Research Article

A New System of Generalized Nonlinear Mixed Variational Inclusions in Banach Spaces

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We introduce and study a new system of generalized nonlinear mixed variational inclusions in real $q$-uniformly smooth Banach spaces. We prove the existence and uniqueness of solution and the convergence of some new $n$-step iterative algorithms with or without mixed errors for this system of generalized nonlinear mixed variational inclusions. The results in this paper unify, extend, and improve some known results in literature.

1. Introduction

Variational inclusion problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium, as well as engineering science. For the past years, many existence results and iterative algorithms for various variational inequality and variational inclusion problems have been studied. For details, see [1–25] and the references therein.

Recently, some new and interesting problems, which are called to be system of variational inequality problems, were introduced and studied. Pang [1], Cohen and Chaplais [2], Bianchi [3], and Ansari and Yao [4] considered a system of scalar variational inequalities, and Pang showed that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium, and the general equilibrium programming problem can be modeled as a system of variational inequalities. Ansari et al. [5] considered a system of vector variational inequalities and obtained its existence results. Allevi et al. [6] considered a system of generalized vector variational inequalities and established some existence results with relative pseudomonotonicity. Kassay and Kolumbán [7] introduced a system of variational inequalities and proved an existence theorem by the Ky Fan lemma. Kassay et al. [8] studied

As generalizations of system of variational inequalities, Agarwal et al. [18] introduced a system of generalized nonlinear mixed quasivariational inclusions and investigated the sensitivity analysis of solutions for this system of generalized nonlinear mixed quasivariational inclusions in Hilbert spaces. Peng and Zhu [19] introduce a new system of generalized nonlinear mixed quasivariational inclusions in \( q \)-uniformly smooth Banach spaces and prove the existence and uniqueness of solutions and the convergence of several new two-step iterative algorithms with or without errors for this system of generalized nonlinear mixed quasivariational inclusions. Kazmi and Bhat [20] introduced a system of nonlinear variational-like inclusions and proved the existence of solutions and the convergence of a new iterative algorithm for this system of nonlinear variational-like inclusions. Fang and Huang [21], Verma [22], and Fang et al. [23] introduced and studied a new system of variational inclusions involving \( H \)-monotone operators, \( A \)-monotone operators and \( (H, \eta) \)-monotone operators, respectively. Yan et al. [24] introduced and studied a system of set-valued variational inclusions which is more general than the model in [21]. Peng and Zhu [25] introduced and studied a system of generalized mixed quasivariational inclusions involving \( (H, \eta) \)-monotone operators which contains those mathematical models in [11–16, 21–24] as special cases.

Inspired and motivated by the results in [1–25], the purpose of this paper is to introduce and study a new system of generalized nonlinear mixed quasivariational inclusions which contains some classes of system of variational inequalities and systems of variational inequalities in the literature as special cases. Using the resolvent technique for the \( m \)-accretive mappings, we prove the existence and uniqueness of solutions for this system of generalized nonlinear mixed quasivariational inclusions. We also prove the convergence of some new \( n \)-step iterative sequences with or without mixed errors to approximation the solution for this system of generalized nonlinear mixed quasivariational inclusions. The results in this paper unifies, extends, and improves some results in [11–16, 19] in several aspects.

2. Preliminaries

Throughout this paper we suppose that \( E \) is a real Banach space with dual space, norm and the generalized dual pair denoted by \( E^* \), \( \| \cdot \| \) and \( \langle \cdot , \cdot \rangle \), respectively, \( 2^E \) is the family of all the nonempty subsets of \( E \), \( \text{dom}(M) \) denotes the domain of the set-valued map \( M : E \to 2^E \), and the generalized duality mapping \( J_q : E \to 2^{E^*} \) is defined by

\[
J_q(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \| f^* \| \cdot \| x \| , \| f^* \| = \| x \|^{q-1} \right\}, \quad \forall x \in E, \tag{2.1}
\]
Lemma 2.1. Let $q > 1$ is a constant. In particular, $J_q$ is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^2 J_2(x)$, for all $x \neq 0$, and $J_q$ is single-valued if $E^*$ is strictly convex.

The modulus of smoothness of $E$ is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup \left\{ \frac{1}{2} \left( \|x + y\| + \|x - y\| - 1 : \|x\| \leq 1, \|y\| \leq t \right) \right\}. \quad (2.2)$$

A Banach space $E$ is called uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0. \quad (2.3)$$

$E$ is called $q$-uniformly smooth if there exists a constant $c > 0$, such that

$$\rho_E(t) \leq ct^q, \quad q > 1. \quad (2.4)$$

Note that $J_q$ is single-valued if $E$ is uniformly smooth.

Xu [26] and Xu and Roach [27] proved the following result.

