( h \) such that

\[ z_{-1} < \left( \frac{q_0 q_1}{p_0 p_1} \right)^h. \quad (3.17) \]

If \( z_2 < q_0/p_0 \), then (3.16) is true for \( q = 1 \).

Now, suppose that

\[ z_2 \geq \frac{q_0}{p_0}. \quad (3.18) \]

Then from (1.3), (3.5), and (3.18), we get \( y_1 > q_0 q_1/p_0 p_1 \) and so from (1.3), (3.3), and (3.5), we have that

\[ z_{-1} > \frac{q_1 q_0}{p_1 p_0^2} > \frac{q_1 q_0}{p_1 p_0}, \quad (3.19) \]

If \( z_5 < q_0/p_0 \), then (3.16) is true for \( q = 2 \).
Now, suppose that
\[ z_5 \geq \frac{q_0}{p_0}. \]  \hfill (3.20)

Using (1.3), (3.3), (3.5), (3.20) and arguing as to prove (3.19) we get
\[ z_{-1} > \left( \frac{q_1 q_0}{p_1 p_0} \right)^2. \]  \hfill (3.21)

Working inductively, we get that
\[ z_{-1+3w} \geq \frac{q_0}{p_0}, \quad w = 1, 2, \ldots, \quad \text{then} \quad z_{-1} > \left( \frac{q_1 q_0}{p_1 p_0} \right)^w. \]  \hfill (3.22)

From (3.22) for \( w = h \), we get \( z_{-1} > (q_1 q_0 / p_1 p_0)^h \) which contradicts (3.17). So \( z_{-1+3h} < q_0/p_0 \) which means that (3.16) holds for \( q = h \).

Arguing as for \( z_{-1} \), we can prove that there exist positive integers \( k, l \) such that
\[ z_{0+3k} < \frac{q_0}{p_0}, \quad z_{1+3l} < \frac{q_0}{p_0}. \]  \hfill (3.23)

From (3.16) and (3.23), we get that there exists a positive integer \( r \) such that
\[ z_r < \frac{q_0}{p_0}, \quad n \geq r. \]  \hfill (3.24)

Finally, from (1.3) and (3.24), we get \( y_{r+2} < 1 \) and so (3.8) is true for \( N = r + 2 \).

Similarly, we can prove that if (3.6) holds, then there exists a positive integer \( N \) such that
\[ z_n < 1, \quad y_n < \frac{q_1}{p_1}, \quad n \geq N. \]  \hfill (3.25)

Finally, suppose that (3.7) hold. From (1.3) and (3.7), we have
\[ y_{n+1} - 1 = \frac{p(z_{n-1} - 1)}{p + y_{n-1}}, \quad z_{n+1} - 1 = \frac{q(y_{n} - 1)}{q + z_{n-1}}, \]  \hfill (3.26)

and so,
\[ y_{n+1} - 1 = \frac{p}{p + y_{n-1}} \frac{q}{q + z_{n-3}} (y_{n-2} - 1). \]  \hfill (3.27)
From (3.27), we get
\[ 0 \leq y_{n+1} - 1 \leq y_{n-2} - 1, \quad \text{or} \quad 0 \geq y_{n+1} - 1 \geq y_{n-2} - 1, \quad (3.28) \]
and so the subsequences \( y_{3n}, y_{3n+1}, y_{3n+2} \) either are bounded from below by 1 and decreasing or bounded from above by 1 and increasing. Hence, \( y_n \) is bounded and persists. Similarly, we can prove that \( z_n \) is bounded and persists. This completes the proof of part (i) of the proposition.

(ii) In statement (i), we have already proved that if (3.7) hold, then every solution of (1.3) is bounded and persists. So, from (3.4), it remains to show that if either
\[ q_0 - 1 < p_0 < q_0, \quad q_1 - 1 < p_1 \leq q_1, \quad (3.29) \]
or
\[ q_0 - 1 < p_0 \leq q_0, \quad q_1 - 1 < p_1 < q_1, \quad (3.30) \]
holds, then the solution \( (y_n, z_n) \) persists. From (3.3), (3.8), (3.25), (3.29), and (3.30), we get that
\[ y_n < \frac{q_1}{p_1}, \quad z_n < \frac{q_0}{p_0}, \quad n \geq N. \quad (3.31) \]

