Research Article

Iterative Approximation of a Common Zero of a Countably Infinite Family of $m$-Accretive Operators in Banach Spaces

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Received 1 September 2007; Accepted 4 February 2008

Recommended by Tomas Dominguez Benavides

Let $E$ be a real reflexive and strictly convex Banach space which has a uniformly Gâteaux differentiable norm and let $C$ be a closed convex nonempty subset of $E$. Strong convergence theorems for approximation of a common zero of a countably infinite family of $m$-accretive mappings from $C$ to $E$ are proved. Consequently, we obtained strong convergence theorems for a countably infinite family of pseudocontractive mappings.

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1. Introduction

Let $E$ be a real Banach space with dual $E^*$. The normalized duality mapping is the mapping $J : E \rightarrow 2^{E^*}$ defined for all $x \in E$ by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2, \|f^*\| = \|x\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between members of $E$ and $E^*$. It is well known that if $E^*$ is strictly convex, then $J$ is single valued. In what follows, the single-valued normalized duality mapping will be denoted by $j$.

Let $(E, \|\cdot\|)$ be a normed linear space. The norm $\|\cdot\|$ is said to be uniformly Gâteaux differentiable if for each $y \in S = \{x \in E : \|x\| = 1\}$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly for $x \in S$. It is well known that $L_p$ spaces, $1 < p < \infty$, have uniformly Gâteaux differentiable norm (see, e.g., [1]). Furthermore, if $E$ has a uniformly Gâteaux differentiable
norm, then the duality mapping is norm-to-weak* uniformly continuous on bounded subsets of $E$.

Let $C$ be a nonempty subset of a normed linear space $E$. A mapping $T : C \to E$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C. \tag{1.3}$$

Most published results on nonexpansive mappings centered on existence theorems for fixed points of these mappings, and iterative approximation of such fixed points.

DeMarr [2] in 1963 studied the problem of existence of common fixed point for a family of nonlinear nonexpansive mappings. He proved the following theorem.

**Theorem 1.1** (DM). Let $E$ be a Banach space and $C$ be a nonempty compact convex subset of $E$. If $\Omega$ is a nonempty commuting family of nonexpansive mappings of $C$ into itself, then the family $\Omega$ has a common fixed point in $C$.

In 1965, Browder [3] proved the result of DeMarr in a uniformly convex Banach space, requiring that $C$ be only bounded, closed, convex, and nonempty. For other fixed-point theorems for families of nonexpansive mappings, the reader may consult Belluce and Kirk [4], Lim [5], and Bruck Jr. [6].

In 1973, Bruck Jr. [7] considered the study of structure of the fixed-point set $F(T) = \{x \in C : Tx = x\}$ of nonexpansive mapping $T$ and established several results.

Kirk [8] introduced an iterative process given by

$$x_{n+1} = \alpha_0 x_n + \alpha_1 Tx_n + \alpha_2 T^2 x_n + \cdots + \alpha_r T^r x_n, \tag{1.4}$$

where $\alpha_i \geq 0$, $\alpha_0 > 0$ and $\sum_{i=0}^r \alpha_i = 1$, for approximating fixed points of nonexpansive mappings on convex subset of uniformly convex Banach spaces. Maiti and Saha [9] worked on and improved the results of Kirk [8].

Considerable research efforts have been devoted to develop iterative methods for approximating common fixed points (when such fixed points exist) of families of several classes of nonlinear mappings (see, e.g., [10–18]).

Let $C$ be a nonempty closed and bounded subset of a real Banach space $E$. Let $T_i : C \to C$, $i = 1, 2, \ldots, r$ be a finite family of nonexpansive mappings and let

$$S = \alpha_0 I + \alpha_1 T_1 + \alpha_2 T_2 + \cdots + \alpha_r T_r, \tag{1.5}$$

where $\alpha_i \geq 0$, $\alpha_1 > 0$, and $\sum_{i=0}^r \alpha_i = 1$. Then the family $\{T_i\}_{i=1}^r$ such that the common fixed-point set $F := \bigcap_{i=1}^r F(T_i) \neq \emptyset$ is said to satisfy condition $A$ (see, e.g., [9, 19, 20]) if there exists a nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0$, $\phi(\varepsilon) > 0$ for all $\varepsilon \in (0, +\infty)$, such that $\|x - Sx\| \geq \phi(d(x, F))$ for all $x \in C$, where $d(x, F) = \inf\{\|x - z\| : z \in F\}$.

Liu et al. [19] introduced the following iteration process:

$$x_0 \in C, \quad x_{n+1} = Sx_n, \quad n \geq 0 \tag{1.6}$$

and showed that $\{x_n\}_{n \geq 0}$ defined by (1.6) converges to a common fixed point of $\{T_i\}_{i=1}^r$ in Banach spaces, provided that $\{T_i\}_{i=1}^r$ satisfy condition $A$. The result of Liu et al. [19] improves
the corresponding results of Kirk [8], Maiti and Saha [9], Senter and Dotson [20] and those of a host of other authors. However, the assumption that the family $\{T_i\}_{i=1}^r$ satisfies condition $A$ is strong.