**Lemma 2.1.** Let $E$ be a real uniformly smooth Banach space. Then, $E$ is $q$-uniformly smooth if and only if there exists a constant $c_q > 0$, such that for all $x, y \in E$,

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x) \rangle + c_q y^q. \quad (2.5)$$

**Definition 2.2 (see [28]).** Let $M : \text{dom}(M) \subseteq E \to 2^E$ be a multivalued mapping:

(i) $M$ is said to be accretive if, for any $x, y \in \text{dom}(M)$, $u \in M(x)$, $v \in M(y)$, there exists $j_q(x - y) \in j_q(x - y)$ such that

$$\langle u - v, j_q(x - y) \rangle \geq 0; \quad (2.6)$$

(ii) $M$ is said to be $m$-accretive if $M$ is accretive and $(I + \rho M)(\text{dom}(M)) = E$ holds for every (equivalently, for some) $\rho > 0$, where $I$ is the identity operator on $E$.

**Remark 2.3.** It is well known that, if $E = \mathcal{H}$ is a Hilbert space, then $M : \text{dom}(M) \subseteq E \to 2^E$ is $m$-accretive if and only if $M$ is maximal monotone (see, e.g., [29]).

We recall some definitions needed later.

**Definition 2.4 (see [28]).** Let the multivalued mapping $M : \text{dom}(M) \subseteq E \to 2^E$ be $m$-accretive, for a constant $\rho > 0$, the mapping $R^M_\rho : E \to \text{dom}(M)$ which is defined by

$$R^M_\rho(u) = (I + \rho M)^{-1}(u), \quad u \in E, \quad (2.7)$$

is called the resolvent operator associated with $M$ and $\rho$. 


Lemma 2.8. If\( R^M \) is single-valued and nonexpansive mapping (see [28]).

Definition 2.6. Let \( E \) be a real uniformly smooth Banach space, and let \( T : E \to E \) be a single-valued operator. However, \( T \) is said to be

(i) \( r \)-strongly accretive if there exists a constant \( r > 0 \) such that

\[
\langle Tx - Ty, J_q(x - y) \rangle \geq r \|x - y\|^q, \quad \forall x, y \in E,
\]

or equivalently,

\[
\langle Tx - Ty, J_2(x - y) \rangle \geq r \|x - y\|^2, \quad \forall x, y \in E;
\]

(ii) \( s \)-Lipschitz continuous if there exists a constant \( s > 0 \) such that

\[
\|T(x) - T(y)\| \leq s \|x - y\|, \quad \forall x, y \in E.
\]

Remark 2.7. If \( T \) is \( r \)-strongly accretive, then \( T \) is \( r \)-expanding, that is,

\[
\|T(x) - T(y)\| \geq r \|x - y\|, \quad \forall x, y \in E.
\]

Lemma 2.8 (see [30]). Let \( \{a_n\}, \{b_n\}, \{c_n\} \) be three real sequences, satisfying

\[
a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad \forall n \geq 0,
\]

where \( t_n \in (0, 1) \), \( \sum_{n=0}^{\infty} t_n = \infty \), for all \( n \geq 0 \), \( b_n = o(t_n) \), \( \sum_{n=0}^{\infty} c_n < \infty \). Then \( a_n \to 0 \).


In this section, we will introduce a new system of generalized nonlinear mixed variational inclusions which contains some classes of system of variational inclusions and systems of variational inequalities in literature as special cases.

In what follows, unless other specified, we always suppose that \( \theta \) is a zero element in \( E \), and for each \( i = 1, 2, \ldots, n \), \( T_i : E \to E \) and \( S_i : E \to E \) are single-valued mappings, \( M_i : E \to 2^E \) is an \( m \)-accretive operator. We consider the following problem: find \( (x^*_1, x^*_2, \ldots, x^*_n) \in E^n \) such that

\[
\theta \in \rho_1 T_1x^*_1 + \rho_1 S_1x^*_2 + x^*_1 - x^*_2 + \rho_1 M_1(x^*_1),
\]

\[
\theta \in \rho_2 T_2x^*_2 + \rho_2 S_2x^*_3 + x^*_2 - x^*_3 + \rho_2 M_2(x^*_2),
\]

\[
\vdots
\]

\[
\theta \in \rho_n T_nx^*_n + \rho_n S_n x^*_{n-1} + x^*_n - x^*_{n-1} + \rho_n M_n(x^*_n),
\]

\[
\theta \in \rho_n T_nx^*_1 + \rho_n S_n x^*_1 + x^*_n - x^*_1 + \rho_n M_n(x^*_n),
\]
which is called the system of generalized nonlinear mixed variational inclusions, where $\rho_i > 0 \ (i = 1, 2, \ldots, n)$ are constants.

In what follows, there are some special cases of the problem (3.1).

(i) If $n = 2$, then problem (3.1) reduces to the system of nonlinear mixed quasivariational inequalities introduced and studied by Peng and Zhu [19].

If $E = \mathcal{H}$ is a Hilbert space and $n = 2$, then problem (3.1) reduces to the system of nonlinear mixed quasivariational inequalities introduced and studied by Agarwal et al. [18].

