Research Article

Common Fixed Point Theorem in Partially Ordered \( L \)-Fuzzy Metric Spaces

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We introduce partially ordered \( L \)-fuzzy metric spaces and prove a common fixed point theorem in these spaces.

1. Introduction

The Banach fixed point theorem for contraction mappings has been generalized and extended in many directions [1–43]. Recently Nieto and Rodríguez-López [27–29] and Ran and Reurings [33] presented some new results for contractions in partially ordered metric spaces. The main idea in [27–33] involves combining the ideas of iterative technique in the contraction mapping principle with those in the monotone technique.

Recall that if \((X,\leq)\) is a partially ordered set and \(F : X \to X\) is such that for \(x, y \in X, x \leq y\) implies \(F(x) \leq F(y)\), then a mapping \(F\) is said to be nondecreasing. The main result of Nieto and Rodríguez-López [27–33] and Ran and Reurings [33] is the following fixed point theorem.

**Theorem 1.1.** Let \((X,\leq)\) be a partially ordered set and suppose that there is a metric \(d\) on \(X\) such that \((X,d)\) is a complete metric space. Suppose that \(F\) is a nondecreasing mapping with

\[
d(F(x), F(y)) \leq kd(x, y)
\]

for all \(x, y \in X, x \leq y\), where \(0 < k < 1\). Also suppose the following.
2. Preliminaries

The notion of fuzzy sets was introduced by Zadeh [44]. Various concepts of fuzzy metric spaces were considered in [15, 16, 22, 45]. Many authors have studied fixed point theory in fuzzy metric spaces; see, for example, [7, 8, 25, 26, 39, 46–48]. In the sequel, we will adopt the usual terminology, notation, and conventions of \( \mathcal{L} \)-fuzzy metric spaces introduced by Saadati et al. [36] which are a generalization of fuzzy metric spaces [49] and intuitionistic fuzzy metric spaces [32, 37].

**Definition 2.1** (see [46]). Let \( \mathcal{L} = (L, \leq_L) \) be a complete lattice, and \( U \) a nonempty set called a universe. An \( \mathcal{L} \)-fuzzy set \( \mathcal{A} \) on \( U \) is defined as a mapping \( \mathcal{A} : U \to L \). For each \( u \in U \), \( \mathcal{A}(u) \) represents the degree (in \( L \)) to which \( u \) satisfies \( \mathcal{A} \).

**Lemma 2.2** (see [13, 14]). Consider the set \( L^* \) and the operation \( \leq_{L^*} \) defined by

\[
L^* = \left\{ (x_1, x_2) : (x_1, x_2) \in [0, 1]^2, \ x_1 + x_2 \leq 1 \right\},
\]

\((x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1 \text{ and } x_2 \geq y_2, \text{ for every } (x_1, x_2), (y_1, y_2) \in L^* \). Then \( (L^*, \leq_{L^*}) \) is a complete lattice.

Classically, a triangular norm \( T \) on \((0, 1], \leq)\) is defined as an increasing, commutative, associative mapping \( T : [0, 1]^2 \to [0, 1] \) satisfying \( T(1, x) = x \), for all \( x \in [0, 1] \). These definitions can be straightforwardly extended to any lattice \( \mathcal{L} = (L, \leq_L) \). Define first \( 0_\mathcal{L} = \inf L \) and \( 1_\mathcal{L} = \sup L \).

**Definition 2.3.** A negation on \( \mathcal{L} \) is any strictly decreasing mapping \( \mathcal{N} : L \to L \) satisfying \( \mathcal{N}(0_\mathcal{L}) = 1_\mathcal{L} \) and \( \mathcal{N}(1_\mathcal{L}) = 0_\mathcal{L} \). If \( \mathcal{N}(\mathcal{N}(x)) = x \), for all \( x \in L \), then \( \mathcal{N} \) is called an involutive negation.

In this paper the negation \( \mathcal{N} : L \to L \) is fixed.