We consider the positive number \( m \) such that
\[ m < \min \{ y_N, z_N, y_{N+1}, z_{N+1}, p_0 + 1 - q_0, p_1 + 1 - q_1 \}. \quad (3.32) \]
Moreover, if
\[ f(y, z) = \frac{p_0 z + y}{q_0 + y}, \quad g(y, z) = \frac{p_1 y + z}{q_1 + z}, \quad (3.33) \]
than it is easy to see that for the functions (3.33), \( f \) is increasing with respect to \( y \) for any \( z, z < q_0/p_0 \) and \( g \) is increasing with respect to \( z \) for any \( y, y < q_1/p_1 \). Therefore, from (1.3), (3.31), and (3.32) we have
\[ y_{N+2} > \frac{(p_0 + 1)m}{q_0 + m} > m, \quad z_{N+2} > \frac{(p_1 + 1)m}{q_1 + m} > m, \quad (3.34) \]
and working inductively, we take
\[ y_{N+s} \geq m, \quad z_{N+s} \geq m, \quad s = 0, 1, \ldots. \quad (3.35) \]

Therefore, \( (y_n, z_n) \) persists and using statement (i), then \( (y_n, z_n) \) is bounded and persists. This completes the proof of the proposition. \( \square \)
Proposition 3.3. One considers the system of difference equations (1.3) where $p_i, q_i, i = 0, 1$ are positive constants, and the initial values $y_i, z_i, i = -1, 0$ are positive numbers. Then, the following statements are true.

(i) If

$$\frac{q_0 q_1}{p_0 p_1} < 1, \quad (3.36)$$

then every solution of (1.3) persists.

(ii) If

$$q_0 \leq p_0 \leq q_0 + 1, \quad q_1 \leq p_1 \leq q_1 + 1, \quad (3.37)$$

then every solution of (1.3) is bounded and persists.

Proof. Let $(y_n, z_n)$ be an arbitrary solution of (1.3).

(i) From (3.36), we have

$$\frac{q_0}{p_0} < 1, \quad (3.38)$$

or

$$\frac{q_1}{p_1} < 1. \quad (3.39)$$

Arguing as in the proof of statement (i) of Proposition 3.2, we can easily prove that if (3.38) holds, then there exists a positive integer $M$ such that

$$y_n > 1, \quad z_n > \frac{q_0}{p_0}, \quad n \geq M, \quad (3.40)$$

and if (3.39) holds, then there exists a positive integer $M$ such that

$$z_n > 1, \quad y_n > \frac{q_1}{p_1}, \quad n \geq M. \quad (3.41)$$

(ii) From Proposition 3.2, we have that if (3.7) holds, then every solution of (1.3) is bounded and persists. So, from (3.37), it remains to show that if either

$$q_0 < p_0 \leq q_0 + 1, \quad q_1 \leq p_1 \leq q_1 + 1, \quad (3.42)$$
or

\[ q_0 \leq p_0 \leq q_0 + 1, \quad q_1 < p_1 \leq q_1 + 1, \quad (3.43) \]

holds, then the solution \((y_n, z_n)\) is bounded and persists.

From (3.36), (3.40), (3.41), (3.42), and (3.43), we get that

\[ y_n > \frac{q_1}{p_1}, \quad z_n > \frac{q_0}{p_0}, \quad n \geq M. \quad (3.44) \]

Suppose that

\[ p_0 \neq q_0 + 1 \quad \text{or} \quad p_1 \neq q_1 + 1. \quad (3.45) \]

From (1.3) and (3.44), we have

\[ z_{M+1} > 1, \quad y_{M+3} > 1. \quad (3.46) \]

We have for the functions (3.33) that \(f\) is decreasing with respect to \(y\) for any \(z, z > q_0/p_0\) and \(g\) is decreasing with respect to \(z\) for any \(y, y > q_1/p_1\). Therefore, relations (1.3), (3.44), and (3.46) imply that

\[ z_{M+3} \leq \frac{p_1 y_{M+2} + 1}{q_1 + 1}, \quad (3.47) \]

and so from (1.3) and (3.46),

\[ y_{M+5} \leq \frac{p_0 p_1}{(q_0 + 1)(q_1 + 1)} y_{M+2} + \frac{p_0}{(q_0 + 1)(q_1 + 1)} + 1. \quad (3.48) \]

Working inductively, we can prove that

\[ y_{n+5} \leq \frac{p_0 p_1}{(q_0 + 1)(q_1 + 1)} y_{n+2} + \frac{p_0}{(q_0 + 1)(q_1 + 1)} + 1, \quad n \geq M. \quad (3.49) \]

Then from (3.42), (3.43), (3.45), and Theorem 3.1, \(y_n\) is bounded. Similarly, we take that \(z_n\) is bounded. Therefore, from (3.44), the solution \((y_n, z_n)\) is bounded and persists.

