EVEN PERFECT NUMBERS AND THEIR EULER’S FUNCTION

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ABSTRACT. The purpose of this article is to prove some results on even perfect numbers and on their Euler’s function. The results obtained are all straightforward deductions from well-known elementary number theory.

KEY WORDS AND PHRASES. Perfect number; triangular number; Euler’s function; number of divisors function.

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1. INTRODUCTION.

A positive integer is called a perfect number if it is equal to the sum of its positive divisors excluding itself.

The nth triangular number is the sum of the first n-positive integers

\[ T(n) = \sum_{k=1}^{n} \frac{k}{2} n(n+1) \]

Euler’s function \( \phi(n) \) is the number of positive integers less than or equal to \( n \) and relatively prime to \( n \).

The number of divisors function \( d(n) \) is the number of positive divisors of \( n \).

2. MAIN RESULTS.

The proof of the following Theorem 1 can be found in many elementary number theory books; see, for example, [1:p. 98].

THEOREM 1. If \( n \) is an even perfect number, there exists a prime \( 2^n-1 \) such that \( n = 2^n-1 \).

THEOREM 2. If \( T(p_i) \) is any even perfect number, where \( p_i \) is prime, and if \( p_k \) is the first prime in the sequence \( \{p_2, p_3, \ldots, p_j, \ldots\} \) where \( p_j = 2p_{j-1} + 1 \), then \( T(p_k) \) is the next even perfect number.

PROOF. It follows from Theorem 1 that an even perfect number is of the form \( 2^{n-1}(2^n-1) \), where \( 2^n-1 \) is prime. Now, \( 2^{n-1}(2^n-1) \) can be written as \( T(p_i) \), where \( p_i = 2^n-1 \). Let \( p_1 \) be any composite term of the sequence \( \{p_2, p_3, \ldots, p_j, \ldots\} \). It can be shown that \( p_1 = 2^{n+1}-1 \), using the facts \( p_1 = 2^n-1 \) and \( p_j = 2p_{j-1}+1 \). Now, it follows from Theorem 1 that \( T(p_1) = 2^{n+1}-2(2^{n+1}-1) \) is
not an even perfect number. Let \( p_k \) be the first prime in the sequence \( \{p_2, p_3, \ldots, p_j, \ldots\} \). As before, \( p_k = 2^{n+k-1} - 1 \). Observe that \( \mathcal{T}(p_k) = 2^{n+k-2}(2^{n+k-1} - 1) \) is of the form \( 2^{m-1}(2^{m}-1) \), where \( 2^{m}-1 \) is prime and thus \( \mathcal{T}(p_k) \) is an even perfect number by Theorem 1.

**EXAMPLE.** \( \mathcal{T}(3) = \frac{1}{2} (3)(4) = 6, \ \mathcal{T}(7) = \frac{1}{2} (7)(8) = 28. \)

\( \mathcal{T}(31) = \frac{1}{2} (31)(32) = 496. \ \mathcal{T}(127) = \frac{1}{2} (127)(128) = 8128, \ldots. \)

**THEOREM 3.** If \( n = 2^{m-1}(2^{m}-1) \), then, \( n = 1^{a} + 3^{a} + \ldots + [2^{(m+1)/2} - 1]^{3} \).

**PROOF.** Observe that \( 2^{(m+1)/2} = 2k \), where \( k = 2^{(m-1)/2} \). Now, consider

\[
1^{a} + 2^{a} + 3^{a} + \ldots + (2k-1)^{a} + (2k)^{a} = \left[ 1 + 2 + 3 + \ldots + (2k-1) + (2k) \right]^{2} = \left[ \frac{1}{2} (2k)(2k+1) \right]^{2}
\]

which implies that 

\[
1^{a} + 2^{a} + 3^{a} + \ldots + (2k-1)^{a}
\]

\[= \frac{k^{2}(2k+1)^{2}}{2} - [2^{a} + 2^{a} + \ldots + (2k)^{a}]
\]

\[= \frac{k^{2}(2k+1)^{2}}{2} - 2^{a}(1^{a} + 2^{a} + \ldots + k^{a})
\]

\[= \frac{k^{2}(2k+1)^{2}}{2} - 8(1 + 2 + \ldots + k)^{2}
\]

\[= \frac{k^{2}(2k+1)^{2}}{2} - 8\left[ \frac{1}{2} k(k+1) \right]^{2}
\]

\[= \frac{k^{2}(2k+1)^{2}}{2} - 2^{2}(k^{2}+1) = k^{2}(2k^{2} - 1).
\]

Since \( k = 2^{(m-1)/2} \), it follows that 

\[1^{a} + 2^{a} + 3^{a} + \ldots + [2^{(m+1)/2} - 1]^{3} = 2^{m-1}(2^{m}-1) = n.
\]

The following Corollary 1 follows from Theorem 3.

**COROLLARY 1.** If \( n \) is an even perfect number \( 2^{p-1}(2^{p}-1) \), then 

\[n = 1^{a} + 3^{a} + \ldots + [2^{(p+1)/2} - 1]^{3}.
\]

**EXAMPLE.** \( 496 = 1^{a} + 3^{a} + 5^{a} + 7^{a}; \ p = 5. \)

The proof of the following Theorem 4 can also be found in many elementary number theory books; see, for example [1: p. 63].

**THEOREM 4.** If \( n = p_{1}^{a_{1}} p_{2}^{a_{2}} \ldots p_{k}^{a_{k}} \)

then \( \phi(n) = n(1 - \frac{1}{p_{1}})(1 - \frac{1}{p_{2}}) \ldots (1 - \frac{1}{p_{k}}) \), where 

\( p_{1}, p_{2}, \ldots, p_{k} \) are distinct primes and \( a_{1}, a_{2}, \ldots, a_{k} \) are positive integers.

