Research Article

Strong Convergence Theorems by Shrinking Projection Methods for Class $T$ Mappings

Qiao-Li Dong, Songnian He, and Fang Su

1 College of Science, Civil Aviation University of China, Tianjin 300300, China
2 Tianjin Key Laboratory for Advanced Signal Processing, Civil Aviation University of China, Tianjin 300300, China
3 Department of Mathematics and Systems Science, National University of Defense Technology, Changsha 410073, China

Correspondence should be addressed to Qiao-Li Dong, dongqiaoli@ymail.com

Received 9 December 2010; Accepted 2 February 2011

Academic Editor: S. Al-Homidan

Copyright © 2011 Qiao-Li Dong et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We prove a strong convergence theorem by a shrinking projection method for the class of $T$ mappings. Using this theorem, we get a new result. We also describe a shrinking projection method for a nonexpansive mapping on Hilbert spaces, which is the same as that of Takahashi et al. (2008).

1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let $C$ be a nonempty closed convex subset of $H$. Recall that a mapping $T : H \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. The set of fixed points of $T$ is $\text{Fix}(T) := \{x \in H : Tx = x\}$.

$T : H \to H$ is said to be quasi-nonexpansive if $\text{Fix}(T)$ is nonempty and $\|Tx - p\| \leq \|x - p\|$ for all $x \in H$ and $p \in \text{Fix}(T)$.

Given $x, y \in H$, let

$$H(x, y) := \{z \in H : \langle z - y, x - y \rangle \leq 0\} \quad (1.1)$$

be the half-space generated by $(x, y)$. A mapping $T : H \to H$ is said to be the class $T$ (or a cutter) if $T \in \mathcal{T} = \{T : H \to H \mid \text{dom}(T) = H \text{ and } \text{Fix}(T) \subset H(x, Tx), \text{ for all } x \in H\}$.

Remark 1.1. The class $\mathcal{T}$ is fundamental because it contains several types of operators commonly found in various areas of applied mathematics and in particular in approximation and optimization theory (see [1] for details).
Combettes [2], Bauschke, and Combettes [1] studied properties of the class $\mathcal{T}$ mappings and presented several algorithms. They introduced an abstract Haugazeau method in [1] as follows: starting $x_0 \in H$,

$$x_{n+1} = P_{H(x_0,x_n)\cap H(x_n,T_n,x_n)}x_0. \quad (1.2)$$

Using Lemma 1.2 given below and the fact that a nonexpansive mapping is quasi-nonexpansive, one can easily obtain hybrid methods introduced by Nakajo and Takahashi [3] for a nonexpansive mapping.

Recently, Takahashi et al. [4] proposed a shrinking projection method for nonexpansive mappings $T_n : C \to C$. Let $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1}x_0$, and

$$y_n = \alpha_n + (1 - \alpha_n)T_nx_n,$$

$$C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\},$$

$$x_{n+1} = P_{C_{n+1}}x_0, \quad n = 1, 2, \ldots,$$  

where $0 \leq \alpha_n \leq a < 1$, $P_K$ denotes the metric projection from $H$ onto a closed convex subset $K$ of $H$.

Inspired by Bauschke and Combettes [1] and Takahashi et al. [4], we present a shrinking projection method for the class of $\mathcal{T}$ mappings. Furthermore, we obtain a shrinking projection method for a nonexpansive mapping on Hilbert spaces, which is the same as presented by Takahashi et al. [4].

We will use the following notations:

(1) $\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence;

(2) $\omega_w(x_n) = \{x : \exists x_n \rightharpoonup x\}$ denotes the weak $\omega$-limit of $\{x_n\}$.

We need some facts and tools in a real Hilbert space $H$ which are listed below.

**Lemma 1.2** (see [1]). Let $H$ be a Hilbert space. Let $I$ be the identity operator of $H$.

(i) If $\text{dom} T = H$, then $2T - I$ is quasi-nonexpansive if and only if $T \in \mathcal{T}$.

(ii) If $T \in \mathcal{T}$, then $\lambda I + (1 - \lambda)T \in \mathcal{T}$, for all $\lambda \in [0,1]$.