(ii) If $E = \mathcal{H}$ is a Hilbert space, and for each $i = 1, 2, \ldots, n$, $M_i(x) = \partial \phi_i(x)$ for all $x \in \mathcal{H}$, where $\phi_i : \mathcal{H} \to R \cup \{+\infty\}$ is a proper, convex, lower semicontinuous functional, and $\partial \phi_i$ denotes the subdifferential operator of $\phi_i$, then problem (3.1) reduces to the following system of generalized nonlinear mixed variational inequalities, which is to find $(x_1^*, x_2^*, \ldots, x_n^*) \in \mathcal{H}^n$ such that

\[
\begin{align*}
\langle \rho_1 T_1 x_2^* + \rho_1 S_1 x_2^* + x_1^* - x_2^*, x - x_1^* \rangle &\geq \rho_1 \phi_1(x_1^*) - \rho_1 \phi_1(x), \quad \forall x \in \mathcal{H}, \\
\langle \rho_2 T_2 x_3^* + \rho_2 S_2 x_3^* + x_2^* - x_3^*, x - x_2^* \rangle &\geq \rho_2 \phi_2(x_2^*) - \rho_2 \phi_2(x), \quad \forall x \in \mathcal{H}, \\
\vdots
\end{align*}
\]

\[
\begin{align*}
\langle \rho_{n-1} T_{n-1} x_n^* + \rho_{n-1} S_{n-1} x_n^* + x_{n-1}^* - x_n^*, x - x_{n-1}^* \rangle &\geq \rho_{n-1} \phi_{n-1}(x_{n-1}^*) - \rho_{n-1} \phi_{n-1}(x), \quad \forall x \in \mathcal{H}, \\
\langle \rho_n T_n x_1^* + \rho_n S_n x_1^* + x_n^* - x_1^*, x - x_n^* \rangle &\geq \rho_n \phi_n(x_n^*) - \rho_n \phi_n(x), \quad \forall x \in \mathcal{H},
\end{align*}
\]

(3.2)

where $\rho_i > 0 \ (i = 1, 2, \ldots, n)$ are constants.

(iii) If $n = 2$, then (3.2) reduces to the problem of finding $(x_1^*, x_2^*) \in \mathcal{H} \times \mathcal{H}$ such that

\[
\begin{align*}
\langle \rho_1 T_1 x_2^* + \rho_1 S_1 x_2^* + x_1^* - x_2^*, x - x_1^* \rangle &\geq \rho_1 \phi_1(x_1^*) - \rho_1 \phi_1(x), \quad \forall x \in \mathcal{H}, \\
\langle \rho_2 T_2 x_1^* + \rho_2 S_2 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle &\geq \rho_2 \phi_2(x_2^*) - \rho_2 \phi_2(x), \quad \forall x \in \mathcal{H}.
\end{align*}
\]

(3.3)

Moreover, if $\phi_1 = \phi_2 = \phi$, then problem (3.3) becomes the system of generalized nonlinear mixed variational inequalities introduced and studied by J. K. Kim and D. S. Kim in [16].

(iv) For $i = 1, 2, \ldots, n$, if $\phi_i = \delta_{K_i}$ (the indicator function of a nonempty closed convex subset $K_i \subset \mathcal{H}$) and $T_i = 0$, then (3.2) reduces to the problem of finding $(x_1^*, x_2^*, \ldots, x_n^*) \in \prod_{i=1}^n K_i$, such that

\[
\begin{align*}
\langle \rho_1 S_1 x_2^* + x_1^* - x_2^*, x - x_1^* \rangle &\geq 0, \quad \forall x \in K_1, \\
\langle \rho_2 S_2 x_3^* + x_2^* - x_3^*, x - x_2^* \rangle &\geq 0, \quad \forall x \in K_1, \\
\vdots
\end{align*}
\]

\[
\begin{align*}
\langle \rho_{n-1} S_{n-1} x_n^* + x_{n-1}^* - x_n^*, x - x_{n-1}^* \rangle &\geq 0, \quad \forall x \in K_{n-1}, \\
\langle \rho_n S_n x_1^* + x_n^* - x_1^*, x - x_n^* \rangle &\geq 0, \quad \forall x \in K_n.
\end{align*}
\]

(3.4)
Problem (3.4) is called the system of nonlinear variational inequalities. Moreover, if \( n = 2 \), then problem (3.4) reduces to the following system of nonlinear variational inequalities, which is to find \((x_1^*, x_2^*) \in K_1 \times K_2\) such that

\[
\begin{align*}
\langle \rho_1 S_1 x_2^* + x_1^* - x_2^*, x - x_1^* \rangle & \geq 0, \quad \forall x \in K_1, \\
\langle \rho_2 S_2 x_2^* + x_2^* - x_1^*, x - x_2^* \rangle & \geq 0, \quad \forall x \in K_2.
\end{align*}
\]

(3.5)

If \( S_1 = S_2 \) and \( K_1 = K_2 = K \), then (3.5) reduces to the problem introduced and researched by Verma [11–13].

