Research Article

On Extremal Self-Dual Ternary Codes of Length 48

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All extremal ternary self-dual codes of length 48 that have some automorphism of prime order \( p \geq 5 \) are equivalent to one of the two known codes, the Pless code or the extended quadratic residue code.

1. Introduction.

The notion of an extremal self-dual code has been introduced in [1]. As Gleason [2] remarks one may use invariance properties of the weight enumerator of a self-dual code to deduce upper bounds on the minimum distance. Extremal codes are self-dual codes that achieve these bounds. The most wanted extremal code is a binary self-dual doubly even code of length 72 and minimum distance 16. One frequently used strategy is to classify extremal codes with a given automorphism, see [3, 4] for the first papers on this subject.

Ternary codes with a given automorphism have been studied in [5]. The minimum distance \( d(C) := \min \{wt(c) \mid 0 \neq c \in C\} \) of a self-dual ternary code \( C = C^\perp \leq \mathbb{F}_3^n \) of length \( n \) is bounded by

\[
d(C) \leq 3 \left\lfloor \frac{n}{12} \right\rfloor + 3. \tag{1.1}\]

Codes achieving equality are called extremal. Of particular interest are extremal ternary codes of length a multiple of 12. There exists a unique extremal code of length 12 (the extended ternary Golay code), two extremal codes of length 24 (the extended quadratic residue code \( Q_{24} := \overline{QR}(23,3) \) and the Pless code \( P_{24} \)). For length 36, the Pless code yields one example of an extremal code. Reference [5] shows that this is the only code with an automorphism of prime order \( p \geq 5 \); a complete classification is yet unknown. The present paper investigates the extremal codes of length 48. There are two such codes known, the extended quadratic
residue code $Q_{48}$ and the Pless code $P_{48}$. The computer calculations described in this paper show that these two codes are the only extremal ternary codes $C$ of length 48 for which the order of the automorphism group is divisible by some prime $p \geq 5$. Theoretical arguments exclude all types of automorphisms that do not occur for the two known examples.

Any extremal ternary self-dual code of length 48 defines an extremal even unimodular lattice of dimension 48 ([6]). A long-term project to find or even classify such lattices was my main motivation for this paper.

2. Automorphisms of Codes

Let $F$ be some finite field, $F^*$ its multiplicative group. For any monomial transformation $\sigma \in \text{Mon}_n(F) := F^* \wr S_n$, the image $\pi(\sigma) \in S_n$ is called the permutational part of $\sigma$. Then $\sigma$ has a unique expression as

$$\sigma = \text{diag}(\alpha_1, \ldots, \alpha_n)\pi(\sigma) = m(\sigma)\pi(\sigma),$$

and $m(\sigma)$ is called the monomial part of $\sigma$. For a code $C \leq F^n$ we let

$$\text{Mon}(C) := \{\sigma \in \text{Mon}_n(F) \mid \sigma(C) = C\}$$

be the full monomial automorphism group of $C$.

We call a code $C \leq F^n$ an orthogonal direct sum, if there are codes $C_i \leq F^{n_i}$ ($1 \leq i \leq s > 1$) of length $n_i$ such that

$$C = \bigoplus_{i=1}^{s} C_i = \left\{ (c_1^{(1)}, \ldots, c_{n_1}^{(1)}, \ldots, c_s^{(1)}, \ldots, c_{n_s}^{(s)}) \mid c_i^{(i)} \in C_i (1 \leq i \leq s) \right\}.$$

Lemma 2.1. Let $C \leq F^n$ not be an orthogonal direct sum. Then the kernel of the restriction of $\pi$ to $\text{Mon}(C)$ is isomorphic to $F^*$.

Proof. Clearly $F^*C = C$ since $C$ is an $F$-subspace. Assume that $\sigma := \text{diag}(\alpha_1, \ldots, \alpha_n) \in \text{Mon}(C)$ with $\alpha_i \in F^*$, not all equal. Let $\{\alpha_1, \ldots, \alpha_n\} = \{\beta_1, \ldots, \beta_s\}$ with pairwise distinct $\beta_i$. Then

$$C = \bigoplus_{i=1}^{s} \ker(\sigma - \beta_i\text{id})$$

is the direct sum of eigenspaces of $\sigma$. Moreover the standard basis is a basis of eigenvectors of $\sigma$ so this is an orthogonal direct sum. \hfill \Box

In the investigation of possible automorphisms of codes, the following strategy has proved to be very fruitful ([4, 7]).