**Definition 2.4.** A triangular norm (\( t \)-norm) on \( \mathcal{L} \) is a mapping \( \mathcal{T} : L^2 \to L \) satisfying the following conditions:

(i) (for all \( x \in L \))(\( \mathcal{T}(x, 1_\mathcal{L}) = x \)) (boundary condition);
(ii) (for all \((x, y) \in L^2\))(\(\Upsilon(x, y) = \Upsilon(y, x)\)) (commutativity);

(iii) (for all \((x, y, z) \in L^3\))((\(\Upsilon(x, \Upsilon(y, z)) = \Upsilon(\Upsilon(x, y), z)\)) (associativity);

(iv) (for all \((x', y', y') \in L^4\))(\(x \leq_L x'\) and \(y \leq_L y' \Rightarrow \Upsilon(x, y) \leq_L \Upsilon(x', y')\)) (monotonicity).

A \(t\)-norm \(\Upsilon\) on \(L\) is said to be continuous if for any \(x, y \in L\) and any sequences \(\{x_n\}\) and \(\{y_n\}\) which converge to \(x\) and \(y\) we have

\[
\lim_{n} \Upsilon(x_n, y_n) = \Upsilon(x, y).
\]  

For example, \(\Upsilon(x, y) = \min(x, y)\) and \(\Upsilon(x, y) = xy\) are two continuous \(t\)-norms on \([0, 1]\). A \(t\)-norm can also be defined recursively as an \((n + 1)\)-ary operation \((n \in \mathbb{N})\) by \(\Upsilon^1 = \Upsilon\) and

\[
\Upsilon^n(x_1, \ldots, x_{n+1}) = \Upsilon(\Upsilon^{n-1}(x_1, \ldots, x_n), x_{n+1})
\]

for \(n \geq 2\) and \(x_i \in L\).

A \(t\)-norm \(\Upsilon\) is said to be of Hadžić type if the family \(\{\Upsilon^n\}_{n \in \mathbb{N}}\) is equicontinuous at \(x = 1_L\), that is,

\[
\forall \varepsilon \in L \setminus \{0_L, 1_L\} \exists \delta \in L \setminus \{0_L, 1_L\} : a >_L \mathcal{N}(\delta) \Rightarrow \Upsilon^n(a) >_L \mathcal{N}(\varepsilon) \quad (n \geq 1).
\]  

\(\mathcal{T}_M\) is a trivial example of a \(t\)-norm of Hadžić type, but there exist \(t\)-norms of Hadžić type weaker than \(\mathcal{T}_M\) [50] where

\[
\mathcal{T}_M(x, y) = \begin{cases} x, & \text{if } x \leq_L y, \\ y, & \text{if } y \leq_L x. \end{cases}
\]

**Definition 2.5.** The 3-tuple \((X, \mathcal{M}, \Upsilon)\) is said to be an \(L\)-fuzzy metric space if \(X\) is an arbitrary (nonempty) set, \(\Upsilon\) is a continuous \(t\)-norm on \(L\) and \(\mathcal{M}\) is an \(L\)-fuzzy set on \(X^2 \times ]0, +\infty[\) satisfying the following conditions for every \(x, y, z\) in \(X\) and \(t, s\) in \([0, +\infty[\):

(a) \(\mathcal{M}(x, y, t) >_L 0_L\);

(b) \(\mathcal{M}(x, y, t) = 1_L\) for all \(t > 0\) if and only if \(x = y\);

(c) \(\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)\);

(d) \(\Upsilon(\mathcal{M}(x, y, t), \mathcal{M}(y, z, s)) \leq_L \mathcal{M}(x, z, t + s)\);

(e) \(\mathcal{M}(x, y, \cdot) : ]0, \infty[ \rightarrow L\) is continuous.
If the $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \tau)$ satisfies the condition:

$$
(f) \lim_{t \rightarrow \infty} \mathcal{M}(x, y, t) = 1_{\mathcal{L}},
$$

(2.6)

then $(X, \mathcal{M}, \tau)$ is said to be Menger $\mathcal{L}$-fuzzy metric space or for short a $\mathbf{M}_{\mathcal{L}}$-fuzzy metric space.