Now, suppose that

\[ p_0 = q_0 + 1, \quad p_1 = q_1 + 1. \quad (3.50) \]
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We claim that $y_n$ is bounded. For the sake of contradiction, we assume that $y_n$ is not bounded. Then, there exists a subsequence $n_i$ such that

$$\lim_{i \to \infty} y_{n_i + 1} = \infty,$$

(3.51)

$$y_{n_i + 1} > \max \{ y_j, j < n_i \}. \quad (3.52)$$

Moreover, from (1.3) and (3.50), we get

$$y_{n_i + 1} < \frac{q_0 + 1}{q_0} z_{n_i - 1} + 1,$$

(3.53)

and so from (3.51),

$$\lim_{i \to \infty} z_{n_i - 1} = \infty. \quad (3.54)$$

Moreover, from (1.3) and (3.50),

$$z_{n_i - 1} < \frac{q_1 + 1}{q_1} y_{n_i - 2} + 1,$$

(3.55)

and so from (3.54),

$$\lim_{i \to \infty} y_{n_i - 2} = \infty. \quad (3.56)$$

Working inductively, we can prove that

$$\lim_{i \to \infty} y_{n_i + 1 - 3s} = \infty, \quad \lim_{i \to \infty} z_{n_i - 1 - 3s} = \infty, \quad s = 0, 1, \ldots \quad (3.57)$$

We claim that $y_{n-6}$ is a bounded sequence. Suppose on the contrary that there exists an unbounded subsequence of $y_{n-6}$ and without loss of generality, we may suppose that

$$\lim_{i \to \infty} y_{n-6} = \infty. \quad (3.58)$$

Arguing as above, we can easily prove that

$$\lim_{i \to \infty} y_{n-9} = \lim_{i \to \infty} y_{n-12} = \infty. \quad (3.59)$$

Also, since from (1.3),

$$y_{n-6} = \frac{(q_0 + 1) \left( \frac{z_{n-8}}{y_{n-8}} \right) + 1}{q_0/y_{n-8} + 1} < \frac{(q_0 + 1) z_{n-8}}{y_{n-8}} + 1, \quad (3.60)$$

$$y_{n-6} = \frac{(q_0 + 1) \left( \frac{z_{n-8}}{y_{n-8}} \right) + 1}{q_0/y_{n-8} + 1} < \frac{(q_0 + 1) z_{n-8}}{y_{n-8}} + 1, \quad (3.60)$$
from (3.58), we have that \( \lim_{i \to \infty} \left( \frac{z_{n-8}}{y_{n-8}} \right) = \infty \) and so eventually,

\[
\frac{z_{n-8}}{y_{n-8}} > 1.
\]

(3.61)

From (1.3), (3.50), and (3.61), we have

\[
y_{n+1} = \frac{(q_0 + 1)z_{n-1} + y_{n-1}}{q_0 + y_{n-1}} < \frac{q_0 + 1}{q_0} z_{n-1} + 1
\]

\[
= \frac{q_0 + 1}{q_0} \left( \frac{(q_1 + 1)y_{n-2} + z_{n-3}}{q_1 + z_{n-3}} \right) + 1
\]

\[
< 1 + \frac{q_0 + 1}{q_0} + \frac{q_0 + 1}{q_0} \frac{q_1 + 1}{q_1} y_{n-2}
\]

\[
< \cdots < A + By_{n-8}
\]

\[
< A + Bz_{n-8},
\]

where

\[
A = 1 + \frac{q_0 + 1}{q_0} + \frac{q_0 + 1}{q_0} \frac{q_1 + 1}{q_1} + \left( \frac{q_0 + 1}{q_0} \right)^2 \frac{q_1 + 1}{q_1}
\]

\[
+ \left( \frac{q_0 + 1}{q_0} \right)^2 \left( \frac{q_1 + 1}{q_1} \right)^2 + \left( \frac{q_0 + 1}{q_0} \right)^3 \left( \frac{q_1 + 1}{q_1} \right)^3,
\]

\[
B = \left( \frac{q_0 + 1}{q_0} \right)^3 \left( \frac{q_1 + 1}{q_1} \right)^3.
\]

Therefore, using (1.3) and (3.50), we get

\[
y_{n+1} < A + B \left( \frac{(q_1 + 1)y_{n-9} + z_{n-10}}{q_1 + z_{n-10}} \right).
\]

(3.64)

and since from (3.57) and (3.59), we have that \( y_{n-9} \to \infty \), \( z_{n-10} \to \infty \) as \( i \to \infty \), we can easily prove that eventually,

\[
y_{n+1} < y_{n-9},
\]