Let $E$ be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let $T_i : E \to E, i = 1, 2, \ldots, r$ be nonexpansive mappings and $\{x_n\}_{n \geq 0}$ a sequence in $E$ defined iteratively by (1.6) and suppose that $f^{-1} : E^* \to E$ is weakly sequentially continuous at 0. If $F := \bigcap_{i=1}^r F(T_i) \neq \emptyset$, then Jung [21] in 2002 proved that, under this situation, $\{x_n\}_{n \geq 0}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^r$. In [22], Gossez and Lami Dozo proved that for any normed linear space $E$, the existence of a weakly sequentially continuous duality mapping implies that the space $E$ satisfies Opial’s condition (that is, for all sequences $\{x_n\}$ in $E$ such that $\{x_n\}$ converges weakly to some $x \in E$, the inequality $\lim inf_{n \to \infty} \|x_n - y\| > \lim inf_{n \to \infty} \|x_n - x\|$ holds for all $y \neq x$, see e.g., [23]). It is well known that $L_p$ spaces, $1 < p < +\infty, p \neq 2$, do not satisfy Opial’s condition. Consequently, the results of Jung [21] are not applicable in $L_p$ spaces $1 < p < +\infty, p \neq 2$.

Another class of nonlinear mappings now studied is the class of accretive operators. Let $E$ be a real normed linear space. A mapping $A : D(A) \subset E \to E$ is said to be accretive if the following inequality holds:

$$\|x - y\| \leq \|x - y + s(Ax - Ay)\| \quad \forall s > 0, \forall x, y \in D(A),$$

(1.7)

where $D(A)$ denotes the domain of the operator $A$. It is not difficult to deduce from (1.7) that the mapping $A$ is accretive if and only if $(I + sA)^{-1}$ is nonexpansive on the range of $(I + sA)$, where $I$ denotes the identity operator defined on $E$. We note that the range, $R(I+sA)$, of $(I+sA)$ needs not be all of $E$. When $A$ is accretive and, in addition, the range of $(I + sA)$ is all of $E$, then $A$ is called $m$-accretive.

Our presentation in this paper is primarily motivated by the study of equations of the form

$$u'(t) + Au(t) = f, \quad u(0) = u_0, \quad f \in E.$$ 

(1.8)

It is well known that many physically significant problems can be modeled by equations of the form (1.8) (where $A$ is accretive), which is generally called Evolution Equation. Typical examples where such evolution equations occur can be found in the heat, wave, and Schrödinger equations (see, e.g., [24]). One of the fundamental results in the theory of accretive operators, due to Browder [25], states that if $A$ is locally Lipschitzian and accretive, then $A$ is $m$-accretive and this implies that (1.8) has a solution $u^* \in D(A)$ for any $f \in E$ (in particular for $f = 0$). This result was subsequently generalized by Martin [26] to continuous accretive operators. If in (1.8), $f = 0$ and $u(t)$ is independent of $t$, then (1.8) reduces to

$$Au = 0$$

(1.9)

whose solutions correspond to the equilibrium points of (1.8). There is no known method to obtain a closed form solution of (1.9). The general approach for approximating a solution of (1.9) is to transform it into a fixed-point problem. Defining $T := I - A$, we observe that $x^*$ is a solution of (1.9) if and only if $x^*$ is a fixed point of $T$ (i.e., $x^* \in Tx^*$). Browder [25] called such an operator $T$ pseudocontractive.

Consequently, the study of methods of approximating fixed points of pseudocontractive maps, which correspond to equilibrium points of the system (1.8), became a flourishing area of research for numerous mathematicians (see, e.g., [27–31]).
Remark 1.2. We observe that a mapping $A := I - T$ is accretive if and only if the mapping $T$ is pseudocontractive. It is, therefore, not difficult to see (using (1.7)) that every nonexpansive mapping is pseudocontractive. The converse, however, does not hold. The following illustrates this fact.

Example 1.3. Let $T : [0, 1] \to (\mathbb{R}, |\cdot|)$ be defined by

$$Tx = \begin{cases} 
\frac{x}{2} & \text{if } x \in \left[0, \frac{1}{2}\right), \\
1 & \text{if } x \in \left[\frac{1}{2}, 1\right].
\end{cases}$$

Clearly, $T$ is not continuous and thus cannot be nonexpansive. Now, let $s > 0$, then for $x, y \in [0, 1/2) \cup (1/2, 1]$ we obtain that $|x - y + s((I - T)x - (I - T)y)| \geq |x - y|$. So, $T$ is pseudocontractive but not nonexpansive. Thus, the class of pseudocontractive mappings properly contains the class of nonexpansive mappings. Moreover, we see in particular that the operator $A$ is accretive, if and only if the mapping $J_A := (I + A)^{-1}$ is a single-valued nonexpansive mapping from $R(I + A)$ to $D(A)$ and that $F(J_A) = N(A)$, where $N(A) = \{x \in D(A) : Ax = 0\}$ and $F(J_A) = \{x \in E : J_Ax = x\}$. (see, e.g., [1]).