Definition 2.2. Let $\sigma \in \text{Mon}(C)$ be an automorphism of $C$. Then $\pi(\sigma) \in S_n$ is a direct product of disjoint cycles of lengths dividing the order of $\sigma$. In particular if the order of $\sigma$ is some prime $p$, then we say that $\sigma$ has cycle type $(t, f)$, if $\pi(\sigma)$ has $t$ cycles of length $p$ and $f$ fixed points (so $pt + f = n$).
Lemma 2.3. Let $\sigma \in \text{Mon}(C)$ have prime order $p$.

(a) If $p$ does not divide $|F^*|$ then there is some element $\tau \in \text{Mon}_n(F)$ such that $m(\tau \sigma \tau^{-1}) = \text{id}$. Replacing $C$ by $\tau(C)$ one hence may assume that $m(\sigma) = 1$.

(b) Assume that $p$ does not divide $\text{char}(F)$, $m(\sigma) = 1$, and $\pi(\sigma) = (1, \ldots, p) \cdots ((t - 1)p + 1, \ldots, tp)(tp + 1) \cdots (n)$. Then $C = C(\sigma) \oplus E$, where

$$C(\sigma) = \{ c \in C \mid c_1 = \cdots = c_p, \ c_{p+1} = \cdots = c_{2p}, \ldots, c_{(t-1)p+1} = \cdots = c_{tp} \}$$

(2.5)

is the fixed code of $\sigma$ and

$$E = \left\{ c \in C \mid \sum_{i=1}^p c_i = \sum_{i=p+1}^{2p} c_i = \cdots = \sum_{i=(t-1)p+1}^{tp} c_i = c_{tp+1} = \cdots = c_n = 0 \right\}$$

(2.6)

is the unique $\sigma$-invariant complement of $C(\sigma)$ in $C$.

(c) Define two projections

$$\pi_i : C(\sigma) \to F^i, \quad \pi_i(c) := (c_p, c_{2p}, \ldots, c_{tp}),$$

$$\pi_f : C(\sigma) \to F^f, \quad \pi_f(c) := (c_{tp+1}, c_{tp+2}, \ldots, c_{tp+f}).$$

(2.7)

So $C(\sigma) \equiv (\pi_1(C(\sigma)), \pi_f(C(\sigma))) =: C(\sigma)^*$. If $C = C^\perp$ is self-dual with respect to $(x, y) := \sum_{i=1}^n x_i \overline{y_i}$, then $C(\sigma)^* \leq F^{d+f}$ is a self-dual code with respect to the inner product $(x, y) := \sum_{i=1}^t p x_i \overline{y_i} + \sum_{j=t+1}^{t+f} x_j \overline{y_j}$.

(d) In particular $\dim(C(\sigma)) = (t + f)/2$ and $\dim(E) = t(p - 1)/2$.

Proof. (a) follows from the Schur-Zassenhaus theorem in finite group theory. For the ternary case, see [5, Lemma 1].

(b) and (c) are similar to [4, Lemma 2].

In the following we will keep the notation of the previous lemma and regard the fixed code $C(\sigma)$.

Remark 2.4. If $f \leq d(C)$ then $t \geq f$.

Proof. Otherwise the kernel $K := \ker(\pi_i) = \{(0, \ldots, 0, c_1, \ldots, c_f) \in C(\sigma)\}$ is a nontrivial subcode of minimum distance $\leq f < d(C)$.

The way to analyse the code $E$ from Lemma 2.3 is based on the following remark.