(3.65)

which contradicts to (3.52).
Therefore, $y_{n-6}$ is a bounded sequence. From (1.3), (3.50), and (3.57), we get

$$z_{n-5} = \frac{(q_1 + 1) y_{n-6} + z_{n-7}}{q_1 + z_{n-7}} = \frac{(q_1 + 1)(y_{n-6}/z_{n-7}) + 1}{q_1/z_{n-7} + 1} \rightarrow 1, \quad i \rightarrow \infty.$$  

(3.66)

Similarly, from (1.3), (3.50) and (3.57) and (3.66) follows,

$$y_{n-3} = \frac{(q_0 + 1) z_{n-5} + y_{n-5}}{q_0 + y_{n-5}} = \frac{(q_0 + 1)(z_{n-5}/y_{n-5}) + 1}{q_0/y_{n-5} + 1} \rightarrow 1, \quad i \rightarrow \infty.$$  

(3.67)

Now, we prove that

$$\liminf_{i \rightarrow \infty} y_{n-1} > 1.$$  

(3.68)

Otherwise, and without loss of generality, we may suppose that $\lim_{i \rightarrow \infty} y_{n-1} \leq 1$. So, relations (1.3), (3.50), and (3.67) imply that

$$\lim_{i \rightarrow \infty} y_{n-1} = \frac{(q_0 + 1) \lim_{i \rightarrow \infty} z_{n-3} + \lim_{i \rightarrow \infty} y_{n-3}}{q_0 + \lim_{i \rightarrow \infty} y_{n-3}} \leq 1,$$  

(3.69)

and so

$$\lim_{i \rightarrow \infty} z_{n-3} \leq \frac{q_0}{q_0 + 1}.$$  

(3.70)

Moreover, from (1.3), (3.44), and (3.50), we get eventually

$$z_{n-3} = \frac{(q_1 + 1) y_{n-4} + z_{n-5}}{q_1 + z_{n-5}} > \frac{(q_1 + 1)(q_1/(q_1 + 1)) + z_{n-5}}{q_1 + z_{n-5}} = 1,$$  

(3.71)

and so from (3.66), $\lim_{i \rightarrow \infty} z_{n-3} \geq 1$ which contradicts to (3.70).

Hence, (3.68) is true.

Similarly, we can prove that

$$\liminf_{i \rightarrow \infty} z_{n-3} > 1.$$  

(3.72)

Therefore, from (3.68) and (3.72), we have eventually

$$y_{n-1} > 1 + k, \quad z_{n-3} > 1 + m,$$  

(3.73)

where $k, m$ are positive real numbers.
Hence, from (1.3), (3.50), and (3.73) we have

\[ y_{n+1} = \frac{(q_0 + 1)[((q_1 + 1)y_{n-2} + z_{n-3})/(q_1 + z_{n-3})] + y_{n-1}}{q_0 + y_{n-1}} < \frac{(q_0 + 1)(q_1 + 1)}{(q_1 + 1 + m)(q_0 + 1 + k)} y_{n-2} + \frac{q_0 + 1}{q_0} + 1. \]  

(3.74)

Then from (3.57), we can prove that eventually

\[ y_{n+1} < y_{n-2}, \]  

(3.75)

which contradicts to (3.52).

Therefore, \( y_n \) is a bounded sequence. Moreover, from (1.3), (3.50), we take that \( z_n \) is bounded. Therefore, the solution \( (y_n, z_n) \) is bounded and persists. This completes the proof of the proposition. \( \square \)

4. Attractivity of the Positive Equilibrium of System (1.3)

In the following propositions, we study the convergency of the solutions of system (1.3) to its positive equilibrium.

**Proposition 4.1.** One considers the system of difference equations (1.3) where \( p_i, q_i, i = 0, 1 \) are positive constants, and the initial values \( y_i, z_i, i = -1, 0 \) are positive numbers. If either (3.29) or (3.30) hold, then every solution of (1.3) tends to the positive equilibrium of (1.3).

**Proof.** Let \( (y_n, z_n) \) be an arbitrary solution of (1.3). From Proposition 3.2, there exist

\[ L_1 = \limsup_{n \to \infty} y_n, \quad L_2 = \limsup_{n \to \infty} z_n, \quad l_1 = \liminf_{n \to \infty} y_n, \quad l_2 = \liminf_{n \to \infty} z_n, \]

\[ 0 < L_1, L_2, l_1, l_2 < \infty. \]  

(4.1)

From (1.3), (3.31), and the monotony of functions (3.33), we have

\[ L_1 \leq \frac{p_0 L_2 + L_1}{q_0 + L_1}, \quad L_2 \leq \frac{p_1 L_1 + L_2}{q_1 + L_2}, \quad l_1 \geq \frac{p_0 l_2 + l_1}{q_0 + l_1}, \quad l_2 \geq \frac{p_1 l_1 + l_2}{q_1 + l_2}, \]  