Let $(X, \mathcal{M}, \tau)$ be an $\mathcal{L}$-fuzzy metric space. For $t \in ]0, +\infty[,$ we define the open ball $B(x, r, t)$ with center $x \in X$ and radius $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, as

$$
B(x, r, t) = \{y \in X : \mathcal{M}(x, y, t) > 1_{\mathcal{M}}(r)\}.
$$

(2.7)

A subset $A \subseteq X$ is called open if for each $x \in A$, there exist $t > 0$ and $r \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $B(x, r, t) \subseteq A$. Let $\tau_{\mathcal{M}}$ denote the family of all open subsets of $X$. Then $\tau_{\mathcal{M}}$ is called the topology induced by the $\mathcal{L}$-fuzzy metric $\mathcal{M}$.

Example 2.6 (see [38]). Let $(X, d)$ be a metric space. Denote $\mathcal{C}(a, b) = (a_1b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $L^*$ and let $M$ and $N$ be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$
\mathcal{M}_{M,N}(x, y, t) = \left(\frac{t}{t + d(x, y)}, \frac{d(x, y)}{t + d(x, y)}\right).
$$

(2.8)

Then $(X, \mathcal{M}_{M,N}, \tau)$ is an intuitionistic fuzzy metric space.

Example 2.7. Let $X = \mathbb{N}$. Define $\mathcal{C}(a, b) = (\max(0, a_1 + b_1 - 1), a_2 + b_2 - a_2b_2)$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $L^*$, and let $\mathcal{M}(x, y, t)$ on $X^2 \times (0, \infty)$ be defined as follows:

$$
\mathcal{M}(x, y, t) = \begin{cases} 
\left(\frac{x}{y}, \frac{y-x}{y}\right) & \text{if } x \leq y, \\
\left(\frac{y}{x}, \frac{x-y}{x}\right) & \text{if } y \leq x
\end{cases}
$$

(2.9)

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}, \tau)$ is an $\mathcal{L}$-fuzzy metric space.

Lemma 2.8 (see [49]). Let $(X, \mathcal{M}, \tau)$ be an $\mathcal{L}$-fuzzy metric space. Then, $\mathcal{M}(x, y, t)$ is nondecreasing with respect to $t$, for all $x, y$ in $X$.

Definition 2.9. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \tau)$ is called a Cauchy sequence, if for each $\varepsilon \in L \setminus \{0_{\mathcal{L}}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m \geq n \geq n_0$ $(n \geq m \geq n_0)$,

$$
\mathcal{M}(x_m, x_n, t) > 1_{\mathcal{M}}(\varepsilon).
$$

(2.10)

The sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to be convergent to $x \in X$ in the $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \tau)$ (denoted by $x_n \xrightarrow{\mathcal{M}} x$) if $\mathcal{M}(x_n, x, t) = \mathcal{M}(x, x_n, t) \rightarrow 1_{\mathcal{L}}$ whenever $n \rightarrow +\infty$ for every $t > 0$. A $\mathcal{L}$-fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.
Definition 2.10. Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. $\mathcal{M}$ is said to be continuous on $X \times X \times ]0, \infty[$ if

$$
\lim_{n \to \infty} \mathcal{M}(x_n, y_n, t_n) = \mathcal{M}(x, y, t) \quad (2.11)
$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X \times X \times ]0, \infty[$ converges to a point $(x, y, t) \in X \times X \times ]0, \infty[$, that is, $\lim_n \mathcal{M}(x_n, x, t) = \lim_n \mathcal{M}(y_n, y, t) = 1_\mathcal{L}$ and $\lim_n \mathcal{M}(x, y, t_n) = \mathcal{M}(x, y, t)$.

Lemma 2.11. Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then $\mathcal{M}$ is continuous function on $X \times X \times ]0, \infty[$.