**Definition 1.3.** Let $T_n \in \mathcal{T}$ for each $n$. The sequence $\{T_n\}$ is called to be coherent if, for every bounded sequence $\{v_n\}$ in $H$, there holds

$$\sum_{n=0}^{\infty} \|v_{n+1} - v_n\|^2 < \infty,$$

$$\Rightarrow \omega_w(v_n) \subset \bigcap_{n=0}^{\infty} \text{Fix}(T_n). \quad (1.4)$$

$$\sum_{n=0}^{\infty} \|v_n - T_nv_n\|^2 < \infty,$$

**Definition 1.4.** $T$ is called demiclosed at $y \in H$ if $Tx = y$ whenever $\{x_n\} \subset H$, $x_n \rightharpoonup x$ and $Tx_n \rightarrow y$.

Next lemma shows that nonexpansive mappings are demeclosed at 0.
Let $T \in \mathcal{T}$ be a closed convex subset of a real Hilbert space $H$, and let $T : C \to C$ be a nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$. If a sequence $\{x_n\}$ in $C$ is such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z = Tz$.

**Lemma 1.6** (see [6]). Let $K$ be a closed convex subset of $H$. Let $\{x_n\}$ be a sequence in $H$ and $u \in H$. Let $q = P_K u$. If $x_n$ is such that $\omega_\omega(x_n) \subset K$ and satisfies the condition

$$\|x_n - u\| \leq \|u - q\|, \quad \forall n,$$  \hfill (1.5)

then $x_n \to q$.

**Lemma 1.7** (Goebel and Kirk [5]). Let $K$ be a closed convex subset of a real Hilbert space $H$, and let $P_k$ be the (metric or nearest point) projection from $H$ onto $K$ (i.e., for $x \in H$, $P_k x$ is the only point in $K$ such that $\|x - P_k x\| = \inf\{\|x - z\| : z \in K\}$). Given $x \in H$ and $z \in K$, then $z = P_k x$ if and only if there holds the relation

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in K.$$  \hfill (1.6)

### 2. Main Results

In this section, we will introduce a shrinking projection method for the class of $\mathcal{T}$ mappings and prove strong convergence theorem.

**Theorem 2.1.** Let $T_n \in \mathcal{T}$ for each $n$ such that $\mathcal{T} := \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset$. Suppose that the sequence $\{T_n\}$ is coherent. Let $x_0 \in H$. For $C_1 = H$ and $x_1 = x_0$, define a sequence $\{x_n\}$ as follows:

$$x_{n+1} = P_{C_n} x_0, \quad n = 1, 2, \ldots,$$

$$C_{n+1} = \{z \in C_n : \langle z - T_n x_n, x_n - T_n x_n \rangle \leq 0\}.$$  \hfill (2.1)

Then, $\{x_n\}$ converges strongly to $P_{\mathcal{T}} x_0$.

**Proof.** We first show by induction that $\mathcal{T} \subset C_n$ for all $n \in \mathbb{N}$. If $\mathcal{T} \subset C_1$ is obvious. Suppose $\mathcal{T} \subset C_k$ for some $k \in \mathbb{N}$. Note that, by the definition of $T_k \in \mathcal{T}$, we always have $\mathcal{T} \subset \text{Fix}(T_k) \subset H(x_k, T_k x_k)$, that is,

$$\langle z - T_k x_k, x_k - T_k x_k \rangle \leq 0, \quad \forall z \in \mathcal{T}.$$  \hfill (2.2)

From the definition of $C_{k+1}$ and $\mathcal{T} \subset C_k$, we obtain $\mathcal{T} \subset C_{k+1}$. This implies that

$$\mathcal{T} \subset C_n, \quad \forall n \in \mathbb{N}.$$  \hfill (2.3)

It is obvious that $C_1 = H$ is closed and convex. So, from the definition, $C_n$ is closed and convex for all $n \in \mathbb{N}$. So we get that $\{x_n\}$ is well defined.

Since $x_n$ is the projection of $x_0$ onto $C_n$ which contains $\mathcal{T}$, we have

$$\|x_0 - x_n\| \leq \|x_0 - y\|, \quad \forall y \in C_n.$$  \hfill (2.4)
Taking \( y = P_{\mathcal{F}}x_0 \in \mathcal{F} \), we get
\[
\|x_0 - x_n\| \leq \|x_0 - P_{\mathcal{F}}x_0\|. \tag{2.5}
\]

The last inequality ensures that \( \{\|x_0 - x_n\|\} \) is bounded. From \( x_n = P_{C_n}x_0 \) and \( x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subseteq C_n \), using Lemma 1.7, we get
\[
\langle x_{n+1} - x_n, x_0 - x_n \rangle \leq 0. \tag{2.6}
\]