(4.2)

and hence

\[ L_1^2 + L_1(q_0 - 1) - p_0 L_2 \leq 0, \quad L_2^2 + L_2(q_1 - 1) - p_1 L_1 \leq 0, \]

\[ l_1^2 + l_1(q_0 - 1) - p_0 l_2 \geq 0, \quad l_2^2 + l_2(q_1 - 1) - p_1 l_1 \geq 0. \]  

(4.3)
The third inequality of (4.3), implies that
\[ l_1 \geq \frac{1 - q_0 + \sqrt{(1 - q_0)^2 + 4p_0l_2}}{2}, \]  
(4.4)
and so from the last inequality of (4.3), we have
\[ 2l_2^2 + 2l_2(q_1 - 1) + (q_0 - 1)p_1 \geq p_1\sqrt{(1 - q_0)^2 + 4p_0l_2}. \]  
(4.5)
Hence, we get
\[ (2l_2^2 + 2l_2(q_1 - 1) + (q_0 - 1)p_1)^2 \geq \left(p_1\sqrt{(1 - q_0)^2 + 4p_0l_2}\right)^2, \]  
(4.6)
or
\[ l_2^3 + 2l_2^2(q_1 - 1) + l_2[ (q_1 - 1)^2 + p_1(q_0 - 1)] + p_1(q_1 - 1)(q_0 - 1) - p_0p_1^2 \geq 0. \]  
(4.7)
The first inequality of (4.3), implies that
\[ 0 < L_1 \leq \frac{1 - q_0 + \sqrt{(1 - q_0)^2 + 4p_0L_2}}{2}, \]  
(4.8)
and so from second inequality of (4.3), we get
\[ 2L_2^2 + 2L_2(q_1 - 1) + (q_0 - 1)p_1 \leq p_1\sqrt{(1 - q_0)^2 + 4p_0L_2}. \]  
(4.9)
Using (4.3), we have
\[ L_1 \geq l_1 > 1 - q_0, \quad L_2 \geq l_2 > 1 - q_1. \]  
(4.10)
Therefore, from (4.5) and (4.10), we get
\[ 2L_2^2 + 2L_2(q_1 - 1) + (q_0 - 1)p_1 = 2L_2(L_2 + q_1 - 1) + (q_0 - 1)p_1 \] 
\[ \geq 2l_2^2 + 2l_2(q_1 - 1) + (q_0 - 1)p_1 \] 
\[ > 0. \]  
(4.11)
Using (4.9) and (4.11), we have
\[ L_2^3 + 2L_2^2(q_1 - 1) + L_2[ (q_1 - 1)^2 + p_1(q_0 - 1)] + p_1(q_1 - 1)(q_0 - 1) - p_0p_1^2 \leq 0. \]  
(4.12)
Proposition 4.2. One considers the system of difference equations (1.3) where \( p_i, q_i, i = 0, 1 \) are positive constants, and the initial values \( y_i, z_i, i = -1, 0 \) are positive numbers. If either (3.42) or (3.43) hold, then every solution of (1.3) tends to the positive equilibrium of (1.3).

Proof. Let \( (y_n, z_n) \) be an arbitrary solution of (1.3). From Proposition 3.3, there exist \( L_i, l_i, \) \( i = 1, 2 \) such that (4.1) are satisfied.

From (1.3), the monotony of functions (3.33) and (3.44), we have

\[
L_1 \leq \frac{p_0 L_2 + l_1}{q_0 + l_1}, \quad L_2 \leq \frac{p_1 L_1 + l_2}{q_1 + l_2}, \quad l_1 \geq \frac{p_0 l_2 + L_1}{q_0 + L_1}, \quad l_2 \geq \frac{p_1 l_1 + L_2}{q_1 + L_2},
\]

and hence

\[
L_1 l_1 + L_1 q_0 \leq p_0 L_2 + l_1, \quad L_1 l_1 + l_1 q_0 \geq p_0 l_2 + L_1, \\
L_2 l_2 + L_2 q_1 \leq p_1 L_1 + l_2, \quad L_2 l_2 + l_2 q_1 \geq p_1 l_1 + L_2.
\]
which implies that

\[(1 + q_0)(L_1 - l_1) \leq p_0(L_2 - l_2), \quad (1 + q_1)(L_2 - l_2) \leq p_1(L_1 - l_1).\]  \(4.20\)

Therefore,

\[\left[(1 + q_0)(1 + q_1) - p_0p_1\right](L_1 - l_1) \leq 0.\]  \(4.21\)

First suppose that \((3.45)\) holds. Then from \((3.42)\) or \((3.43)\), and \((3.45)\), we get \(L_1 - l_1 \leq 0\), which means that

\[L_1 = l_1.\]  \(4.22\)

Using \((4.20)\), it is obvious that

\[L_2 = l_2.\]  \(4.23\)

So if \((3.45)\) holds, the proof is completed.