**Lemma 3.1.** For any given \( x_i^* \in E(i = 1, 2, \ldots, n) \), \((x_1^*, x_2^*, \ldots, x_n^*)\) is a solution of the problem (3.1) if and only if

\[
\begin{align*}
x_1^* & = R_{\rho_1}^{M_1} [x_2^* - \rho_1 (T_1 x_2^* + S_1 x_2^*)], \\
x_2^* & = R_{\rho_2}^{M_2} [x_3^* - \rho_2 (T_2 x_3^* + S_2 x_3^*)], \\
& \vdots \\
x_{n-1}^* & = R_{\rho_{n-1}}^{M_{n-1}} [x_n^* - \rho_{n-1} (T_{n-1} x_n^* + S_{n-1} x_n^*)], \\
x_n^* & = R_{\rho_n}^{M_n} [x_1^* - \rho_n (T_n x_1^* + S_n x_1^*)],
\end{align*}
\]

(3.6)

where \( R_{\rho_i}^{M_i} = (I + \rho_i M_i)^{-1} \) is the resolvent operators of \( M_i \) for \( i = 1, 2, \ldots, n \).

**Proof.** It is easy to know that Lemma 3.1 follows from Definition 2.4 and so the proof is omitted. \( \square \)

4. Existence and Uniqueness

In this section, we will show the existence and uniqueness of solution for problems (3.1).

**Theorem 4.1.** Let \( E \) be a real \( q \)-uniformly smooth Banach spaces. For \( i = 1, 2, \ldots, n \), let \( S_i : E \to E \) be strongly accretive and Lipschitz continuous with constants \( k_i \) and \( \mu_i \), respectively, let \( T_i : E \to E \) be Lipschitz continuous with constant \( \nu_i \), and let \( M_i : E \to E \) be an \( m \)-accretive mapping. If for each \( i = 1, 2, \ldots, n \),

\[
0 < \sqrt{1 - q \rho_i k_i + c_0 \rho_i^q \mu_i^q + \rho_i \nu_i} < 1,
\]

(4.1)

then (3.1) has a unique solution \((x_1^*, x_2^*, \ldots, x_n^*) \in E^n\).
Proof. First, we prove the existence of the solution. Define a mapping \( F : E \to E \) as follows:

\[
F(x) = R_{p_1}^{M_1} \left[ x_2 - \rho_1(T_1x_2 + S_1x_2) \right],
\]

\[
x_2 = R_{p_2}^{M_2} \left[ x_3 - \rho_2(T_2x_3 + S_2x_3) \right],
\]

\[
\vdots
\]

\[
x_n = R_{p_n}^{M_n} \left[ x - \rho_n(T_nx + S_nx) \right].
\]

(4.2)

For \( i = 1, 2, \ldots, n \), since \( R_{p_i}^{M_i} \) is a nonexpansive mapping, \( S_i \) is strongly accretive and Lipschitz continuous with constants \( k_i \) and \( \mu_i \), respectively, and \( T_i \) is Lipschitz continuous with constant \( \nu_i \), for any \( x, y \in E \), we have

\[
\| F(x) - F(y) \| = \left\| R_{p_1}^{M_1} \left[ x_2 - \rho_1(T_1x_2 + S_1x_2) \right] - R_{p_1}^{M_1} \left[ y_2 - \rho_1(T_1y_2 + S_1y_2) \right] \right\|
\]

\[
\leq \| (x_2 - y_2) - \rho_1((T_1x_2 + S_1x_2) - (T_1y_2 + S_1y_2)) \|
\]

\[
\leq \| (x_2 - y_2) - \rho_1(S_1x_2 - S_1y_2) \| + \rho_1 \| T_1x_2 - T_1y_2 \|
\]

\[
\leq \sqrt{\| x_2 - y_2 \|^q - q\rho_1\langle S_1x_2 - S_1y_2, I_q(x_2 - y_2) \rangle + c_q\rho_1^q \| S_1x_2 - S_1y_2 \|^q + \rho_1\nu_i \| x_2 - y_2 \|
\]

\[
= \left( \sqrt{1 - q\rho_1k_1 + c_q\rho_1^q\mu_1^q + \rho_1\nu_i} \right) \| x_2 - y_2 \|
\]

\[
= \left( \sqrt{1 - q\rho_1k_1 + c_q\rho_1^q\mu_1^q + \rho_1\nu_i} \right) \left\| R_{p_2}^{M_2} \left[ x_3 - \rho_2(T_2x_3 + S_2x_3) \right] - R_{p_2}^{M_2} \left[ y_3 - \rho_2(T_2y_3 + S_2y_3) \right] \right\|
\]

\[
\leq \left( \sqrt{1 - q\rho_1k_1 + c_q\rho_1^q\mu_1^q + \rho_1\nu_i} \right) \| x_3 - y_3 - \rho_2\left( (S_2x_3 - S_2y_3) + (T_2x_3 - T_2y_3) \right) \|
\]