It follows that
\[
\|x_0 - x_{n+1}\|^2 = \|(x_0 - x_n) - (x_{n+1} - x_n)\|^2
\]
\[
= \|x_0 - x_n\|^2 - 2\langle x_0 - x_n, x_{n+1} - x_n \rangle + \|x_{n+1} - x_n\|^2
\]
\[
\geq \|x_0 - x_n\|^2 + \|x_{n+1} - x_n\|^2
\]
\[
\geq \|x_0 - x_n\|^2. \tag{2.7}
\]

Thus \( \{\|x_n - x_0\|\} \) is increasing. Since \( \{\|x_n - x_0\|\} \) is bounded, \( \lim_{n \to \infty} \|x_n - x_0\| \) exists. From (2.7), it follows that
\[
\|x_{n+1} - x_n\|^2 \leq \|x_0 - x_{n+1}\|^2 - \|x_0 - x_n\|^2, \tag{2.8}
\]

and \( \sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < \infty \). On the other hand, by \( x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \), we have
\[
\langle x_{n+1} - T_nx_n, x_n - T_nx_n \rangle \leq 0. \tag{2.9}
\]

Hence,
\[
\|x_{n+1} - x_n\|^2 = \|(x_{n+1} - T_nx_n) - (x_n - T_nx_n)\|^2
\]
\[
= \|x_{n+1} - T_nx_n\|^2 - 2\langle x_{n+1} - T_nx_n, x_n - T_nx_n \rangle + \|x_n - T_nx_n\|^2
\]
\[
\geq \|x_{n+1} - T_nx_n\|^2 + \|x_n - T_nx_n\|^2. \tag{2.10}
\]

We therefore get \( \sum_{n=1}^{\infty} \|x_n - T_nx_n\|^2 < \infty \). Since the sequence \( \{T_n\} \) is coherent, we have \( \omega_{w^*}(x_n) \subset \mathcal{F} \). From Lemma 1.6 and (2.5), the result holds. \( \square \)

**Remark 2.2.** We take \( C_1 = H \) so that \( \mathcal{F} \subset C_1 \) is satisfied.
Theorem 2.3. Let $T_n \in \mathcal{I}$ for each $n$ such that $\mathcal{F} := \bigcap_{n=1}^{\infty} \text{Fix}(T_n) \neq \emptyset$. Suppose that the sequence $\{T_n\}$ is coherent. Let $x_0 \in H$. For $C_1 = H$ and $x_1 = x_0$, define a sequence $\{x_n\}$ as follows:

$$y_n = \alpha_n x_n + (1 - \alpha_n)T_n x_n,$$

$$C_{n+1} = \{z \in C_n : \langle z - y_n, x_n - y_n \rangle \leq 0\},$$

(2.11)

$$x_{n+1} = P_{C_{n+1}} x_0, \quad n = 1, 2, \ldots,$$

where $0 \leq \alpha_n \leq a < 1$. Then, $\{x_n\}$ converges strongly to $P_{\mathcal{F}} x_0$.

Proof. Set $S_n = \alpha_n I + (1 - \alpha_n)T_n$. By Lemma 1.2(ii), we have that $S_n \in \mathcal{I}$. From $\|x_n - S_n x_n\| = (1 - \alpha_n)\|x_n - T_n x_n\|$, it follows that $(1 - a)\|x_n - T_n x_n\| \leq \|x_n - S_n x_n\| \leq \|x_n - T_n x_n\|$ which implies that the sequence $\{S_n\}$ is coherent. It is obvious that $\text{Fix}(S_n) = \text{Fix}(T_n)$, for all $n \in \mathbb{N}$. Hence $\mathcal{F} = \bigcap_{n=1}^{\infty} \text{Fix}(S_n) = \bigcap_{n=1}^{\infty} \text{Fix}(T_n)$. Using Theorem 2.1, we get the desired result. \hfill \Box

3. Deduced Results

In this section, using Theorem 2.3, we obtain some new strong convergence results for the class of $\mathcal{I}$ mappings, a quasi-nonexpansive mapping and a nonexpansive mapping in a Hilbert space.

Theorem 3.1. Let $T \in \mathcal{I}$ such that $\text{Fix}(T) \neq \emptyset$ and satisfying that $I - T$ is demiclosed at 0. Let $x_0 \in H$. For $C_1 = H$ and $x_1 = x_0$, define a sequence $\{x_n\}$ as follows:

$$y_n = \alpha_n x_n + (1 - \alpha_n)T x_n,$$

$$C_{n+1} = \{z \in C_n : \langle z - y_n, x_n - y_n \rangle \leq 0\},$$

(3.1)

$$x_{n+1} = P_{C_{n+1}} x_0, \quad n = 1, 2, \ldots,$$

where $0 \leq \alpha_n \leq a < 1$. Then, $\{x_n\}$ converges strongly to $P_{\text{Fix}(T)} x_0$.