Now, suppose that \((3.50)\) hold. Then from \((4.20)\), we have

\[L_2 - l_2 = L_1 - l_1.\]  \(4.24\)

Moreover, from \((4.24)\), it follows that

\[(q_0 + 1)l_2 + L_1 - l_1q_0 = (q_0 + 1)L_2 + l_1 - L_1q_0.\]  \(4.25\)

In addition, from \((3.50)\), the first and the second inequalities of \((4.19)\), we get

\[(q_0 + 1)l_2 + L_1 - l_1q_0 \leq L_1l_1 \leq (q_0 + 1)L_2 + l_1 - L_1q_0.\]  \(4.26\)

Therefore, from \((4.25)\) and \((4.26)\), we have

\[L_1 = \frac{(q_0 + 1)L_2 + l_1}{q_0 + l_1}.\]  \(4.27\)

We may assume that there exists a positive integer \(n_i\) such that

\[\lim_{i \to \infty} y_{n_i-j} = A_j, \quad \lim_{i \to \infty} z_{n_i-j} = B_j, \quad \lim_{i \to \infty} y_{n_i+1} = L_1.\]  \(4.28\)

Moreover, from \((1.3)\), \((3.50)\), and \((4.28)\), we get

\[L_1 = \frac{(q_0 + 1)B_1 + A_1}{q_0 + A_1}.\]  \(4.29\)
Since \( f(x, y) = ((q_0 + 1)x + y)/(q_0 + y) \) is decreasing with respect to \( y \), for any \( x > (q_0/(q_0 + 1)) \), if \( B_1 < L_2 \) or \( l_1 < A_1 \), then from (3.44), and (3.50), we get

\[
L_1 < \frac{(q_0 + 1)L_2 + l_1}{q_0 + l_1},
\]

which contradicts to (4.27). So,

\[
B_1 = L_2, \quad l_1 = A_1.
\]

Using the same argument, we can prove that

\begin{align*}
A_2 &= L_1, \quad B_3 = l_2, \\
B_3 &= l_2, \quad A_3 = L_1, \\
B_4 &= L_2, \quad A_4 = l_1, \\
A_4 &= l_1, \quad B_5 = L_2, \\
B_5 &= L_2, \quad A_5 = l_1, \\
A_5 &= l_1, \quad B_6 = l_2,
\end{align*}

and so \( L_1 = l_1 = A \). Also, from (4.24), we have \( L_2 = l_2 = B \). Therefore,

\[
\lim_{n \to \infty} y_n = A, \quad \lim_{n \to \infty} z_n = B,
\]

where obviously \( A = B = 2 \). This completes the proof of the proposition.

**Proposition 4.3.** One considers the system of difference equations (1.3) where \( p_i, q_i, i = 0, 1 \) are positive constants, and the initial values \( y_i, z_i, i = -1, 0 \) are positive numbers. If relations (3.7) hold, then every solution of (1.3) tends to the positive equilibrium \((1, 1)\) of (1.3).

**Proof.** Let \((y_n, z_n)\) be an arbitrary solution of (1.3). From the proof of Proposition 3.2, the subsequences \( y_{3n}, y_{3n+1}, y_{3n+2}, z_{3n}, z_{3n+1}, \) and \( z_{3n+2} \) are monotone and \( y_n, z_n \) are bounded and persist. So, there exist positive numbers \( L_1, L_2, L_3, M_1, M_2, \) and \( M_3 \) such that

\[
L_1 = \lim_{n \to \infty} y_{3n}, \quad L_2 = \lim_{n \to \infty} y_{3n+1}, \quad L_3 = \lim_{n \to \infty} y_{3n+2}, \\
M_1 = \lim_{n \to \infty} z_{3n}, \quad M_2 = \lim_{n \to \infty} z_{3n+1}, \quad M_3 = \lim_{n \to \infty} z_{3n+2}.
\]
and from (1.3) and (3.7), we get

\[
L_1 = \frac{pM_2 + L_2}{p + L_2}, \quad M_1 = \frac{qL_3 + M_2}{q + M_2},
\]
\[
L_2 = \frac{pM_3 + L_3}{p + L_3}, \quad M_2 = \frac{qL_1 + M_3}{q + M_3},
\]
\[
L_3 = \frac{pM_1 + L_1}{p + L_1}, \quad M_3 = \frac{qL_2 + M_1}{q + M_1}.
\]