\[
\leq \left( \sqrt{1 - q\rho_1k_1 + c_q\rho_1^q\mu_1^q + \rho_1\nu_i} \right) \| x_3 - y_3 - \rho_2\left( S_2x_3 - S_2y_3 \right) \| + \rho_2 \| T_2x_3 - T_2y_3 \|
\]

\[
\leq \left( \sqrt{1 - q\rho_1k_1 + c_q\rho_1^q\mu_1^q + \rho_1\nu_i} \right)
\times \left[ \sqrt{\| x_3 - y_3 \|^q - q\rho_2\langle S_2x_3 - S_2y_3, I_q(x_3 - y_3) \rangle + c_q\rho_2^q \| S_2x_3 - S_2y_3 \|^q + \rho_2\nu_2 \| x_3 - y_3 \|} \right]
\]

\[
\leq \left( \sqrt{1 - q\rho_1k_1 + c_q\rho_1^q\mu_1^q + \rho_1\nu_i} \right) \left( \sqrt{1 - q\rho_2k_2 + c_q\rho_2^q\mu_2^q + \rho_2\nu_2} \right) \| x_3 - y_3 \|
\]

\[
\leq \cdots \leq \prod_{i=1}^{n-1} \left( \sqrt{1 - q\rho_i k_i + c_q\rho_i^q \mu_i^q + \rho_i \nu_i} \right) \| x_n - y_n \|
\]
\[
\begin{align*}
&= \prod_{i=1}^{n-1} \left\| R_{\rho_i}^{M_i} [x - \rho_n (T_i x + S_i x)] - R_{\rho_n}^{M_n} [y - \rho_n (T_n y + S_n y)] \right\| \\
&\leq \prod_{i=1}^{n-1} \left( \sqrt[n-1]{1 - q \rho_i k_i + c q \rho_i^q \mu_i^q + \rho_i v_i} \right) \left\| x - \rho_n (S_n x) - S_n y \right\| + \rho_n \left\| T_n x - T_n y \right\| \\
&\leq \prod_{i=1}^{n-1} \left( \sqrt[n-1]{1 - q \rho_i k_i + c q \rho_i^q \mu_i^q + \rho_i v_i} \right) \\
&\quad \times \left[ \sqrt[n-1]{\| x - y \|} - q \rho_n (S_n x - S_n y, J_q (x - y)) + c q \rho_n^q \| S_n x - S_n y \| + \rho_n v_n \| x - y \| \right] \\
&\leq \prod_{i=1}^{n} \left( \sqrt[n-1]{1 - q \rho_i k_i + c q \rho_i^q \mu_i^q + \rho_i v_i} \right) \| x - y \|.
\end{align*}
\]

(4.3)

It follows from (4.1) that

\[
0 < \prod_{i=1}^{n} \left( \sqrt[n-1]{1 - q \rho_i k_i + c q \rho_i^q \mu_i^q + \rho_i v_i} \right) < 1.
\]

(4.4)

Thus, (4.3) implies that \( F \) is a contractive mapping and so there exists a point \( x_1^* \in E \) such that

\[
x_1^* = F(x_1^*).
\]

(4.5)

Let

\[
\begin{align*}
x_i^* &= R_{\rho_i}^{M_i} [x_{i+1}^* - \rho_i (T_i (x_{i+1}^*) + S_i (x_{i+1}^*))] , \quad i = 1, 2, \ldots, n - 1 \\
x_n^* &= R_{\rho_n}^{M_n} [x_1^* - \rho_n (T_n (x_1^*) + S_n (x_1^*))]
\end{align*}
\]

(4.6)

then by the definition of \( F \), we have

\[
\begin{align*}
x_1^* &= R_{\rho_1}^{M_1} [x_2^* - \rho_1 (T_1 x_2^* + S_1 x_2^*)], \\
x_2^* &= R_{\rho_2}^{M_2} [x_3^* - \rho_2 (T_2 x_3^* + S_2 x_3^*)], \\
&\vdots \\
x_{n-1}^* &= R_{\rho_{n-1}}^{M_{n-1}} [x_n^* - \rho_{n-1} (T_{n-1} x_n^* + S_{n-1} x_n^*)], \\
x_n^* &= R_{\rho_n}^{M_n} [x_1^* - \rho_n (T_n x_1^* + S_n x_1^*)],
\end{align*}
\]