Let $C$ be a nonempty closed convex subset of a real reflexive and strictly convex Banach space $E$ which has a uniformly Gâteaux differentiable norm. Let $A_i : C \to E$, $i = 1, 2, \ldots, r$ be a finite family of $m$-accretive mappings with $N = \bigcap_{i=1}^r N(A_i) \neq \emptyset$. Suppose that every bounded closed convex nonempty subset of $E$ has the fixed-point property for nonexpansive mappings; Zegeye and Shahzad [32] constructed an iterative sequence which converges strongly to a common solution of the equations $A_ix = 0$, $i = 1, 2, \ldots, r$.

It is our purpose in this paper to construct an iterative algorithm for the approximation of a common zero of a countably infinite family of $m$-accretive operators in Banach spaces. As a result, we obtain strong convergence theorems for approximation of a common fixed point of a countably infinite family $\{T_k\}_{k \in \mathbb{N}}$ of pseudocontractive mappings, provided that $I - T_k$ is $m$-accretive for all $k \in \mathbb{N}$. Our theorems improve, generalize, and extend the corresponding results of Zegeye and Shahzad [32] and several other results recently announced (see Remark 3.18 of this paper) from a finite family $\{A_i\}_{i=1}^r$ of $m$-accretive mappings to a countably infinite family $\{A_k\}_{k \in \mathbb{N}}$ of $m$-accretive mappings. Furthermore, our theorems are applicable, in particular in $L_p$ spaces $1 < p < +\infty$, and our method of proof is of independent interest.

2. Preliminaries

In the sequel, the following Lemmas and Theorems will be used.

Lemma 2.1 (see, e.g., [18, 27, 33]). Let $\{\lambda_n\}_{n \geq 1}$ be a sequence of nonnegative real numbers satisfying the condition

$$\lambda_{n+1} \leq (1 - \alpha_n)\lambda_n + \sigma_n, \quad n \geq 0, \quad (2.1)$$

where $\{\alpha_n\}_{n \geq 0}$ and $\{\sigma_n\}_{n \geq 0}$ are sequences of real numbers such that $\{\alpha_n\}_{n \geq 1} \subset [0, 1]$, $\sum_{n=1}^\infty \alpha_n = +\infty$. Suppose that $\sigma_n = o(\alpha_n)$, $n \geq 0$ (i.e., $\lim_{n \to \infty} (\sigma_n/\alpha_n) = 0$) or $\sum_{n=1}^\infty |\sigma_n| < +\infty$ or $\lim_{n \to \infty} (\sigma_n/\alpha_n) \leq 0$, then $\lambda_n \to 0$ as $n \to \infty$. 

Lemma 2.2. Let $E$ be a real normed linear space. Then the following inequality holds: for all $x, y \in E$, for all $j(x + y) \in j(x + y)$,
\[\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle.\] (2.2)

Lemma 2.3 (see [7, Lemma 3, page 257]). Let $C$ be a nonempty closed and convex subset of a real strictly convex Banach space $E$. Let $\{T_k\}_{k \in \mathbb{N}}$ be a sequence of nonself nonexpansive mappings $T_k : C \to E$. Then there exists a nonexpansive mapping $T : C \to E$ such that $F(T) = \bigcap_{k=1}^{\infty} F(T_k)$. Proof. If the sequence $\{T_k\}_{k \in \mathbb{N}}$ does not have a common fixed point, we can assume $T$ to be translation by any nonzero vector in which case $F(T) = \bigcap_{k=1}^{\infty} F(T_k) = \emptyset$. Otherwise, let $x^*$ be a common fixed point of $\{T_k\}_{k \in \mathbb{N}}$. Let $\{\xi_k\}_{k \geq 1}$ be any sequence of positive real numbers such that $\sum_{k=1}^{\infty} \xi_k = 1$ and set $T := \sum_{k=1}^{\infty} \xi_k T_k$. Then the mapping $T$ is well defined, since
\[\sum_{k=1}^{\infty} \xi_k T_k x \leq \sum_{k=1}^{\infty} \|T_k x - T_k x^*\| + \|T_k x^*\| \leq \|x - x^*\| + \|x^*\|.\] (2.3)

Thus, $\sum_{k=1}^{\infty} \xi_k T_k x$ converges absolutely for each $x \in C$. It is easy to see that $T$ is nonexpansive and maps $C$ into $E$. Next, we claim that $F(T) = \bigcap_{k=1}^{\infty} F(T_k)$. The inclusion $\bigcap_{k=1}^{\infty} F(T_k) \subset F(T)$ is obvious. We prove the reverse inclusion only. Suppose that $T x_0 = x_0$. Then
\[\|x_0 - x^*\| = \|T x_0 - x^*\| = \left\| \sum_{k=1}^{\infty} \xi_k T_k x_0 - x^* \right\| = \left\| \sum_{k=1}^{\infty} \xi_k (T_k x_0 - x^*) \right\| \leq \sum_{k=1}^{\infty} \xi_k \|T_k x_0 - x^*\|.\] (2.4)