Proof. The proof is the same as that for fuzzy spaces (see [35, Proposition 1]). \qed

Lemma 2.12. If an $\mathcal{M}_\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ satisfies the following condition:

$$
\mathcal{M}(x, y, t) = C, \quad \forall t > 0,
$$

then one has $C = 1_\mathcal{L}$ and $x = y$.

Proof. Let $\mathcal{M}(x, y, t) = C$ for all $t > 0$. Then by (f) of Definition 2.5, we have $C = 1_\mathcal{L}$ and by (b) of Definition 2.5, we conclude that $x = y$. \qed

Lemma 2.13 (see [50]). Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{M}_\mathcal{L}$-fuzzy metric space in which $\mathcal{T}$ is Hadzic' type. Suppose

$$
\mathcal{M}(x_n, x_{n+1}, t) \geq \mathcal{M}\left(x_0, x_1, \frac{t}{k^n}\right) \quad (2.13)
$$

for some $0 < k < 1$ and $n \in \mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence.

3. Main Results

Definition 3.1. Suppose that $(X, \preceq)$ is a partially ordered set and $F, h : X \to X$ are mappings of $X$ into itself. We say that $F$ is $h$-nondecreasing if for $x, y \in X$,

$$
h(x) \preceq h(y) \quad \text{implies} \quad F(x) \preceq F(y). \quad (3.1)
$$

Now we present the main result in this paper.

Theorem 3.2. Let $(X, \preceq)$ be a partially ordered set and suppose that there is an $\mathcal{L}$-fuzzy metric $\mathcal{M}$ on $X$ such that $(X, \mathcal{M}, \mathcal{T})$ is a complete $\mathcal{M}_\mathcal{L}$-fuzzy metric space in which $\mathcal{T}$ is Hadzic' type. Let $F, h : X \to X$ be two self-mappings of $X$ such that there exist $k \in (0,1)$ and $q \in (0,1)$ such that
$F(X) \subseteq h(X)$, $F$ is a $h$-nondecreasing mapping and

$$\mathcal{M}(F(x), F(y), kt) \geq \mathcal{C}_M \{ \mathcal{M}(h(x), h(y), t), \mathcal{M}(h(x), \mathcal{M}(F(x), t), \mathcal{M}(h(y), F(y), t),$$

$$\mathcal{M}(h(x), F(y), (1 + q)t), \mathcal{M}(h(y), F(x), (1 - q)t) \} \quad (3.2)$$

for all $x, y \in X$ for which $h(x) \leq h(y)$ and all $t > 0$.

Also suppose that

if $\{h(x_n)\} \subset X$ is a nondecreasing sequence with $h(x_n) \rightarrow h(z)$ in $h(X)$,

then $h(z) \leq h(h(z))$ and $h(x_n) \leq h(z)$ $\forall n$ hold. \quad (3.3)

Also suppose that $h(X)$ is closed. If there exists an $x_0 \in X$ with $h(x_0) \leq F(x_0)$, then $F$ and $h$ have a coincidence. Further, if $F$ and $h$ commute at their coincidence points, then $F$ and $h$ have a common fixed point.

Proof. Let $x_0 \in X$ be such that $h(x_0) \leq F(x_0)$. Since $F(X) \subseteq h(X)$, we can choose $x_1 \in X$ such that $h(x_1) = F(x_0)$. Again from $F(X) \subseteq h(X)$ we can choose $x_2 \in X$ such that $h(x_2) = F(x_1)$. Continuing this process we can choose a sequence $\{x_n\}$ in $X$ such that

$h(x_{n+1}) = F(x_n)$ $\forall n \geq 0$. \quad (3.4)

Since $h(x_0) \leq F(x_0)$ and $h(x_1) = F(x_0)$, we have $h(x_0) \leq h(x_1)$. Then from (3.1),

$$F(x_0) \leq F(x_1), \quad (3.5)$$

that is, by (3.4), $h(x_1) \leq h(x_2)$. Again from (3.1),

$$F(x_1) \leq F(x_2), \quad (3.6)$$

that is, $h(x_2) \leq h(x_3)$. Continuing we obtain

$$F(x_0) \leq F(x_1) \leq F(x_2) \leq F(x_3) \leq \cdots \leq F(x_n) \leq F(x_{n+1}) \leq \cdots. \quad (3.7)$$