(4.7)

that is, \( (x_1^*, x_2^*, \ldots, x_n^*) \) is a solution of problem (3.1).
Then, we show the uniqueness of the solution. Let \((\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)\) be another solution of problem (3.1). It follows from Lemma 3.1 that

\[
\begin{align*}
\bar{x}_1 &= R_{\rho_1}^{M_1} [\bar{x}_2 - \rho_1 (T_1 \bar{x}_2 + S_1 \bar{x}_2)], \\
\bar{x}_2 &= R_{\rho_2}^{M_2} [\bar{x}_3 - \rho_2 (T_2 \bar{x}_3 + S_2 \bar{x}_3)], \\
&\vdots \\
\bar{x}_{n-1} &= R_{\rho_{n-1}}^{M_{n-1}} [\bar{x}_n - \rho_{n-1} (T_{n-1} \bar{x}_n + S_{n-1} \bar{x}_n)], \\
\bar{x}_n &= R_{\rho_n}^{M_n} [\bar{x}_1 - \rho_n (T_n \bar{x}_1 + S_n \bar{x}_1)].
\end{align*}
\] (4.8)

As the proof of (4.3), we have

\[
\|x_1^* - \bar{x}_1\| \leq \prod_{i=1}^n \left( \sqrt[1-q]{1 - q \rho_i k_i} + c q \rho_i^{\frac{q}{2}} \mu_i^{\frac{q}{2}} + \rho_i \nu_i \right) \|x_1^* - \bar{x}_1\|. \tag{4.9}
\]

It follows from (4.1) that

\[
0 < \prod_{i=1}^n \left( \sqrt[1-q]{1 - q \rho_i k_i} + c q \rho_i^{\frac{q}{2}} \mu_i^{\frac{q}{2}} + \rho_i \nu_i \right) < 1. \tag{4.10}
\]

Hence,

\[
x_1^* = \bar{x}_1, \tag{4.11}
\]

and so for \(i = 2, 3, \ldots, n\), we have

\[
x_i^* = \bar{x}_i. \tag{4.12}
\]

This completes the proof. \(\square\)

Remark 4.2. (i) If \(E\) is a 2-uniformly smooth space, and there exist \(\rho_i > 0\) \((i = 1, 2, \ldots, n)\) such that

\[
0 < \rho_i < \min \left\{ \frac{2(k_i - \nu_i)}{c_2 \mu_i^2 - \nu_i^2}, \frac{1}{\nu_i} \right\}, \quad \nu_i < c_2 \mu_i. \tag{4.13}
\]

Then (4.1) holds. We note that the Hilbert spaces and \(L_p\) (or \(l_q\)) spaces \((2 \leq q < \infty)\) are 2-uniformly smooth.

(ii) Let \(n = 2\), by Theorem 4.1, we recover [19, Theorem 3.1]. So Theorem 4.1 unifies, extends, and improves [19, Theorems 3.1, Corollaries 3.2 and 3.3], [16, Theorems 2.1–2.4] and the main results in [13].
5. Algorithms and Convergence

This section deals with an introduction of some \( n \)-step iterative sequences with or without mixed errors for problem (3.1) that can be applied to the convergence analysis of the iterative sequences generated by the algorithms.

**Algorithm 5.1.** For any given point \( x_0 \in E \), define the generalized \( N \)-step iterative sequences \( \{x_{1,k}\}, \{x_{2,k}\}, \ldots, \{x_{n,k}\} \) as follows:

\[
x_{1,k+1} = (1 - \alpha_k)x_{1,k} + \alpha_kR_{\rho_1}^{M_1}\left[x_{2,k} - \rho_1(T_1x_{2,k} + S_1x_{2,k})\right] + \alpha_ku_{1,k} + w_k,
\]

\[
x_{2,k} = R_{\rho_2}^{M_2}\left[x_{3,k} - \rho_2(T_2x_{3,k} + S_2x_{3,k})\right] + u_{2,k},
\]

\[
\vdots
\]

\[
x_{n-1,k} = R_{\rho_{n-1}}^{M_{n-1}}\left[x_{n,k} - \rho_{n-1}(T_{n-1}x_{n,k} + S_{n-1}x_{n,k})\right] + u_{n-1,k},
\]

\[
x_{n,k} = R_{\rho_n}^{M_n}\left[x_{1,k} - \rho_n(T_nx_{1,k} + S_nx_{1,k})\right] + u_{n,k},
\]

where \( x_{1,1} = x_0 \), \( \{\alpha_k\} \) is a sequence in \([0,1]\), and \( \{u_{i,k}\} \subset E \) \( (i = 1, 2, \ldots, n) \), \( \{w_k\} \subset E \) are the sequences satisfying the following conditions:

\[
\sum_{k=1}^{\infty} \alpha_k = +\infty; \quad \sum_{k=1}^{\infty} \|w_k\| < +\infty; \quad \lim_{k \to \infty} \|u_{i,k}\| = 0, \quad i = 1, 2, \ldots, n. \tag{5.2}
\]

**Theorem 5.2.** Let \( T_i, S_i, \) and \( M_i \) be the same as in Theorem 4.1, and suppose that the sequences \( \{x_{1,k}\}, \{x_{2,k}\}, \ldots, \{x_{n,k}\} \) are generated by Algorithm 5.1. If the condition (4.1) holds, then \( (x_{1,k}, x_{2,k}, \ldots, x_{n,k}) \) converges strongly to the unique solution \( (x_1^*, x_2^*, \ldots, x_n^*) \) of the problem (3.1).