But $T_k x^* = x^*$ and $T_k$ are nonexpansive for all $k \in \mathbb{N}$, so $\|T_k x_0 - x^*\| \leq \|x_0 - x^*\|$. Since $\sum_{k=1}^{\infty} \xi_k = 1$, (2.4) implies that
\[\left\| \sum_{k=1}^{\infty} \xi_k T_k x_0 - x^* \right\| = \|x_0 - x^*\|,\] (2.5)
\[\|T_k x_0 - x^*\| = \|x_0 - x^*\| \quad \forall k \in \mathbb{N}.
\]
Since $E$ is strictly convex and each $\xi_k > 0$ while $\sum_{k=1}^{\infty} \xi_k = 1$, (2.5) implies that $T_k x_0 - x^* = T_m x_0 - x^*$ for all $k, m \in \mathbb{N}$, that is, $T_k x_0 = T_m x_0$ for all $k, m \in \mathbb{N}$. Hence,
\[x_0 = T x_0 = \sum_{k=1}^{\infty} \xi_k T_k x_0 = \sum_{k=1}^{\infty} \xi_k T_m x_0 = T_m x_0 \quad \forall m \in \mathbb{N}.\] (2.6)

Thus, $x_0 \in \bigcap_{m=1}^{\infty} F(T_m)$. This completes the proof.

\[\square\]

Remark 2.4. The proof of Lemma 2.3 is as given by Bruck Jr. [7]. We included it here for completeness of our presentation in this paper.
Theorem 2.5 (I). (see e.g., [1]). Let \( A \) be a continuous accretive operator defined on a real Banach space \( E \) with \( D(A) = E \). Then \( A \) is \( m \)-accretive.

Theorem 2.6 (M). (see [34]). Let \( C \) be a closed convex nonempty subset of a real reflexive Banach space \( E \) which has uniformly Gâteaux differentiable norm and \( T : C \to E \) a nonexpansive mapping with \( F(T) \neq \emptyset \). Suppose that every bounded closed convex nonempty subset of \( C \) has the fixed-point property for nonexpansive mappings, then there exists a continuous path \( t \to z_t, 0 < t < 1 \) satisfying \( z_t = tu + (1 - t)Tz_t \), for arbitrary but fixed \( u \in C \), which converges strongly to a fixed point of \( T \).

3. Main results

For the rest of this paper, \( \{\alpha_n\}_{n \geq 1} \) is a real sequence such that \( \{\alpha_n\}_{n \geq 1} \subset [0, 1] \) and satisfies (i) \( \lim_{n \to \infty} \alpha_n = 0 \); (ii) \( \sum_{n=1}^{\infty} \alpha_n = \infty \) and either (iii) \( \lim_{n \to \infty} |\alpha_n - \alpha_{n-1}| / \alpha_n = 0 \) or (iii) \( \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty \). The sequence \( \{\xi_k\}_{k=1}^{\infty} \) is a sequence of positive real numbers such that \( \sum_{k=1}^{\infty} \xi_k = 1 \).

We now state and prove our main theorems.

3.1. Strong convergence theorems for a countably infinite family of \( m \)-accretive mappings

Theorem 3.1. Let \( C \) be a closed convex nonempty subset of a real reflexive and strictly convex Banach space \( E \) which has a uniformly Gâteaux differentiable norm. Let \( A_k : C \to E, k \in \mathbb{N} \) be a countably infinite family of \( m \)-accretive mappings such that \( N' = \bigcap_{k=1}^{\infty} N(A_k) \neq \emptyset \). Suppose that every bounded closed convex nonempty subset of \( C \) has the fixed point property for nonexpansive mappings. For arbitrary \( u, x_1 \in C \), let \( \{x_n\}_{n \geq 1} \) be iteratively generated by

\[
    x_{n+1} = \alpha_n u + (1 - \alpha_n)Sx_n, \quad n \geq 1,
\]

where \( S = \sum_{k=1}^{\infty} \xi_k J_{A_k} \); \( J_{A_k} = (I + A_k)^{-1}, k \in \mathbb{N} \). Then, \( \{x_n\}_{n \geq 1} \) converges strongly to a common zero of \( \{A_k\}_{k \in \mathbb{N}} \).

Proof. Since \( J_{A_k} = (I + A_k)^{-1} \) is nonexpansive for each \( k \in \mathbb{N} \), we obtain, by Lemma 2.3, that \( S = \sum_{k=1}^{\infty} \xi_k J_{A_k} \) is well defined, nonexpansive, and \( F(S) = \bigcap_{k=1}^{\infty} F(J_{A_k}) = N' \). Now, let \( q \in F(S) \), then we obtain by induction (using (3.1)) that

\[
    \|x_n - q\| \leq \max\{\|x_1 - q\|, \|u - q\|\}
\]

(3.2)

for all \( n \in \mathbb{N} \); hence \( \{x_n\}_{n \geq 1} \) and \( \{Sx_n\}_{n \geq 1} \) are bounded. This implies that for some \( M_0 > 0 \),

\[
    \|x_{n+1} - Sx_n\| = \alpha_n \|u - Sx_n\| \leq \alpha_n M_0 \to 0 \quad \text{as} \quad n \to \infty.
\]