Now we will show that a sequence $\{\mathcal{M}(F(x_n), F(x_{n+1}), t)\}$ converges to $1_{\mathcal{L}}$ for each $t > 0$. If $\mathcal{M}(F(x_n), F(x_{n+1}), t) = 1_{\mathcal{L}}$ for some $n$ and for each $t > 0$, then it is easily to show that $\mathcal{M}(F(x_{n+k}), F(x_{n+k+1}), t) = 1_{\mathcal{L}}$ for all $k \geq 0$. So we suppose that $\mathcal{M}(F(x_n), F(x_{n+1}), t) < 1_{\mathcal{L}}$ for all $n$. We show that for each $t > 0$,

$$\mathcal{M}(F(x_n), F(x_{n+1}), kt) \geq \mathcal{M}(F(x_n-1), F(x_n), t) \quad \forall n \geq 1. \quad (3.8)$$
Since from (3.4) and (3.7) we have $h(x_{n-1}) \leq h(x_n)$, from (3.1) with $x = x_n$ and $y = x_{n+1}$,

$$\mathcal{M}(F(x), F(x_{n+1}), k) \geq L \mathcal{M} \{ \mathcal{M}(h(x), h(x_{n+1}), t), \mathcal{M}(h(x), F(x_{n+1}), t), \mathcal{M}(h(x_{n+1}), F(x_{n+1}), t), \mathcal{M}(h(x_{n+1}), F(x), (1 - q)t) \}. \quad (3.9)$$

So by (3.4),

$$\mathcal{M}(F(x), F(x_{n+1}), k) \geq L \mathcal{M} \{ \mathcal{M}(F(x), F(x_{n+1}), (1 + q)t), \mathcal{M}(F(x), F(x_{n+1}), t), \mathcal{M}(F(x_{n+1}), F(x_{n+1}), (1 + q)t), 1_L \}. \quad (3.10)$$

Since by (d) of Definition 2.5

$$\mathcal{M}(F(x_{n-1}), F(x_{n+1}), (1 + q)t) \geq L \mathcal{M} \{ \mathcal{M}(F(x_{n-1}), F(x_{n+1}), t), \mathcal{M}(F(x_{n-1}), F(x_{n+1}), q t) \}, \quad (3.11)$$

we have

$$\mathcal{M}(F(x), F(x_{n+1}), k) \geq L \mathcal{M} \{ \mathcal{M}(F(x), F(x_{n+1}), t), \mathcal{M}(F(x_{n+1}), F(x_{n+1}), (1 + q)t), \mathcal{M}(F(x), F(x_{n+1}), q t) \}. \quad (3.12)$$

As $t$-norm is continuous, letting $q \to 1_L$ we get

$$\mathcal{M}(F(x), F(x_{n+1}), k) \geq L \mathcal{M} \{ \mathcal{M}(F(x), F(x_{n+1}), t), \mathcal{M}(F(x_{n-1}), F(x_{n+1}), t) \}. \quad (3.13)$$

Consequently,

$$\mathcal{M}(F(x), F(x_{n+1}), t) \geq L \mathcal{M} \left\{ \mathcal{M} \left( F(x_{n-1}), F(x_{n+1}), \frac{1}{k} t \right), \mathcal{M} \left( F(x_{n+1}), F(x_{n+1}), \frac{1}{k} t \right) \right\}. \quad (3.14)$$

By repeating the above inequality, we obtain

$$\mathcal{M}(F(x), F(x_{n+1}), t) \geq L \mathcal{M} \left\{ \mathcal{M} \left( F(x_{n-1}), F(x_{n+1}), \frac{1}{k^p} t \right), \mathcal{M} \left( F(x_{n+1}), F(x_{n+1}), \frac{1}{k^p} t \right) \right\}. \quad (3.15)$$