Then, we have

\[
L_1 p + L_1 L_2 = pM_2 + L_2, \quad M_1 q + M_1 M_2 = qL_3 + M_2,
\]
\[
L_2 p + L_2 L_3 = pM_3 + L_3, \quad M_2 q + M_2 M_3 = qL_1 + M_3,
\]
\[
L_3 p + L_1 L_3 = pM_1 + L_1, \quad M_3 q + M_3 M_1 = qL_2 + M_1,
\]

and hence,

\[
(L_1 - M_2) p = L_2 (1 - L_1), \quad (M_1 - L_3) q = M_2 (1 - M_1),
\]
\[
(L_2 - M_3) p = L_3 (1 - L_2), \quad (M_2 - L_1) q = M_3 (1 - M_2),
\]
\[
(L_3 - M_1) p = L_1 (1 - L_3), \quad (M_3 - L_2) q = M_1 (1 - M_3).
\]

Therefore, we take

\[
\frac{1}{p} L_2 (1 - L_1) = \frac{1}{q} M_3 (M_2 - 1),
\]
\[
\frac{1}{p} L_3 (1 - L_2) = \frac{1}{q} M_1 (M_3 - 1),
\]
\[
\frac{1}{p} L_1 (1 - L_3) = \frac{1}{q} M_2 (M_1 - 1).
\]

So,

\[
\text{if } L_1 \geq 1 \text{ (resp., } L_1 \leq 1), \quad \text{then } M_2 \leq 1 \text{ (resp., } M_2 \geq 1),
\]
\[
\text{if } L_2 \geq 1 \text{ (resp., } L_2 \leq 1), \quad \text{then } M_3 \leq 1 \text{ (resp., } M_3 \geq 1),
\]
\[
\text{if } L_3 \geq 1 \text{ (resp., } L_3 \leq 1), \quad \text{then } M_1 \leq 1 \text{ (resp., } M_1 \geq 1).
\]

Therefore, if \( L_1 \geq 1, \ M_2 \leq 1 \) (resp., \( L_1 \leq 1, \ M_2 \geq 1 \)), we have \( L_1 - M_2 \geq 0 \) (resp., \( L_1 - M_2 \leq 0 \)) and so from (4.37), \( L_1 \leq 1 \) (resp., \( L_1 \geq 1 \)). Hence, \( L_1 = 1 \) and from (4.37), \( M_2 = 1 \). Similarly, we can prove that \( L_2 = 1, \ L_3 = 1, \ M_1 = 1, \ M_3 = 1 \). This completes the proof of the proposition. \( \square \)
5. Stability of System (1.3)

In this section we find conditions so that the positive equilibrium \((y, z)\) and the zero equilibrium of (1.3) are stable.

**Proposition 5.1.** Consider system (1.3) where \(p_i, q_i, i = 0, 1\) are positive constants and the initial values \(y_i, z_i, i = -1, 0\) are positive numbers. Then, the following statements are true.

(i) If

\[
q_0 - 1 < p_0 \leq q_0, \quad q_1 - 1 < p_1 \leq q_1, \quad q_0 + q_1 + p_0p_1 + q_0q_1 < 1, \tag{5.1}
\]

then the unique positive equilibrium \((y, z)\) of (1.3) is globally asymptotically stable.

(ii) If

\[
q_0 + q_1 + p_0p_1 + 1 < q_0q_1, \tag{5.2}
\]

then the zero equilibrium of (1.3) is locally asymptotically stable.

**Proof.** (i) Since \((y, z)\) is the unique positive positive equilibrium of (1.3), we have

\[
y = \frac{p_0z + y}{q_0 + y}, \quad z = \frac{p_1y + z}{q_1 + z}. \tag{5.3}
\]

Then from (5.1) and (5.3), we get

\[
y \leq \frac{q_0z + y}{q_0 + y}, \quad z \leq \frac{q_1y + z}{q_1 + z}. \tag{5.4}
\]

Without loss of generality we assume that \(z \leq y\). Then from (5.4), it results that

\[
y \leq \frac{q_0y + y}{q_0 + y}, \tag{5.5}
\]

which means that

\[
y \leq 1. \tag{5.6}
\]

Moreover, from (5.4) and (5.6), we get

\[
z \leq \frac{q_1 + z}{q_1 + z} = 1. \tag{5.7}
\]

In addition, from (5.3), we have

\[
y > \frac{y}{q_0 + y}, \quad z > \frac{z}{q_1 + z}, \tag{5.8}
\]

and so

\[ y > 1 - q_0, \quad z > 1 - q_1. \]  \hfill (5.9)