Moreover, from (3.1) we obtain that

\[
    \|x_{n+1} - x_n\| = \|\alpha_n u + (1 - \alpha_n)Sx_n - \alpha_{n-1} u - (1 - \alpha_{n-1}) Sx_{n-1}\|
    = \|((\alpha_n - \alpha_{n-1}) u - Sx_{n-1}) + (1 - \alpha_n) (Sx_n - Sx_{n-1})\|
    \leq (1 - \alpha_n) \|x_n - x_{n-1}\| + \alpha_n \|\alpha_n - \alpha_{n-1}\| M_0.
\]

(3.4)

This results in the following two cases.
Case 1. Condition (iii) is satisfied. In this case, \( \|x_{n+1} - x_n\| \leq (1 - \alpha_n)\|x_n - x_{n-1}\| + \sigma_n \), where 
\( \sigma_n = \alpha_n \beta_n ; \beta_n = (|\alpha_n - \alpha_{n-1}|M_0)/\alpha_n \) so that \( \sigma_n = o(\alpha_n) \) (since \( \lim_{n \to \infty} |\alpha_n - \alpha_{n-1}|/\alpha_n = 0 \)).

Case 2. Condition (iii)' is satisfied. In this case, \( \|x_{n+1} - x_n\| \leq (1 - \alpha_n)\|x_n - x_{n-1}\| + \sigma_n \), where 
\( \sigma_n = |\alpha_n - \alpha_{n-1}|M_0 \), so that \( \sum_{n=0}^{\infty} \sigma_n < \infty \).

In either case, we obtain (by Lemma 2.1) that \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \). This implies that 
\( \lim_{n \to \infty} \|x_n - Sx_n\| = 0 \) (since \( \|x_n - Sx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - Sx_n\| \to 0 \) as \( n \to \infty \)). For all \( t \in (0, 1) \), define the mapping \( G_t : E \to E \) by

\[
G_t x := tu + (1 - t)Sx, \quad x \in E.
\]

It is easy to see that \( G_t \) is a contraction for each \( t \in (0, 1) \), and so has for each \( t \in (0, 1) \) a unique fixed point \( z_t \in C \); using Theorem 2.6, we have that \( z_t \to z^* \in F(S) \) as \( t \to 0 \). Now,

\[
z_t - x_n = t(u - x_n) + (1 - t)(Sx_t - x_n).
\]

So, by Lemma 2.2 we have that

\[
\|z_t - x_n\|^2 \leq (1 - t)^2\|Sx_t - x_n\|^2 + 2t\langle u - x_n, j(z_t - x_n) \rangle \\
\leq (1 - t)^2\|Sx_t - Sx_n\|^2 + 2\|Sx_t - x_n\|^2 + 2\|z_t - x_n\|^2 + 2\langle u - z_t, j(z_t - x_n) \rangle \\
\leq (1 + 2t^2)\|z_t - x_n\|^2 + 2t\langle u - z_t, j(z_t - x_n) \rangle + \|Sx_t - x_n\|\|z_t - x_n\| + \|Sx_t - x_n\|.
\]

This implies that

\[
\langle u - z_t, j(x_n - z_t) \rangle \leq \left( \frac{t}{2} + \frac{\|Sx_n - x_n\|}{2t} \right)M,
\]

for some \( M > 0 \). Thus,

\[
\limsup_{n \to \infty} \langle u - z_t, j(x_n - z_t) \rangle \leq \frac{t}{2}M.
\]

Moreover, we have that

\[
\langle u - z_t, j(x_n - z_t) \rangle = \langle u - z, j(x_n - z) \rangle + \langle u - z, j(x_n - z_t) - j(x_n - z) \rangle + \langle z - z_t, j(x_n - z_t) \rangle
\]

(3.10)

Thus, since \( \{x_n\}_{n \geq 1} \) is bounded, we have that \( \langle z^* - z_t, j(x_n - z_t) \rangle \to 0 \) as \( t \to 0 \). Also, \( \langle u - z^*, j(x_n - z_t) - j(x_n - z^*) \rangle \to 0 \) as \( t \to 0 \) since the normalized duality mapping \( j \) is norm-to-
weak* uniformly continuous on bounded subsets of \( E \). Thus as \( t \to 0 \), we obtain from (3.9) and (3.10) that

\[
\limsup_{n \to \infty} \langle u - z^*, j(x_n - z^*) \rangle \leq 0.
\]

(3.11)
Now, put

\[ \mu_n := \max \{ 0, \langle u - z^*, j(x_n - z^*) \rangle \}. \]  

(3.12)

Then, \( 0 \leq \mu_n \) for all \( n \geq 0 \). It is easy to see that \( \mu_n \to 0 \) as \( n \to \infty \) since by (3.11), if \( \varepsilon > 0 \) is given, there exists \( n_\varepsilon \in \mathbb{N} \) such that \( \langle u - z^*, j(x_n - z^*) \rangle < \varepsilon \) for all \( n \geq n_\varepsilon \). Thus, \( 0 \leq \mu_n < \varepsilon \) for all \( n \geq n_\varepsilon \).

So, \( \lim_{n \to \infty} \mu_n = 0 \).