Since $\mathcal{M}(F(x), F(x_{n+1}), (1/k^p)t) \to 1_L$ as $p \to \infty$, it follows that

$$\mathcal{M}(F(x), F(x_{n+1}), t) \geq L \mathcal{M} \left( F(x_{n-1}), F(x_{n+1}), \frac{1}{k^p} t \right). \quad (3.16)$$

Thus we proved (3.7). By repeating the above inequality (3.7), we get

$$\mathcal{M}(F(x), F(x_{n+1}), t) \geq L \mathcal{M} \left( F(x_{n}), F(x_{n+1}), \frac{1}{k^n} t \right). \quad (3.17)$$
Since $\mathcal{M}(x, y, t) \to 1_{\mathcal{L}}$ as $t \to +\infty$ and $k < 1$, letting $n \to \infty$ in (3.17) we get

$$\lim_{n \to \infty} \mathcal{M}(F(x_n), F(x_{n+1}), t) = 1_{\mathcal{L}} \quad \text{for each } t > 0. \quad (3.18)$$

Now we will prove that $\{F(x_n)\}$ is a Cauchy sequence which means that for every $\delta > 0$ and $\epsilon \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ there exists $n(\delta, \epsilon) \in \mathbb{N}$ such that

$$M(F(x_n), F(x_{n+p}), \delta) > L \cdot \mathcal{M}(\epsilon) \quad \text{for every } n \geq n(\delta, \epsilon) \text{ and every } p \in \mathbb{N}. \quad (3.19)$$

Let $\epsilon \in L \setminus \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $\delta > 0$ be arbitrary. For any $p \geq 1$ we have

$$\delta = \delta(1 - k)(1 + k + \cdots + k^p + \cdots) > \delta(1 - k)\left(1 + k + \cdots + k^{p-1}\right). \quad (3.20)$$

Since $M(x, y, t)$ is nondecreasing with respect to $t$, for all $x, y$ in $X$,

$$\mathcal{M}(F(x_n), F(x_{n+p}), \delta) \geq L \cdot \mathcal{M}\left(F(x_n), F(x_{n+p}), \delta(1 - k)\left(1 + k^n + \cdots + k^{p-1}\right)\right) \quad (3.21)$$

and hence, by (d) of Definition 2.5,

$$\mathcal{M}(F(x_n), F(x_{n+p}), \delta) \geq L \cdot \mathcal{T}^p \left\{ \mathcal{M}(F(x_n), F(x_{n+1}), (1 - k)\delta), \mathcal{M}(F(x_{n+1}), F(x_{n+2}), (1 - k)\delta k), \ldots, \mathcal{M}(F(x_{n+p-1}), F(x_{n+p}), (1 - k)\delta k^{p-1}) \right\}. \quad (3.22)$$

From (3.17) it follows that

$$\mathcal{M}(F(x_{n+i}), F(x_{n+i+1}), t) \geq L \cdot \mathcal{M}\left(F(x_n), F(x_{n+1}), \frac{t}{k^i}\right) \quad \text{for each } i \geq 1_{\mathcal{L}}. \quad (3.23)$$

From (3.23) with $t = (1 - k)\delta k^i$ we get

$$\mathcal{M}\left(F(x_{n+i}), F(x_{n+i+1}), (1 - k)\delta k^i\right) \geq L \cdot \mathcal{M}(F(x_n), F(x_{n+1}), (1 - k)\delta). \quad (3.24)$$

Thus by (3.22),

$$\mathcal{M}(F(x_n), F(x_{n+p}), \delta) \geq L \cdot \mathcal{T}^p \left\{ \mathcal{M}(F(x_n), F(x_{n+1}), (1 - k)\delta), \mathcal{M}(F(x_{n+1}), F(x_{n+2}), (1 - k)\delta), \ldots, \mathcal{M}(F(x_{n+p-1}), F(x_{n+p}), (1 - k)\delta) \right\}. \quad (3.25)$$