Remark 2.5. Let $p \neq \text{char}(F)$ be some prime and $\sigma \in \text{Mon}_n(F)$ an element of order $p$. Let

$$X^p - 1 = (X - 1)g_1 \cdots g_m \in F[X]$$

(2.8)
be the factorization of $X^p - 1$ into irreducible polynomials. Then all factors $g_i$ have the same degree $d = |(\mathbb{F} + p\mathbb{Z})|$, the order of $|\mathbb{F}|$ mod $p$. There are polynomials $a_i \in \mathbb{F}[X]$ (0 ≤ $i$ ≤ $m$) such that

$$1 = a_0g_1 \cdots g_m + (X - 1)\sum_{i=1}^{m}a_i\prod_{j \neq i}g_j.$$  

(2.9)

Then the primitive idempotents in $\mathbb{F}[X]/(X^p - 1)$ are given by the classes of

$$\bar{e}_0 = a_0g_1 \cdots g_m, \quad \bar{e}_i = a_i\prod_{j \neq i}g_j(X - 1), \quad 1 \leq i \leq m.$$  

(2.10)

Let $L$ be the extension field of $\mathbb{F}$ with $[L : \mathbb{F}] = d$. Then the group ring

$$\frac{\mathbb{F}[X]}{(X^p - 1)} = \mathbb{F}(\sigma) \cong \mathbb{F} \oplus L \oplus \cdots \oplus L$$  

(2.11)

is a commutative semisimple $\mathbb{F}$-algebra. Any code $C \subseteq \mathbb{F}^n$ with an automorphism $\sigma \in \text{Mon}(C)$ is a module for this algebra. Put $e_i := \bar{e}_i(\sigma) \in \mathbb{F}[\sigma]$. Then $C = Ce_0 \oplus Ce_1 \oplus \cdots \oplus Ce_m$ with $Ce_0 = C(\sigma)$, $E = Ce_1 \oplus \cdots \oplus Ce_m$. Omitting the coordinates of $E$ that correspond to the fixed points of $\sigma$, the codes $Ce_i$ are $L$-linear codes of length $t$. Clearly $\dim_L(E) = d \sum_{i=1}^{m} \dim_L(Ce_i)$. If $C$ is self-dual then $\dim(E) = (p - 1)/2$.

3. Extremal Ternary Codes of Length 48

Let $C = C^1 \subseteq \mathbb{F}^{48}_3$ be an extremal self-dual ternary code of length 48, so $d(C) = 15$.

3.1. Large Primes

In this section we prove the main result of this paper.

**Theorem 3.1.** Let $C = C^1 \subseteq \mathbb{F}^{48}_3$ be an extremal self-dual code with an automorphism of prime order $p \geq 5$. Then $C$ is one of the two known codes. So either $C = Q_{48}$ is the extended quadratic residue code of length 48 with automorphism group

$$\text{Mon}(Q_{48}) = C_2 \times PSL_2(47) \text{ of order } 2^5 \cdot 3 \cdot 23 \cdot 47$$  

(3.1)

or $C = P_{48}$ is the Pless code with automorphism group

$$\text{Mon}(P_{48}) = C_2 \times SL_2(23) \cdot 2 \text{ of order } 2^6 \cdot 3 \cdot 11 \cdot 23.$$  

(3.2)

**Lemma 3.2.** Let $\sigma \in \text{Mon}(C)$ be an automorphism of prime order $p \geq 5$. Then either $p = 47$ and $(t, f) = (1, 1)$ or $p = 23$ and $(t, f) = (2, 2)$ or $p = 11$ and $(t, f) = (4, 4)$. 
Proof. For the proof we use the notation of Lemma 2.3. In particular we let \( K := \ker(\pi) = \{ (0, \ldots, 0, c_1, \ldots, c_f) \in C(\sigma) \} \) and put \( K^* := \{ (c_1, \ldots, c_f) \mid (0, \ldots, 0, c_1, \ldots, c_f) \in C(\sigma) \} \). Then

\[
K^* \subseteq \mathbb{F}_3^f, \quad d(K^*) \geq 15, \quad \dim(K^*) \geq \frac{f - t}{2}.
\] (3.3)

Moreover \( tp + f = 48 \).

(1) If \( t = 1 \), then \( p = 47 \). If \( p = 47 \), then \( t = f = 1 \). So assume that \( p < 47 \) and \( t = 1 \). Then the code \( E \) has length \( p \) and dimension \( (p - 1)/2 \), therefore \( p \geq d(C) = 15 \). So \( p \geq 17 \) and \( f \leq 48 - 17 = 31 \).

Then \( K^* \subseteq \mathbb{F}_3^f \) has dimension \( (f - 1)/2 \) and minimum distance \( d(K^*) \geq 15 \). From the bounds given in [8] there is no such possibility for \( f \leq 31 \).