Next, we obtain from the recursion formula (3.1) that

\[ x_{n+1} - z^* = \alpha_n (u - z^*) + (1 - \alpha_n) (S x_n - z^*). \]  

(3.13)

It follows that

\[
\|x_{n+1} - z^*\|^2 \leq (1 - \alpha_n)^2 \|S x_n - z^*\|^2 + 2\alpha_n \langle u - z^*, j(x_{n+1} - z^*) \rangle \\
\quad \leq (1 - \alpha_n) \|x_n - z^*\|^2 + 2\alpha_n \mu_{n+1} \\
= (1 - \alpha_n) \|x_n - z^*\|^2 + \gamma_n, 
\]

(3.14)

where \( \gamma_n = 2\alpha_n \mu_{n+1} \). Therefore, \( \gamma_n = o(\alpha_n) \) and by Lemma 2.1, we obtain that \( \{x_n\}_{n \geq 1} \) converges strongly to \( z^* \in F(S) \). But \( F(S) = \bigcap_{k=1}^{\infty} F(J_{A_k}) = \bigcap_{k=1}^{\infty} N(A_k) = N' \). Hence, \( \{x_n\}_{n \geq 1} \) converges strongly to the common zero of the family \( \{A_k\}_{k \in \mathbb{N}} \) of \( m \)-accretive operators. This completes the proof.

**Corollary 3.2.** Let \( C \) be a closed convex nonempty subset of a real reflexive and strictly convex Banach space \( E \) which has a uniformly Gâteaux differentiable norm. Let \( A_k : C \to E, \ k = 1, 2, \ldots, r \) be a finite family of \( m \)-accretive mappings such that \( N' = \bigcap_{k=1}^{\infty} N(A_k) \neq \emptyset \). Suppose that every bounded closed convex nonempty subset of \( C \) has the fixed point property for nonexpansive mappings. For arbitrary \( u, x_1 \in C \), let \( \{x_n\}_{n \geq 1} \) be iteratively generated by

\[ x_{n+1} = \alpha_n u + (1 - \alpha_n) S x_n, \quad n \geq 1, \]  

(3.15)

where \( S = \sum_{k=1}^{r} \alpha_k J_{A_k}; J_{A_k} = (I + A_k)^{-1}; \{\alpha_k\}_{k=1}^{r} \) is a finite collection of positive real numbers such that \( \sum_{k=1}^{r} \alpha_k = 1 \). Then, \( \{x_n\}_{n \geq 1} \) converges strongly to a common zero of \( \{A_k\}_{k=1}^{r} \).

**Proof.** The mapping \( S = \sum_{k=1}^{r} \alpha_k J_{A_k} \) is clearly nonexpansive. Following the argument of the proof of Lemma 2.3 we get that \( F(S) = \bigcap_{k=1}^{r} F(J_{A_k}) \). The rest follows from Theorem 3.1. This completes the proof.

**Remark 3.3.** If, in particular, we consider a single \( m \)-accretive operator \( A \), the requirement that \( E \) be strictly convex will be dispensed, in this case, with \( r = 1 \) and \( S \) in Corollary 3.2 coincides with \( J_A = (I + A)^{-1} \).

**Remark 3.4.** We note that if \( E \) is smooth, then \( E \) is reflexive and has a uniformly Gâteaux differentiable norm and with property that every bounded closed convex nonempty subset of \( E \) has the fixed point property for nonexpansive mappings (see e.g., [1]).
Thus, we have the following corollary.

**Corollary 3.5.** Let \( C \) be a closed convex nonempty subset of a real uniformly smooth Banach space \( E \). Let \( A : C \to E \) be an \( m \)-accretive operator with \( N(A) \neq \emptyset \). For arbitrary \( u, x_1 \in C \), let the sequence \( \{x_n\}_{n \geq 1} \) be iteratively generated by

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n) J_A x_n, \quad n \geq 1,
\]

where \( J_A = (I + A)^{-1} \). Then \( \{x_n\}_{n \geq 1} \) converges strongly to some \( x^* \in N(A) \).

**Remark 3.6.** If in Theorem 3.1 we consider \( C = E \), then the condition that \( A_k \) is \( m \)-accretive for each \( k \in \mathbb{N} \) could be replaced with the continuity of each \( A_k \).

Thus, we have the following theorem.

**Theorem 3.7.** Let \( E \) be a real reflexive and strictly convex Banach space which has a uniformly Gâteaux differentiable norm. Let \( A_k : E \to E \), \( k \in \mathbb{N} \) be a countably infinite family of continuous accretive operators such that \( N' = \bigcap_{k=1}^{\infty} N(A_k) \neq \emptyset \). Suppose that every bounded closed convex nonempty subset of \( E \) has the fixed point property for nonexpansive mappings. For arbitrary \( u, x_1 \in E \), let \( \{x_n\}_{n \geq 1} \) be iteratively generated by

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n) S x_n, \quad n \geq 1,
\]

where \( S = \sum_{k=1}^{\infty} \alpha_k J_{A_k} \), \( J_{A_k} = (I + A_k)^{-1} \). Then, \( \{x_n\}_{n \geq 1} \) converges strongly to a common zero of \( \{A_k\}_{k \in \mathbb{N}} \).