Proof. Let $T_n = T$ in (2.11) for all $n \in \mathbb{N}$. Following the proof of Theorem 2.1, we can easily get (2.5) and $\sum_{n=1}^{\infty} \|x_n - T x_n\|^2 < \infty$. By (2.5), we obtain that $\{x_n\}$ is bounded and $\omega_{w}(x_n)$ is nonempty. For any $\bar{x} \in \omega_{w}(x_n)$, there exists a subsequence $\{x_{n_i}\}$ of the sequence $\{x_n\}$ such that $x_{n_i} \rightharpoonup \bar{x}$. From $\sum_{n=1}^{\infty} \|x_n - T x_n\|^2 < \infty$, it follows that $\|x_n - T x_n\| \to 0$. Since $I - T$ is demiclosed at 0, we get $\bar{x} \in \text{Fix}(T)$. Thus $\omega_{w}(x_n) \subset \text{Fix}(T)$ which together with Lemma 1.6 and (2.5) implies that $x_n \to P_{\text{Fix}(T)} x_0$. \hfill \Box

Theorem 3.2. Let $H$ be a Hilbert space. Let $S$ be a quasi-nonexpansive mapping on $H$ such that $\text{Fix}(S) \neq \emptyset$ and satisfying that $I - S$ is demiclosed at 0. Let $x_0 \in H$. For $C_1 = H$ and $x_1 = x_0$, define a sequence $\{x_n\}$ as follows:

$$u_n = \alpha_n x_n + (1 - \alpha_n)S x_n,$$

$$C_{n+1} = \{z \in C_n : \|z - u_n\| \leq \|x_n - z\|\},$$

(3.2)

$$x_{n+1} = P_{C_{n+1}} x_0, \quad n = 1, 2, \ldots,$$

where $0 \leq \alpha_n \leq a < 1$. Then, $\{x_n\}$ converges strongly to $P_{\text{Fix}(S)} x_0$. 
Proof. By Lemma 1.2(i), $(S + I)/2 \in \mathcal{I}$. Substitute $T$ in (3.1) by $(S + I)/2$. Then $y_n = ((1 + \alpha_n)/2)x_n + ((1 - \alpha_n)/2)Sx_n$, Set $u_n = 2y_n - x_n = \alpha_n x_n + (1 - \alpha_n)Sx_n$, then $y_n = (u_n + x_n)/2$. So, we have

$$
C_{n+1} = \{ z \in C_n : \langle z - y_n, x_n - y_n \rangle \leq 0 \} \\
= \{ z \in C_n : \langle 2z - (x_n + u_n), x_n - u_n \rangle \leq 0 \} \\
= \{ z \in C_n : \|z - u_n\| \leq \|x_n - z\| \}. 
$$

(3.3)

Since $I - S$ is demiclosed at 0, $(S + I)/2 = (I - S)/2$ is demiclosed at 0. So we can obtain the result by using Theorem 3.1.

Since a nonexpansive mapping is quasi-nonexpansive, using Lemma 1.5 and Theorem 3.2, we have following corollary.

**Corollary 3.3.** Let $H$ be a Hilbert space. Let $S$ be a nonexpansive mapping $H$ such that $\text{Fix}(S) \neq \emptyset$. Let $x_0 \in H$. For $C_1 = H$ and $x_1 = x_0$, define a sequence $\{x_n\}$ as follows:

$$
u_n = \alpha_n x_n + (1 - \alpha_n)Sx_n, \\
C_{n+1} = \{ z \in C_n : \|z - u_n\| \leq \|x_n - z\| \}, \\
x_{n+1} = P_{C_{n+1}}x_0, \quad n = 1, 2, \ldots,
$$

(3.4)

where $0 \leq \alpha_n \leq a < 1$. Then, $\{x_n\}$ converges strongly to $P_{\text{Fix}(S)}x_0$.

**Remark 3.4.** Corollary 3.3 is a special case of Theorem 4.1 in [4] when $C_1 = H$.

**Acknowledgments**

The authors would like to express their thanks to the referee for the valuable comments and suggestions for improving this paper. This paper is supported by Research Funds of Civil Aviation University of China Grant (08QD10X) and Fundamental Research Funds for the Central Universities Grant (ZXH2009D021).

**References**