(2) If \( t = 2 \), then \( p = 23 \). Assume that \( t = 2 \). Since \( 2 \cdot p \leq 48 \) we get \( p \leq 23 \), and if \( p = 23 \), then \( (t, f) = (2, 2) \).

So assume that \( p < 23 \). The code \( E \) is a nonzero code of length \( 2p \) and minimum distance \( \geq 15 \), so \( 2p \geq 15 \) and \( p \) is one of \( 11, 13, 17, 19 \), and \( f = 26, 22, 14, 10 \). The code \( K^* \subseteq \mathbb{F}_3^f \) has dimension \( \geq f/2 - 1 \) and minimum distance \( \geq 15 \). Again by [8] there is no such code.

(3) \( p \neq 13 \). For \( p = 13 \) one now only has the possibility \( t = 3 \) and \( f = 9 \). The same argument as above constructs a code \( K^* \subseteq \mathbb{F}_3^f \) of dimension at least \((f + t)/2 - t = 3 \) of minimum distance \( \geq 15 > f \) which is absurd.

(4) If \( p = 11 \), then \( t = f = 4 \). Otherwise \( t = 3 \) and \( f = 15 \) and the code \( K^* \) as above has length \( 15 \), dimension \( \geq 6 \), and minimum distance \( \geq 15 \) which is impossible.

(5) If \( p = 7 \), then \( t = f = 6 \). Otherwise \( t = 3, 4, 5 \) and \( f = 27, 20, 13 \) and the code \( K^* \) as above has dimension \( \geq (f + t)/2 - t = 12, 8, 4 \), length \( f \), and minimum distance \( \geq 15 \) which is impossible by [8].

(6) \( p \neq 7 \). Assume that \( p = 7 \), then \( t = f = 6 \) and the kernel \( K \) of the projection of \( C(\sigma) \) onto the first 42 components is trivial. So the image of the projection is \( \mathbb{F}_3^5 \oplus \langle (1, 1, 1, 1, 1, 1) \rangle \); in particular it contains the vector \( (1^7, 0^{35}) \) of weight 7. So \( C(\sigma) \) contains some word \( (1^7, 0^{35}, a_1, \ldots, a_6) \) of weight \( \leq 13 \) which is a contradiction.

(7) If \( p = 5 \), then \( t = f = 8 \) or \( t = 9 \) and \( f = 3 \). Otherwise \( t = 3, 4, 5, 6, 7 \) and \( f = 33, 28, 23, 18, 13 \) and the code \( K^* \subseteq \mathbb{F}_3^f \) has dimension \( \geq (f + t)/2 - t = 15, 12, 9, 6, 3 \) and minimum distance \( \geq 15 \) which is impossible by [8].

(8) \( p \neq 5 \). Assume that \( p = 5 \). Then one possibility is that \( t = 8 \) and the projection of \( C(\sigma) \) onto the first 8 \cdot 5 coordinates is \( \mathbb{F}_3^5 \oplus \langle (1, 1, 1, 1, 1) \rangle \) and contains a word of weight 5. But then \( C(\sigma) \) has a word of weight \( w \) with \( 5 < w \leq 5 + 8 = 13 \) a contradiction.

The other possibility is \( t = 9 \). Then the code \( E = E^\perp \) is a Hermitian self-dual code of length 9 over the field with \( \mathbb{F}_3^d \) = 81 elements, which is impossible, since the length of such a code is 2 times the dimension and hence even. \( \square \)

Lemma 3.3. If \( p = 11 \), then \( C \cong \mathbb{F}_3^{48} \).

Proof. Let \( \sigma \in \text{Mon}(C) \) be of order 11. Since \( (x^{11} - 1) = (x - 1)gh \in \mathbb{F}_3[x] \) for irreducible polynomials \( g, h \) of degree 5,

\[
\mathbb{F}_3(\sigma) \cong \mathbb{F}_3 \oplus \mathbb{F}_3^5 \oplus \mathbb{F}_3^{48}.
\] (3.4)
Let $e_1$, $e_2$, $e_3 \in \mathbb{F}_3^3(\sigma)$ denote the primitive idempotents. Then $C = C e_1 \oplus C e_2 \oplus C e_3$ with $C(\sigma) = C e_1 = C e_2$ of dimension 4 and $C e_3 = C e_3^T \leq (\mathbb{F}_3^3 \oplus \mathbb{F}_3^3)^4$. Clearly the projection of $C(\sigma)$ onto the first 44 coordinates is injective. Since all weights of $C$ are multiples of 3 and $\geq 15$, this leaves just one possibility for $C(\sigma)$:

$$
G_0 = (L_0 \mid R_0) := \begin{pmatrix}
1^{11} & 0^{11} & 0^{11} & 0^{11} \\
0^{11} & 1^{11} & 0^{11} & 0^{11} \\
0^{11} & 0^{11} & 1^{11} & 0^{11} \\
0^{11} & 0^{11} & 0^{11} & 1^{11}
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}.
\tag{3.5}
$$

The cyclic code $Z$ of length 11 with generator polynomial $(x - 1)^g$ (and similarly the one with generator polynomial $(x - 1)^h$) has weight enumerator

$$
x^{11} + 132x^5 y^6 + 110x^2 y^9.
\tag{3.6}
$$

In particular it contains more words of weight 6 than of weight 9. This shows that the dimension of $C e_i$ over $\mathbb{F}_3^3$ is 2 for both $i = 2, 3$, since otherwise one of them has dimension $\geq 3$ and therefore contains all words $(0, 0, c, ac)$ for all $c \in Z$ and some $\alpha \in \mathbb{F}_3^3$. Not all of them can have weight $\geq 15$. Similarly one sees that the codes $C e_i \leq \mathbb{F}_3^{3^i}$ have minimum distance 3 for $i = 2, 3$. So we may choose generator matrices

$$
G_1 := \begin{pmatrix} 1 & 0 & a & b \\
0 & 1 & c & d \end{pmatrix}, \quad G_2 := \begin{pmatrix} 1 & 0 & a' & b' \\
0 & 1 & c' & d' \end{pmatrix}
\tag{3.7}
$$

with $(a \ b) \in GL_2(\mathbb{F}_3^3)$ and $(a' \ b') = -((a \ b)^{-1})^T$. To obtain $\mathbb{F}_3^3$-generator matrices for the corresponding codes $C e_2$ and $C e_3^3$ of length 48, we choose a generator matrix $g_1 \in \mathbb{F}_3^{3 \times 48}$ of the cyclic code $Z$ of length 11 with generator polynomial $(x - 1)^g$ and the corresponding dual basis $g_2 \in \mathbb{F}_3^{3 \times 11}$ of the cyclic code with generator polynomial $(x - 1)^h$. We compute the action of $\sigma$ (the multiplication with $x$) and represent this as left multiplication with $z_{11} \in \mathbb{F}_3^{3 \times 5}$ on the basis $g_1$. If $a = \sum_{i=0}^4 a_i z_{11}^i \in \mathbb{F}_3^5$ with $a_i \in \mathbb{F}_3$, then the entry $a$ in $G_1$ is replaced by $\sum_{i=0}^4 a_i z_{11}^i g_1 \in \mathbb{F}_3^{3 \times 11}$ and analogously for $G_2$, where we use of course the matrix $g_2$ instead of $g_1$. Replacing the code by an equivalent one we may choose $a, b, c$ as orbit representatives of the action of $-z_{11}$ on $\mathbb{F}_3^5$.

A generator matrix of $C$ is then given by

$$
\begin{pmatrix}
L_0 & R_0 \\
G_1 & 0 \\
G_2 & 0
\end{pmatrix}.
\tag{3.8}
$$

All codes obtained this way are equivalent to the Pless code $P_{48}$. \qed
Lemma 3.4. If \( p = 23 \), then \( C \equiv P_{48} \) or \( C \equiv Q_{48} \).