Hence we get

$$\mathcal{M}(F(x_n), F(x_{n+p}), \delta) \geq L \cdot \mathcal{M}(F(x_n), F(x_{n+1}), (1 - k)\delta). \quad (3.26)$$
From (3.26) and (3.17),

$$\mathcal{M}(F(x_n), F(x_{n+p}), \delta) \geq L \mathcal{M}
\left(F(x_0), F(x_1), \frac{(1-k)\delta}{k^n}\right).$$

(3.27)

Hence we conclude, as $\mathcal{M}(x, y, t) \to 1_L$ as $t \to +\infty$ and $k < 1$, that there exists $n(\delta, e) \in \mathbb{N}$ such that

$$\mathcal{M}(F(x_n), F(x_{n+p}), \delta) > 1_L \mathcal{M}(e) \quad \text{for every } n \geq n(\delta, e) \text{ and every } p \in \mathbb{N}.$$  

(3.28)

Thus we proved that $\{F(x_n)\}$ is a Cauchy sequence.

Since $h(X)$ is closed and as $F(x_n) = h(x_{n+1})$, there is some $z \in X$ such that

$$\lim_{n \to -\infty} h(x_n) = h(z).$$

(3.29)

Now we show that $z$ is a coincidence of $F$ and $h$. Since from (3.3) and (3.29) we have $h(x_n) \leq h(z)$ for all $n$, then from (3.2) and by (d) of Definition 2.5 we have

$$\mathcal{M}(F(x_n), F(z), kt) \geq L \mathcal{M}
\{\mathcal{M}(h(x_n), h(z), t), \mathcal{M}(h(x_n), F(x_n), t), \mathcal{M}(h(z), F(z), t),$$

$$\mathcal{M}(h(x_n), F(z), (1+q)t), M(h(z), F(x_n), (1-q)t)\}.$$ 

(3.30)

Letting $n \to \infty$ we get

$$\mathcal{M}(h(z), F(z), kt) \geq L \mathcal{M}
\{\mathcal{M}(h(z), h(z), t), \mathcal{M}(h(z), h(z), t), \mathcal{M}(h(z), F(z), t),$$

$$\mathcal{M}(h(z), F(z), (1+q)t), \mathcal{M}(h(z), h(z), (1-q)t)\} $$

(3.31)

for all $t > 0$. Therefore,

$$\mathcal{M}(h(z), F(z), t) \geq L \mathcal{M}
\left(h(z), F(z), \frac{1}{k}t\right).$$

(3.32)

Hence we get

$$\mathcal{M}(h(z), F(z), t) \geq L \mathcal{M}
\left(h(z), F(z), \frac{1}{k^n}t\right) \to 1_L \quad \text{as } n \to \infty \forall t > 0.$$  

(3.33)

Hence we conclude that $\mathcal{M}(h(z), F(z), t) = 1_L$ for all $t > 0$. Then by (b) of Definition 2.5 we have $F(z) = h(z)$. Thus we proved that $F$ and $h$ have a coincidence.

Suppose now that $F$ and $h$ commute at $z$. Set $w = h(z) = F(z)$. Then

$$F(w) = F(h(z)) = h(F(z)) = h(w).$$

(3.34)
Since from (3.3) we have \( h(z) \leq h(h(z)) = h(w) \) and as \( h(z) = F(z) \) and \( h(w) = F(w) \), from (3.2) we get

\[
\mathcal{M}(w, F(w), k) = \mathcal{M}(F(z), F(w), k)
\]

\[
\geq L \mathcal{T}_M \{ \mathcal{M}(h(z), h(w), t), \mathcal{M}(h(z), F(z), t), \mathcal{M}(h(w), F(w), t), \mathcal{M}(h(w), F(z), (1 + q)t), \mathcal{M}(h(z), F(w), (1 - q)t) \}
\]

\[
= \mathcal{M}(F(z), F(z), (1 - q)t).
\]