**Proof.** By the Theorem 4.1, we know that problem (3.1) has a unique solution \( (x_1^*, x_2^*, \ldots, x_n^*) \), it follows from Lemma 3.1 that

\[
x_1^* = R_{\rho_1}^{M_1}\left[x_2^* - \rho_1(T_1x_2^* + S_1x_2^*)\right],
\]

\[
x_2^* = R_{\rho_2}^{M_2}\left[x_3^* - \rho_2(T_2x_3^* + S_2x_3^*)\right],
\]

\[
\vdots
\]

\[
x_{n-1}^* = R_{\rho_{n-1}}^{M_{n-1}}\left[x_n^* - \rho_{n-1}(T_{n-1}x_n^* + S_{n-1}x_n^*)\right],
\]

\[
x_n^* = R_{\rho_n}^{M_n}\left[x_1^* - \rho_n(T_nx_1^* + S_nx_1^*)\right].
\]

(5.3)
By (5.1) and (5.3), we have

\[
\|x_{1,k+1} - x_1^*\|
\]

\[
= \| (1 - \alpha_k)x_{1,k} + \alpha_k R_{p_1}^{M_1}[x_{2,k} - \rho_1(T_1x_{2,k} + S_1x_{2,k})] + \alpha_k u_{1,k} + w_k - x_1^* \|
\]

\[
\leq (1 - \alpha_k)\|x_{1,k} - x_1^*\| + \alpha_k\| R_{p_1}^{M_1}[x_{2,k} - \rho_1(T_1x_{2,k} + S_1x_{2,k})] - R_{p_1}^{M_1}[x_1^* - \rho_1(T_1x_1^* + S_1x_1^*)] \|
\]

\[
+ \alpha_k\|u_{1,k}\| + \|w_k\|
\]

\[
\leq (1 - \alpha_k)\|x_{1,k} - x_1^*\| + \alpha_k\| (x_{2,k} - x_1^*) - \rho_1 [ (T_1x_{2,k} + S_1x_{2,k}) - (T_1x_1^* + S_1x_1^*) ] \|
\]

\[
+ \alpha_k\|u_{1,k}\| + \|w_k\|.
\]

(5.4)

For \( i = 1, 2, \ldots, n \), since \( S_i \) is strongly monotone and Lipschitz continuous with constants \( k_i \) and \( \mu_i \) respectively, and \( T_i \) is Lipschitz continuous with constant \( \nu_i \), we get for \( i = 1, 2, \ldots, n - 1 \),

\[
\| (x_{i+1,k} - x_{i+1}^*) \|
\]

\[
\leq \| (x_{i+1,k} - x_{i+1}^*) \| - \rho_i \| (S_i x_{i+1,k} - S_i x_{i+1}^*) \| + \rho_i \| T_i x_{i+1,k} - T_i x_{i+1}^* \|
\]

\[
\leq \sqrt{\| (x_{i+1,k} - x_{i+1}^*) \|^q - q \rho_i \| (S_i x_{i+1,k} - S_i x_{i+1}^*) \|^q} + c_i \rho_i^q \| S_i x_{i+1,k} - S_i x_{i+1}^* \|^q
\]

\[
+ \rho_i \nu_i \| x_{i+1,k} - x_{i+1}^* \|
\]

\[
\leq \xi_i \| x_{i+1,k} - x_{i+1}^* \|,
\]

(5.5)

where \( \xi_i = \sqrt{1 - q \rho_i k_i + c_i \rho_i^q \mu_i^q} + \rho_i \nu_i \).

It follows from (5.4) and (5.5) that

\[
\| x_{1,k+1} - x_1^* \| \leq (1 - \alpha_k)\|x_{1,k} - x_1^*\| + \alpha_k \xi_k \|x_{2,k} - x_2^*\| + \alpha_k\|u_{1,k}\| + \|w_k\|.
\]

(5.6)

By (5.1), (5.3), and (5.5), we have

\[
\| x_{2,k} - x_2^* \|
\]

\[
= \| R_{p_2}^{M_2}[x_{3,k} - \rho_2(T_2x_{3,k} + S_2x_{3,k})] - R_{p_2}^{M_2}[x_2^* - \rho_2(T_2x_2^* + S_2x_2^*)] + u_{2,k} \|
\]

\[
\leq \| (x_{3,k} - x_2^*) - \rho_2 [(T_2x_{3,k} + S_2x_{3,k}) - (T_2x_2^* + S_2x_2^*)] \| + \|u_{2,k}\|
\]

\[
\leq \xi_2 \| x_{3,k} - x_2^* \| + \|u_{2,k}\|.
\]
\[ \|x_{n,k} - x^*_n\| \]

\[ = \left\| R^{M_n}_{\rho_n} \left[ x_{1,k} - \rho_n (T_n x_{1,k} + S_n x_{1,k}) \right] - R^{M_n}_{\rho_n} \left[ x^*_1 - \rho_n (T_n x^*_1 + S_n x^*_1) \right] + u_{n,k} \right\| \]