Proof. Let \( \sigma \in \text{Mon}(C) \) be of order 23. Since \( (x^{23} - 1) = (x - 1)gh \in F_3[x] \) for irreducible polynomials \( g, h \) of degree 11,

\[
F_3(\sigma) \cong F_3 \oplus F_{311} \oplus F_{311}.
\]

(3.9)

Let \( e_1, e_2, e_3 \in F_3(\sigma) \) denote the primitive idempotents. Then \( C = Ce_1 \oplus Ce_2 \oplus Ce_3 \) with \( C(\sigma) = Ce_1 \oplus Ce_3 \) of dimension 2 and \( Ce_2 = Ce_3 \leq (F_{311} \oplus F_{311})^2 \). Since all weights of \( C \) are multiples of 3, this leaves just one possibility for \( C(\sigma) \) (up to equivalence):

\[
C(\sigma) = \left\langle (1^{23},0^{23},1,0),(0^{23},1^{23},0,1) \right\rangle.
\]

(3.10)

The codes \( Ce_2 \) and \( Ce_3 \) are codes of length 2 over \( F_{311} \) such that \( \dim(Ce_2) + \dim(Ce_3) = 2 \). Note that the alphabet \( F_{311} \) is identified with the cyclic code of length 23 with generator polynomial \( (x - 1)g \) (resp., \( (x - 1)h \)). These codes have minimum distance \( 9 < 15 \), so \( \dim(Ce_2) = \dim(Ce_3) = 1 \) and both codes have a generator matrix of the form \( (1,t) \) (resp., \( (1,1-t^{-1}) \)) for \( t \in F_{311}^* \). Going through all possibilities for \( t \) (up to the action of the subgroup of \( F_{311}^* \) of order 23) the only codes \( C \) for which \( (C(\sigma) \oplus Ce_2 \oplus Ce_3 \) have minimum distance \( \geq 15 \) are the two known extremal codes \( P_{48} \) and \( Q_{48} \).

Lemma 3.5. If \( p = 47 \), then \( C \equiv Q_{48} \).

Proof. The subcode \( C_0 := \{ c \in F^{47}_3 | (c,0) \in C \} \) is a cyclic code of length 47, dimension 23, and minimum distance \( \geq 15 \). Since \( x^{47} - 1 = (x - 1)gh \in F_3[x] \) for irreducible polynomials \( g, h \) of degree 23, \( C_0 \) is the cyclic code with generator polynomial \( (x - 1)g \) (or equivalently \( (x - 1)h \)) and \( C = \langle (C_0,0), 1 \rangle \leq F_{48}^3 \) is the extended quadratic residue code.

\[ \square \]

3.2. Automorphisms of Order 2

As above let \( C = C^\perp \leq F_{48}^3 \) be an extremal self-dual ternary code. Assume that \( \sigma \in \text{Mon}(C) \) such that the permutational part \( \pi(\sigma) \) has order 2. Then \( \sigma^2 = \pm 1 \) because of Lemma 2.1. If \( \sigma^2 = -1 \), then \( \sigma \) is conjugate to a block diagonal matrix with all blocks \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \): \( f \) and \( C \) is a Hermitian self-dual code of length 24 over \( F_9 \). Such automorphisms \( \sigma \) with \( \sigma^2 = -1 \) occur for both known extremal codes.

If \( \sigma^2 = 1 \), then \( \sigma \) is conjugate to a block diagonal matrix

\[
\sigma \sim \text{diag}\left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^t, 1^t, (-1)^a \right)
\]

(3.11)

for \( t, a, f \in \mathbb{N}_0, 2t + a + f = 48 \).

Proposition 3.6. Assume that \( \sigma \in \text{Mon}(C), \sigma^2 = 1 \) and \( \pi(\sigma) \neq 1 \). Then either \( (t,a,f) = (24,0,0) \) or \( (t,a,f) = (22,2,2) \). Automorphisms of both kinds are contained in \( \text{Aut}(P_{48}) \).
Proof. (1) Wlog \( f \leq a \): Replacing \( \sigma \) by \( -\sigma \) we may assume without loss of generality that \( f \leq a \).

(2) \( f - t \in 4\mathbb{Z} \): By Lemma 2.3 the code \( C(\sigma)^* \leq \mathbb{F}_3^{f+t-f} \) is a self-dual code with respect to the inner product \( \langle x, y \rangle = -\sum_{i=1}^f x_i y_i + \sum_{j=1}^f x_j y_j \). This space only contains a self-dual code if \( f - t \) is a multiple of 4.

(3) \( t + f \in \{22, 24\} \): The code \( C(\sigma)^* \leq \mathbb{F}_3^{f+t-f} \) has dimension \((t + f)/2\) and minimum distance \( \geq 15/2 \) and hence minimum distance \( \geq 8 \). By [8] this implies that \( t + f \geq 22 \). Since \( t + a \geq t + f \) and \((t + a) + (t + f) = 48 \) this only leaves these two possibilities.