**Proof.** By Theorem 2.5, we have that \( A_k \) is \( m \)-accretive for each \( k \in \mathbb{N} \). The rest follows from Theorem 3.1. \( \square \)

**Corollary 3.8.** Let \( E \) be a real reflexive and strictly convex Banach space which has a uniformly Gâteaux differentiable norm. Let \( A_k : E \to E \), \( k = 1, 2, \ldots, r \) be a finite family of continuous accretive operators such that \( N' = \bigcap_{k=1}^{r} N(A_k) \neq \emptyset \). Suppose that every bounded closed convex nonempty subset of \( E \) has the fixed point property for nonexpansive mappings. For arbitrary \( u, x_1 \in E \), let \( \{x_n\}_{n \geq 1} \) be iteratively generated by

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n) S x_n, \quad n \geq 1,
\]

where \( S = \sum_{k=1}^{r} \alpha_k J_{A_k} \), \( J_{A_k} = (I + A_k)^{-1} \), where \( \{\alpha_k\}_{k=1}^{r} \) is a finite collection of positive real numbers such that \( \sum_{k=1}^{r} \alpha_k = 1 \). Then, \( \{x_n\}_{n \geq 1} \) converges strongly to a common zero of \( \{A_k\}_{k=1}^{r} \).

### 3.2. Strong convergence theorem for countably infinite family of pseudocontractive mappings

**Theorem 3.9.** Let \( C \) be a closed convex nonempty subset of a real reflexive and strictly convex Banach space \( E \) which has a uniformly Gâteaux differentiable norm. Let \( T_k : C \to E \), \( k \in \mathbb{N} \) be a countably infinite family of pseudocontractive mappings such that for each \( k \in \mathbb{N} \), \( (I - T_k) \) is \( m \)-accretive on \( C \) and \( F' = \bigcap_{k=1}^{\infty} F(T_k) \neq \emptyset \). Let \( J_{T_k} = (I + (I - T_k)^{-1}) = (2I - T_k)^{-1} \) for each \( k \in \mathbb{N} \). Suppose that every
bounded closed convex nonempty subset of \( C \) has the fixed-point property for nonexpansive mappings. For arbitrary \( u, x_1 \in C \), let \( \{x_n\}_{n \geq 1} \) be iteratively generated by

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 1,
\]

where \( T = \sum_{k=1}^{\infty} \xi_k J_{T_k} \). Then, \( \{x_n\}_{n \geq 1} \) converges strongly to a common fixed point of \( \{T_k\}_{k \in \mathbb{N}} \).

\[\tag{3.19}\]

Proof. Put \( A_k := (I - T_k) \) for each \( k \in \mathbb{N} \). It is then obvious that \( N(A_k) = F(T_k) \) and hence \( \bigcap_{k=1}^{\infty} N(A_k) = F' = \bigcap_{k=1}^{\infty} F(T_k) \). Besides, \( A_k \) is \( m \)-accretive for each \( k \in \mathbb{N} \). Thus, the proof follows from Theorem 3.1.

\(\square\)

**Corollary 3.10.** Let \( C \) be a closed convex nonempty subset of a real reflexive and strictly convex Banach space \( E \) which has a uniformly Gâteaux differentiable norm. Let \( T_k : C \to E, k = 1, 2, \ldots, r \) be a finite family of pseudocontractive mappings such that for each \( k = 1, 2, \ldots, r \), \( (I - T_k) \) is \( m \)-accretive on \( C \) and \( F = \bigcap_{k=1}^{r} F(T_k) \neq \emptyset \). Let \( J_{T_k} = (I + (I - T_k)^{-1}) = (2I - T_k)^{-1} \). Suppose that every nonempty closed convex subset of \( C \) has the fixed-point property for nonexpansive mappings. For arbitrary \( u, x_1 \in C \), let \( \{x_n\}_{n \geq 1} \) be iteratively generated by

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 1,
\]

where \( T = \sum_{k=1}^{r} \alpha_k J_{T_k} \) and \( \{\alpha_k\}_{k=1}^{r} \) is a finite collection of positive numbers such that \( \sum_{k=1}^{r} \alpha_k = 1 \). Then, \( \{x_n\}_{n \geq 1} \) converges strongly to a common fixed point of \( \{T_k\}_{k=1}^{r} \).

**Corollary 3.11.** Let \( C \) be a nonempty closed convex subset of a real uniformly smooth Banach space \( E \). Let \( T : C \to E \) be pseudocontractive mappings such that \( (I - T) \) is \( m \)-accretive on \( C \) and \( F(T) \neq \emptyset \). Let \( J_T = (I + (I - T)^{-1}) = (2I - T)^{-1} \). For arbitrary \( u, x_1 \in C \), let \( \{x_n\}_{n \geq 1} \) be iteratively generated by

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n) J_T x_n, \quad n \geq 1.
\]

Then, \( \{x_n\}_{n \geq 1} \) converges strongly to a fixed point of \( T \).