Then the linearized system of (1.3) about the positive equilibrium \((y, z)\) is

\[ Z_{n+1} = AZ_n, \]  \hfill (5.10)

where

\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & \frac{q_0 - p_0z}{(q_0 + y)^2} & \frac{p_0}{q_0 + y} & 0 \\
\frac{q_1 - p_1y}{(q_1 + z)^2} & \frac{p_1}{q_1 + z} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad Z_n = \begin{pmatrix}
w_{n-1} \\
v_{n-1} \\
v_n
\end{pmatrix}. \tag{5.11}
\]

The characteristic equation of \(A\) is

\[
\lambda^4 - \left( \frac{q_0 - p_0z}{(q_0 + y)^2} + \frac{q_1 - p_1y}{(q_1 + z)^2} \right) \lambda^3 - \frac{p_1p_0}{(q_0 + y)(q_1 + z)} \lambda^2 + \left( \frac{q_0 - p_0z}{(q_0 + y)^2} \right) \left( \frac{q_1 - p_1y}{(q_1 + z)^2} \right) = 0. \tag{5.12}
\]

According to Remark 1.3.1 of [7], all the roots of (5.12) are of modulus less than 1 if and only if

\[
\left| \frac{q_0 - p_0z}{(q_0 + y)^2} + \frac{q_1 - p_1y}{(q_1 + z)^2} \right| + \frac{p_1p_0}{(q_0 + y)(q_1 + z)} + \left| \frac{q_0 - p_0z}{(q_0 + y)^2} \right| < 1. \tag{5.13}
\]

From (5.3), we get

\[
q_0 - p_0z = (1 - y)(y + q_0), \quad q_1 - p_1y = (1 - z)(z + q_1). \tag{5.14}
\]

Then from (5.6), (5.7), and (5.14), inequality (5.13) is equivalent to

\[
\frac{1 - y}{q_0 + y} + \frac{1 - z}{q_1 + z} + \frac{p_1p_0}{(q_0 + y)(q_1 + z)} + \frac{(1 - y)(1 - z)}{(q_0 + y)(q_1 + z)} < 1. \tag{5.15}
\]

Using (5.9), inequality (5.15) holds if (5.1) are satisfied. Using Propositions 4.1 and 4.3, we have that the unique positive equilibrium \((y, z)\) of (1.3) is globally asymptotically stable.

(ii) Arguing as above, we can prove that the linearized system of (1.3) about the zero equilibrium is

\[ Z_{n+1} = AZ_n, \]  \hfill (5.16)
where

\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{1}{q_0} & \frac{p_0}{q_0} & 0 & 0 \\
0 & \frac{1}{q_1} & \frac{p_1}{q_1} & 0
\end{pmatrix}, \quad Z_n = \begin{pmatrix}
\omega_{n-1} \\
\nu_{n-1} \\
\omega_n \\
\nu_n
\end{pmatrix}
\]

(5.17)

The characteristic equation of \(A\) is

\[
\lambda^4 - \left( \frac{1}{q_0} + \frac{1}{q_1} \right) \lambda^2 - \frac{p_1 p_0}{q_0 q_1} \lambda + \frac{1}{q_0 q_1} = 0. 
\]

(5.18)

Using [7, Remark 1.3.1], all the roots of (5.18) are of modulus less than 1 if and only if relation (5.2) holds. This completes the proof of the proposition.

\[ \square \]

6. Conclusion

In this paper, in order to investigate (1.2), we study the equivalent system (1.3). Summarizing the results of Sections 2, 3, 4, we get the following statements, concerning (1.2).

(i) If (2.1) hold, then (1.2) has a unique positive periodic solution of period 2.

(ii) If either (3.4) or (3.37) holds, then every positive solution of (1.2) is bounded and persists and tends to the unique positive periodic solution.

(iii) If (5.1) hold, then the unique periodic solution of (1.2) is globally asymptotically stable and if (5.2) holds, then the zero solution of (1.2) is locally asymptotically stable.

Open Problem

Consider the difference equation (1.2) where \(p_n, q_n, n = 0, 1, \ldots\) are positive sequences of period 2, and the initial values \(x_i, i = -3, -2, -1, 0\) are positive numbers. Prove that

(i) if

\[
q_0 - 1 < p_0 \leq q_0 + 1, \quad q_1 - 1 < p_1 \leq q_1 + 1
\]

(6.1)

are satisfied, then every positive solution of (1.2) is bounded and persists;

(ii) if relations (6.1) are satisfied, then every positive solution of (1.2) tends to the unique positive equilibrium \((y, z)\) of (1.2) as \(n \to \infty\).

Acknowledgment

The authors would like to thank the referees for their helpful suggestions.
References