\[ \leq \|(x_{1,k} - x^*_1) - \rho_n [(T_n x_{1,k} + S_n x_{1,k}) - (T_n x^*_1 + S_n x^*_1)]\| + \|u_{n,k}\| \]

\[ \leq \|x_{1,k} - x^*_1\| + \|u_{n,k}\|, \quad (5.7) \]

since \(S_n\) is strongly accretive and Lipschitz continuous with constants \(k_n\) and \(\mu_n\), respectively, and \(T_n\) is Lipschitz continuous with constant \(\nu_n\), we get

\[ \|x_{n,k} - x^*_n\| \]

\[ = \left\| R^{M_n}_{\rho_n} \left[ x_{1,k} - \rho_n (T_n x_{1,k} + S_n x_{1,k}) \right] - R^{M_n}_{\rho_n} \left[ x^*_1 - \rho_n (T_n x^*_1 + S_n x^*_1) \right] + u_{n,k} \right\| \]

\[ \leq \|(x_{1,k} - x^*_1) - \rho_n [(T_n x_{1,k} + S_n x_{1,k}) - (T_n x^*_1 + S_n x^*_1)]\| + \|u_{n,k}\| \]

\[ \leq \sqrt{\|x_{1,k} - x^*_1\|^2 - q \rho_n (S_n x_{1,k} - S_n x^*_1, J_q (x_{1,k} - x^*_1)) + c_q \rho_n^\alpha n \|S_n x_{1,k} - S_n x^*_1\|^q} \]

\[ + \rho_n \|T_n x_{1,k} - T_n x^*_1\| \]

\[ \leq \xi_n \|x_{1,k} - x^*_1\| + \|u_{n,k}\|, \quad (5.8) \]

where \(\xi_n = \sqrt{1 - q \rho_n k_n + c_q \rho_n^q n^q} + \rho_n\nu_n\).

It follows from (5.6)–(5.8) that

\[ \|x_{1,k+1} - x^*_1\| \]

\[ \leq (1 - \alpha_k) \|x_{1,k} - x^*_1\| + \alpha_k \xi_1 \|x_{2,k} - x^*_2\| + \alpha_k \|u_{1,k}\| + \|w_k\| \]

\[ \leq (1 - \alpha_k) \|x_{1,k} - x^*_1\| + \alpha k \xi_1 \|x_{3,k} - x^*_3\| + \|u_{2,k}\| + \|u_{1,k}\| + \|w_k\| \]

\[ \leq (1 - \alpha_k) \|x_{1,k} - x^*_1\| + \alpha k \xi_3 \|x_{3,k} - x^*_3\| + \|u_{2,k}\| + \|u_{1,k}\| + \|w_k\| \]

\[ \leq (1 - \alpha_k) \|x_{1,k} - x^*_1\| + \alpha k \xi_2 \|x_{3,k} - x^*_3\| + \alpha k \xi_3 \|u_{2,k}\| + \|u_{1,k}\| + \|w_k\| \]

\[ \leq \cdots \leq (1 - \alpha_k) \|x_{n,k} - x^*_n\| + \alpha k \xi_2 \|x_{n-1,k} - x^*_n\| + \alpha k \xi_2 \cdot \xi_{n-1} \|x_{n-2,k} - x^*_n\| + \|u_{1,k}\| + \|w_k\| \]

\[ + \cdots + \alpha k \xi_{n-1} \|u_{2,k}\| + \alpha k \|u_{1,k}\| + \|w_k\| \]
For any given point, we obtain the following Algorithm 5.3 and Theorem 5.4.

\[
\begin{align*}
&\xi_{1,k} \rightarrow \xi_{1} \\
&\xi_{2,k} \rightarrow \xi_{2} \\
&\xi_{3,k} \rightarrow \xi_{3} \\
&\cdots \\
&\xi_{n-1,k} \rightarrow \xi_{n-1} \\
&\xi_{n,k} \rightarrow \xi_{n} 
\end{align*}
\]

where \( \xi_{i} \) is a sequence in \([0,1]\), satisfying \( \sum_{k=1}^{\infty} \alpha_{k} = +\infty \).

**Theorem 5.4.** Let \( T_{i} \), \( S_{i} \), and \( M_{i} \) be the same as in Theorem 4.1, and suppose that the sequences \( \{x_{1,k}\}, \{x_{2,k}\}, \ldots, \{x_{n,k}\} \) are generated by Algorithm 5.3. If (4.1) holds, then \( \{x_{1,k}, x_{2,k}, \ldots, x_{n,k}\} \) converges strongly to the unique solution \( \{x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*}\} \) of (3.1).
Remark 5.5. Theorem 5.4 unifies and generalizes [19, Theorems 4.3 and 4.4] and the main results in [11, 12]. So Theorem 5.2 unifies, extends, and improves the corresponding results in [11–14, 16, 19].

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References