As a consequence of Theorem 4, one can easily obtain Theorem 5, Corollary 2, and Corollary 3

**THEOREM 5.** \( n = 2^{p-1}(2^{p}-1) \) is an even perfect number if and only if

\( \phi(n) = 2^{p-1}(2^{p-1} - 1) \), where \( 2^{p-1} \) is prime.

**COROLLARY 2.** If \( n \) is an even perfect number, then \( \phi(n) = n - 4^{p-1} \).

**EXAMPLE.** \( \phi(8128) = \phi(2^{6}) \phi(127) = 4032 = 8128 - 4^{6}. \)

**COROLLARY 3.** If \( n \) is an even perfect number, then \( \phi(n) = \frac{n}{2} - 2^{p-2} \).

**THEOREM 6.** If \( n_{1}, n_{2}, \ldots, n_{k} \) are \( k \)-distinct even perfect numbers, then

\( \phi(n_{1} n_{2} \ldots n_{k}) = 2^{k-1} \phi(n_{1}) \phi(n_{2}) \ldots \phi(n_{k}). \)

**PROOF.** \( \phi(n_{1} n_{2} \ldots n_{k}) = \phi[2^{p_{1}-1}(2^{p_{1}-1} - 1) \ldots 2^{p_{k}-1}(2^{p_{k}-1} - 1)] = \phi[2^{p_{1}+p_{2}+\ldots+p_{k}-k+p_{1}+p_{2}+\ldots+p_{k}-k}] = \phi[2^{p_{1}+p_{2}+\ldots+p_{k}}] \phi[2^{p_{1}+p_{2}+\ldots+p_{k}}] = \phi[2^{p_{1}+p_{2}+\ldots+p_{k}}] \phi[2^{p_{1}} \phi[2^{p_{2}} \ldots \phi[2^{p_{k}}]). \)
The following Theorem 7 is proved in many books on elementary number theory; see, for example, [1: p. 96].

**THEOREM 7.** If \( n = \Pi_{i=1}^{k} p_i^{\alpha_i} \) then \( d(n) = \Pi_{i=1}^{k} (1 + \alpha_i) \), where \( p_i, \alpha_i \geq 1 \) are distinct primes and \( \alpha_i \geq 1 \) are positive integers, and \( d(n) \) is the number of divisors function.

**THEOREM 8.** If \( n = \Pi_{i=1}^{k} p_i^{\alpha_i} \) and \( d(n) \) is an even perfect number \( 2^{p-1}(2^p - 1) \), then

\[
\begin{align*}
&1) \quad p \geq k. \\
&2) \quad \alpha_j = 2^{\mu_j} (2^p - 1) - 1 \text{ for exactly one } j \text{ such that } 1 \leq j \leq k \text{ and } \\
&\quad \mu_j \geq 0. \\
&3) \quad \alpha_i = 2^{\mu_i} - 1, \text{ where } \mu_i > 0, \ 1 \leq i \leq k, \ i \neq j. \\
&4) \quad \sum_{i=1}^{k} \mu_i = p - 1.
\end{align*}
\]

**PROOF.** From Theorem 5, one obtains \( d(n) = \Pi_{i=1}^{k} (1 + \alpha_i) = 2^{p-1}(2^p - 1) \), which implies that \( (2^p - 1) \) divides exactly one of the factors \( (1 + \alpha_i), 1 \leq i \leq k \), say \( (1 + \alpha_j) \). Thus \( (1 + \alpha_j) = (2^p - 1) \cdot \lambda \) for some \( \lambda \) and exactly one \( j \) such that

\[
1 \leq j \leq k, \text{ and } (2^p - 1) \cdot \lambda \cdot \Pi_{i=1}^{k} (1 + \alpha_i) = 2^{p-1}(2^p - 1), \text{ that is,}
\]

\[
\lambda \cdot \Pi_{i=1}^{k} (1 + \alpha_i) = 2^{p-1}, \text{ which implies that } 1 + \alpha_i = 2^{\mu_i}, \ 1 \leq i \leq k.
\]

\[
1 \neq j, \mu_i > 0; \ \lambda = 2^{\mu_j}, \mu_j \geq 0 \ \text{ and } \sum_{i=1}^{k} \mu_i = p - 1, \text{ which is (iv).}
\]

Observe that \( \sum_{i=1}^{k} \mu_i \geq k - 1 \) since \( \mu_i > 0 \) for \( i \neq j \) and \( \mu_j \geq 0 \). Thus, \( p - 1 \geq k - 1 \) or \( p \geq k \), which is (i). Now, \( (1 + \alpha_j) = (2^p - 1) \cdot \lambda = (2^p - 1) \cdot 2^{\mu_j} \) for exactly one \( j \), such that \( 1 \leq j \leq k \) and \( \mu_j \geq 0 \) implies that \( \alpha_j = 2^{\mu_j} \cdot (2^p - 1) - 1 \) for exactly one \( j \) such that \( 1 \leq j \leq k \) and \( \mu_j \geq 0 \), which proves (ii).
Finally, $1 + \alpha_i = 2^{\mu_i}, \ 1 \leq i \leq k, \ i \neq j, \ \mu_i > 0$ implies that 

$\alpha_i = 2^{\mu_i} - 1, \ 1 \leq i \leq k, \ i \neq j, \ \mu_i > 0$, which proves (tit).

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REFERENCES

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