(4) \( t + f \neq 22 \): We first treat the case \( f \leq 14 \). Then \( K^* \equiv \ker(\pi_t) \) is a code of length \( f \leq 14 \) and minimum distance \( \geq 15 \) and hence trivial. So \( \pi_t \) is injective and

\[
C(\sigma) \equiv D := \pi_t(C(\sigma)) \leq \mathbb{F}_3^t, \quad \dim(D) = 11, \quad d(D) \geq \left\lfloor \frac{15 - f}{2} \right\rfloor.
\]  

(3.12)

Using [8] and the fact that \( f - t \) is a multiple of 4, this only leaves the cases \((t, f) \in \{(19, 3), (21, 1)\}\). To rule out these two cases we use the fact that \( D \) is the dual of the self-orthogonal ternary code \( D^\perp = \pi_t(\ker(\pi_f)) \). The bounds in [9] give \( d(D) \leq 5 < (15 - 3)/2 \) for \( t = 19 \) and \( d(D) \leq 6 < (15 - 1)/2 \) for \( t = 21 \).

If \( f \geq 15 \), then \( t \leq 7 \) and \( K^* \equiv \ker(\pi_t) \) has dimension \( f - t > 0 \) and minimum distance \( \geq 15 \). This is easily ruled out by the known bounds (see [8]).

(5) If \( t + f = 24 \) then either \((t, f) = (24, 0)\) or \((t, f) = (22, 2)\). Again the case \( f > t \) is easily ruled out using dimension and minimum distance of \( K^* \) as before.

So assume that \( f \leq t \), and let \( D = \pi_t(C(\sigma)) \) as before. Then \( \dim(D) = 12 \) and using [8] one gets that

\[
(t, f) \in \{(24, 0), (22, 2), (20, 4)\}.
\]  

(3.13)

Assume that \( t = 20 \). Then there is some self-dual code \( \Lambda = \Lambda^\perp \leq \mathbb{F}_3^{20} \) such that

\[
D^\perp = \pi_t(\ker(\pi_f)) \leq \Lambda = \Lambda^\perp \leq D.
\]  

(3.14)

Clearly also \( d(\Lambda) \geq d(D) \geq 6 \), so \( \Lambda \) is an extremal ternary code of length 20. There are 6 such codes, and none of them has a proper overcode with minimum distance 6.

Remark 3.7. If \( \sigma \in \text{Mon}(C) \) is some automorphism of order 4, then \( \sigma^2 = -1 \) or \( \sigma^2 \) has type \((24, 0, 0)\) in the notation of Proposition 3.6.

Proof. Assume that \( \sigma \in \text{Mon}(C) \) has order 4 but \( \sigma^2 \neq -1 \). Then \( \tau = \sigma^2 \) is one of the automorphisms from Proposition 3.6 and so \( \sigma \) is conjugate to some block diagonal matrix

\[
\sigma \sim \text{diag}\left(\left(\begin{array}{ccc}0 & 1 & 0 \\0 & 0 & 1 \\0 & 0 & 0\end{array}\right)^{t/2}, \left(\begin{array}{c}0 \1 \0\end{array}\right)^{f/2}, \left(\begin{array}{c}0 \0 \0\end{array}\right)^{a/2}\right)\).
\]  

(3.15)
If \( t = 22 \) and \( f = 2 \) then the fixed code of \( \sigma \) is a self-dual code in \( \langle (1, 1, 1, 1) \rangle^{t/2} \oplus \langle (1, 1) \rangle^{f/2} \) and \( C(\sigma)^* \leq \mathbb{F}_3^{t/2+f/2} \) is a self-dual code with respect to the form \( (x, y) := \sum_{i=1}^{t/2} x_i y_i - \sum_{i=t/2+1}^{t/2+f/2} x_i y_i \) which implies that \( t/2 - f/2 \) is a multiple of 4, a contradiction.

For the two known extremal codes all automorphisms \( \sigma \) of order 4 satisfy \( \sigma^2 = -1 \). It would be nice to have some argument to exclude the other possibility.

References