**Theorem 3.12.** Let \( E \) be a real reflexive and strictly convex Banach space \( E \) which has a uniformly Gâteaux differentiable norm. Let \( T_k : E \to E, k \in \mathbb{N} \) be a countably infinite family of continuous pseudocontractive mappings such that \( F = \bigcap_{k=1}^{\infty} F(T_k) \neq \emptyset \). Let \( J_{T_k} = (I + (I - T_k)^{-1}) = (2I - T_k)^{-1} \) for each \( k \in \mathbb{N} \). Suppose that every bounded closed convex nonempty subset of \( E \) has the fixed-point property for nonexpansive mappings. For arbitrary \( u, x_1 \in E \), let \( \{x_n\}_{n \geq 1} \) be iteratively generated by

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 1,
\]

where \( T = \sum_{k=1}^{\infty} \xi_k J_{T_k} \). Then, \( \{x_n\}_{n \geq 1} \) converges strongly to a common fixed point of \( \{T_k\}_{k \in \mathbb{N}} \).

\[\tag{3.22}\]

Proof. The proof follows from Theorem 3.9.

\(\square\)

**Corollary 3.13.** Let \( E \) be a real reflexive and strictly convex Banach space \( E \) which has a uniformly Gâteaux differentiable norm. Let \( T_k : E \to E, k = 1, 2, \ldots, r \) be a finite family of continuous pseudocontractive mappings such that \( F = \bigcap_{k=1}^{r} F(T_k) \neq \emptyset \). Let \( J_{T_k} = (I + (I - T_k)^{-1}) = (2I - T_k)^{-1} \) for each \( k = 1, 2, \ldots, r \). Suppose that every bounded closed convex nonempty subset of \( E \) has the fixed-point property for nonexpansive mappings. For arbitrary \( u, x_1 \in E \), let \( \{x_n\}_{n \geq 1} \) be iteratively generated by

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 1,
\]

where \( T = \sum_{k=1}^{r} \alpha_k J_{T_k} \). Then, \( \{x_n\}_{n \geq 1} \) converges strongly to a common fixed point of \( \{T_k\}_{k=1}^{r} \).

\[\tag{3.23}\]
where \( T = \sum_{k=1}^r \alpha_k J_{T_k}; \ J_{T_k} = (I + T_k)^{-1}, \) where \( \{\alpha_k\}_{k=1}^r \) is a finite collection of positive numbers such that \( \sum_{k=1}^r \alpha_k = 1. \) Then, \( \{x_n\}_{n \geq 1} \) converges strongly to a common fixed point of \( \{T_k\}_{k=1}^r. \)

**Corollary 3.14.** Let \( E \) be a real uniformly smooth Banach space. Let \( T: E \to E \) be continuous pseudo-contractive mappings such that \( F(T) \neq \emptyset. \) Let \( J_T = (I + (I - T)^{-1}) = (2I - T)^{-1}. \) For arbitrary \( u, x_1 \in E, \) let \( \{x_n\}_{n \geq 1} \) be iteratively generated by

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n) J_T x_n, \quad n \geq 1.
\]

Then, \( \{x_n\}_{n \geq 1} \) converges strongly to fixed point of \( T. \)

**Remark 3.15.** A prototype for the sequence \( \{\alpha_n\}_{n \geq 1} \) satisfying the conditions on our iteration parameter is the sequence \( \{1/(n + 1)\}_{n \geq 1}. \) We note that conditions (iii) and (iii)' are not comparable, since (e.g.) the sequence \( \{\beta_n\}_{n \geq 1} \) given by

\[
\beta_n = \begin{cases} 
\frac{1}{\sqrt{n}}, & \text{if } n \text{ is odd} \\
\frac{1}{\sqrt{n} - 1}, & \text{if } n \text{ is even}
\end{cases}
\]

satisfies (iii) but does not satisfy (iii)' (see e.g., [33]).

**Remark 3.16.** The addition of bounded error terms to our recursion formulas leads to no further generalization.

**Remark 3.17.** If \( f: K \to K \) is a contraction mapping and we replace \( u \) by \( f(x_n) \) in the recursion formulas of our theorems, we obtain what some authors now call viscosity iteration process. We observe that all our theorems in this paper carry over trivially to the so-called viscosity process. One simply replaces \( u \) by \( f(x_n), \) repeats the argument of this paper, using the fact that \( f \) is a contraction map.

**Remark 3.18.** Our theorems improve, extend, and generalize the corresponding results of Zegye and Shahzad [32] and that of a host of other authors from approximation of a common zero (common fixed point) of a finite family of accretive (pseudocontractive) operators to approximation of a common zero (common fixed point) of a countably infinite family of accretive (pseudocontractive) operators. Furthermore, Theorem 3.12 extends the corresponding results of Liu et al. [19], Maiti and Saha [9], Senter and Dotson [20], Jung [17] from approximation of a common fixed point of a finite family of nonexpansive mappings to the approximation of common fixed points of a countably infinite family of continuous pseudocontractive mappings, without assuming that our operators satisfy the so-called condition \( A. \) Our theorems are applicable, in particular, in \( L_p \) spaces, \( 1 < p < +\infty. \)

**References**