Letting \( q \to 0 \) we get

\[
\mathcal{M}(F(z), F(w), k) \geq L \mathcal{T}_M (F(z), F(w), t).
\]

Hence, similarly as above, we get

\[
\mathcal{M}(F(z), F_w, t) \geq L \mathcal{T}_M \left( F(z), F_w, \frac{1}{k^n t} \right) \to 1_L \quad \text{as } n \to \infty \ \forall t > 0. \tag{3.37}
\]

Hence we conclude that \( F(w) = F(z) \). Since \( F(z) = h(z) = w \), we have

\[
F(w) = h(w) = w. \tag{3.38}
\]

Thus we proved that \( F \) and \( h \) have a common fixed point.

\[ \square \]

Remark 3.3. Note that \( F \) is \( h \)-nondecreasing and can be replaced by \( F \) which is \( h \)-non-increasing in Theorem 3.2 provided that \( h(x_0) \leq F(x_0) \) is replaced by \( F(x_0) \geq h(x_0) \) in Theorem 3.2.

Corollary 3.4. Let \( (X, \leq) \) be a partially ordered set and suppose that there is an \( L \)-fuzzy metric \( \mathcal{M} \) on \( X \) such that \( (X, \mathcal{M}, \mathcal{T}) \) is a complete \( M \)-\( L \)-fuzzy metric space in which \( \mathcal{T} \) is Hadzic’ type. Let \( F : X \to X \) be a nondecreasing self-mappings of \( X \) such that there exist \( k \in (0, 1) \) and \( q \in (0, 1) \) such that

\[
\mathcal{M}(F(x), F(y), kt) \geq L \mathcal{T}_M \{ \mathcal{M}(x, y, t), \mathcal{M}(x, F(x), t), \mathcal{M}(y, F(y), t), \mathcal{M}(x, F(y), (1 + q)t), \mathcal{M}(y, F(x), (1 - q)t) \} \tag{3.39}
\]

for all \( x, y \in X \) for which \( x \leq y \) and all \( t > 0 \). Also suppose the following.

(i) If \( \{x_n\} \subset X \) is a nondecreasing sequence with \( x_n \to z \) in \( X \), then \( x_n \leq z \) for all \( n \) hold.

(ii) \( F \) is continuous.

If there exists an \( x_0 \in X \) with \( x_0 \leq F(x_0) \), then \( F \) has a fixed point.

Proof. Taking \( h = I \) (\( I \) = the identity mapping) in Theorem 3.2, then (3.3) reduces to the hypothesis (i).
Suppose now that $F$ is continuous. Since from (3.4) we have $x_{n+1} = F(x_n)$ for all $n \geq 0$, and as from (3.29), $x_n \to z$, then

$$F(z) = F\left(\lim_{n \to \infty} x_n\right) = \lim_{n \to \infty} F(x_n) = z. \quad (3.40)$$

**Corollary 3.5.** Let $(X, \leq)$ be a partially ordered set and suppose that there is an $L$-fuzzy metric $\mathcal{M}$ on $X$ such that $(X, \mathcal{M}, \mathcal{T})$ is a complete $\mathcal{M}$-$L$-fuzzy metric space in which $\mathcal{T}$ is Hadžić’ type. Let $F : X \to X$ be a nondecreasing self-mappings of $X$ such that there exist $k \in (0, 1)$ and $q \in (0, 1)$ such that

$$\mathcal{M}(F(x), F(y), kt) \geq L \mathcal{T}_M\{\mathcal{M}(x, y, t), \mathcal{M}(x, F(x), t), \mathcal{M}(y, F(y), t)\} \quad (3.41)$$

for all $x, y \in X$ for which $x \leq y$ and all $t > 0$. Also suppose the following.

(i) If $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \to z$ in $X$, then $x_n \leq z$ for all $n$ hold.

(ii) $F$ is continuous.

If there exists an $x_0 \in X$ with $x_0 \leq F(x_0)$, then $F$ has a fixed point.

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**References**


